ANTI-RAMSEY NUMBERS FOR DISJOINT COPIES OF GRAPHS

Izolda Gorgol and Agnieszka Görlich

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Abstract. A subgraph of an edge-colored graph is called rainbow if all of its edges have different colors. For a graph $G$ and a positive integer $n$, the anti-Ramsey number $ar(n, G)$ is the maximum number of colors in an edge-coloring of $K_n$ with no rainbow copy of $H$. Anti-Ramsey numbers were introduced by Erdős, Simonovits and Sós and studied in numerous papers. Let $G$ be a graph with anti-Ramsey number $ar(n, G)$. In this paper we show the lower bound for $ar(n, pG)$, where $pG$ denotes $p$ vertex-disjoint copies of $G$. Moreover, we prove that in some special cases this bound is sharp.

Keywords: anti-Ramsey number, rainbow number, disjoint copies.

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1. INTRODUCTION

A subgraph of an edge-colored graph is called rainbow if all of its edges have different colors. For a graph $G$ and a positive integer $n$, the anti-Ramsey number $ar(n, G)$ is the maximum number of colors in an edge-coloring of $K_n$ with no rainbow copy of $G$. Anti-Ramsey numbers were introduced by Erdős et al. [4]. They showed that these are closely related to Turán numbers. Since then numerous results were established for a variety of graphs $H$, including among others cycles [1, 11, 13], matchings [5, 9, 17], trees [10, 12] and cycles with an edge added [8, 15]. The paper of Fujita, Magnant and Ozeki [6] presents the survey of results of that type.

In this paper we consider the following problem. Given a connected graph $G$, the anti-Ramsey number $ar(n, G)$, we ask what can be said about $ar(m, pG)$, where $pG$ denotes $p$ vertex-disjoint copies of $G$. We give the lower bound for this number and discuss the sharpness of it. As far as we know the only considered graphs of this type were matchings.
2. PRELIMINARIES

Graphs considered below will always be simple. Throughout the paper we use the standard graph theory notation. For a graph $G$ the order of $G$ is denoted by $|G|$ and the size is denoted by $\|G\|$. $K_n$ and $pG$ stand for, respectively, the complete graph on $n$ vertices and the disjoint union of $p$ copies of a graph $G$. A degree of a vertex $v$ in a graph $G$ is denoted by $d_G(v)$ and by $N_G(v)$ and $N_G[v]$ its open and closed neighborhoods, respectively. For a graph $G$ and its subgraph $H$ by $G - H$ we mean a graph obtained from $G$ by deleting all vertices of $H$ with all incident edges. If $W \subseteq V(G)$, then $G[W]$ denotes the subgraph of $G$ induced by $W$. For a set $S$, by $|S|$ we denote the cardinality of $S$.

Additionally, we introduce the following notation. $C(G)$ is a set of colors used on the edges of a graph $G$, $C(v)$ is a set of colors used on the edges incident to a vertex $v$ and $c(e)$ denotes a color of the edge $e$. For a given coloring of the edges of $K_n$ we choose exactly one edge in each color. A subgraph $F$ such that $V(F) = V(K_n)$ induced by these edges we call a selective subgraph.

We will need the following theorems.

**Theorem 2.1** ([4]). $ar(m, K_3) = m - 1$ for $m \geq 3$.

**Theorem 2.2** ([16]). If $G$ is a graph with $n \geq 3$ vertices such that $\|G\| > \binom{n-1}{2} + 1$, then $G$ has a Hamiltonian cycle.

**Theorem 2.3** ([13]). If $m \geq k \geq 3$ and $r$ is the reminder of the division $m$ by $k - 1$, then

$$ar(m, C_k) = \left\lfloor \frac{m}{k-1} \right\rfloor \binom{k-1}{2} + \binom{r}{2} + \left\lceil \frac{m}{k-1} \right\rceil.$$

**Theorem 2.4** ([11, 14]). $ar(m, K_{1,3}) = \left\lfloor \frac{mp}{2} \right\rfloor + 1, m \geq 4$.

**Theorem 2.5** ([11, 14]). $ar(m, K_{1,4}) = m + 1, m \geq 5$.

3. LOWER BOUND

**Theorem 3.1.** Let $G$ be an arbitrary connected graph on $n \geq 3$ vertices and $m \geq p|V(G)|$. Then

$$ar(m, pG) \geq \max \left\{ \left\lfloor \frac{pm-2}{2} \right\rfloor + 1, ar(m - p + 1, G) + (p - 1)m - \left\lfloor \frac{p}{2} \right\rfloor \right\}.$$

**Proof.** We color the edges of $K_m$ as follows. To obtain the first number we choose $K_{pm-2}$ and color it rainbowly and we color the remaining edges with one extra color. In such a way we do not obtain any rainbow $pG$ and use exactly $\left\lfloor \frac{pm-2}{2} \right\rfloor + 1$ colors.

To obtain the second number we choose $K_{p-1}$ and color it rainbowly, then we color the edges of $K_m - K_{p-1}$ with next $ar(m - p + 1, G)$ colors without producing
rainbow $G$ and finally we color the remaining edges each with next distinct colors. In such a way we do not obtain any rainbow $pG$ and use exactly
\[ ar(m - p + 1, G) + (p - 1)(m - p + 1) + \binom{p - 1}{2} = ar(m - p + 1, G) + (p - 1)m - \binom{p}{2} \]
colors, so the theorem is proved.

It is worth to pay attention to the fact that the lower bound from Theorem 3.1 is not appropriate for a matching. In this case assuming $G = K_2$ it is reasonable to put $ar(m, K_2) = 0$. But in the construction of the coloring we must not use 0 colors on the rest of the graph. Similarly the colorings are based on the fact that by adding one new color we do not produce any copy of $G$ which is not true for $G = K_2$. That is why matchings need a different treating. It is done in [5, 9, 17] by using an appropriate Turán number.

From this point of view $G = P_3$ is the smallest graph to consider.

In our paper we are interested in selecting graphs for which this lower bound can be sharp. We do not focus on cases when
\[ \max \left\{ \frac{pm - 2}{2} + 1, \ ar(m - p + 1, G) + (p - 1)m - \binom{p}{2} \right\} = \frac{pm - 2}{2} + 1, \]
since this can happen only for finitely many values of $m$. It is so, as the first expression is a constant and the second one is at least linear in $m$.

We state the following conjecture.

**Conjecture 3.2.** Let $G$ be a connected graph on $n \geq 3$ vertices and $m \geq p|V(G)|$. Then
\[ ar(m, pG) = \begin{cases} 7 & \text{for } m = 6, \\ m & \text{for } m \geq 7. \end{cases} \]
if and only if $G$ is a tree.

In the next paragraphs we give the reasons which motivated us to state such a conjecture.

### 3.1. DISJOINT PATHS

It is easy to see that $ar(m, P_3) = 1$. By Theorem 3.1, it can be obtained that $ar(m, 2P_3) \geq m$ for $m \geq 7$ and $ar(6, 2P_3) \geq 7$. The next theorem shows that this lower bound is sharp. The result was also achieved by Bialostocki, Gilboa and Roditty [2], but with a different method of the proof, so we put the theorem into the paper.

**Theorem 3.3.**
\[ ar(m, 2P_3) = \begin{cases} 7 & \text{for } m = 6, \\ m & \text{for } m \geq 7. \end{cases} \]
Proof. The lower bound for \( m \geq 7 \) results from Theorem 3.1. For \( m = 6 \) we color the edges of a subgraph \( K_m = K \) with \( m + 1 \) colors and assume that there is no rainbow \( 2P_3 \). By Theorem 2.1, there is a rainbow triangle \( T \) with vertices \( \{u, v, w\} \). Let \( C_R = C(K) \setminus C(T) \) and \( V(K - T) = \{x_1, x_2, \ldots, x_{m-3}\} \). Note that if there is an edge \( e \in E(K - T) \) with \( c(e) \in C_R \), then we obtain a rainbow \( 2P_3 \) consisting of the edge \( e \), an edge \( e' \in E(K - T) \) incident to it and to edges from \( E(T) \) of colors different from \( c(e') \). A contradiction. Hence we can assume that all edges of colors from \( C_R \) are placed between \( T \) and \( K - T \).

Since \( |C_R| = m - 2 \), at least one vertex from \( T \) is joined to at least two vertices from \( K - T \) with edges of distinct colors from \( C_R \). Let \( u \) be this vertex, \( c(uw_1), c(uw_2) \in C_R \), \( c(uw_1) \neq c(uw_2) \) and \( C'_R = C_R \setminus \{c(uw_1), c(uw_2)\} \).

Note that we can assume that for each \( i \in \{3, \ldots, m - 3\} \) we have that \( c(x_i, v) \notin C'_R \) and \( c(x_i, w) \notin C'_R \), since otherwise there would be a rainbow \( 2P_3 \): \( x_1uw_2, x_jvw (x_iw, wv, \text{respectively}) \) for a certain \( j \in \{3, \ldots, m - 3\} \). Since \( |C'_R| = m - 4 \), there is an edge of color from \( C'_R \) between \( \{x_1, x_2\} \) and \( \{v, w\} \). Let \( x_1v \) be this edge. Similarly as above we obtain that \( c(x_i, u) \notin C'_R \setminus \{c(x_i, v)\} \) for each \( i \in \{3, \ldots, m - 3\} \), otherwise \( x_2ux_j \), \( x_1vw \) is a rainbow \( 2P_3 \) for certain \( j \in \{3, \ldots, m - 3\} \). Now there are only two edges left \( (x_2v \text{ and } x_2w) \) which are allowed to be colored with colors from \( C'_R \setminus \{c(x_1, v)\} \). But \( |C'_R \setminus \{c(x_1, v)\}| = m - 3 \). A contradiction. Hence we have that \( c(x_3x_3) \in C(T) \).

The next theorem deals with three copies of \( P_3 \). It is a special case of a more general result obtained by Gilboa and Roditty [7], namely \( ar(m, pP_3) = (p - 1)(m - \frac{p}{2}) + 1 \) for \( m > 5p + 1 \). By a different method of the proof, we managed to decrease the constraint for \( m \) from 16 to 12 for \( p = 3 \).

Theorem 3.4. \( ar(m, 3P_3) = 2m - 2 \) for \( m > 12 \).

Proof. The lower bound results from Theorem 3.1. To show the upper bound we color the edges of a complete graph \( K_m = K \) with \( 2m - 1 \) colors arbitrarily and assume that there is no rainbow \( 3P_3 \).

Let \( F \) be a selective subgraph of \( K \) containing the longest cycle and \( l \) denote its length. Since \( |V(F)| = m \) and \( |E(F)| = 2m - 1 \), such a selective subgraph can be chosen. Moreover, there are at most two vertex-disjoint cycles in \( F \), since otherwise a rainbow \( 3P_3 \) is in \( K \).

Note that if \( l \geq 9 \), then obviously a rainbow \( 3P_3 \) is contained in \( K \). Moreover, by Theorem 2.3, \( l \geq 5 \). Therefore \( l \in \{5, 6, 7, 8\} \). Let \( C_l \) be the subgraph of \( F \) being the longest cycle.

Let \( F_l = F[V(C_l)], B = F - C_l, R = \{vw : v \in V(C_l), w \in V(B)\} \) and \( N = \{w \in V(B) : \text{there exists } v \in V(C_l) \text{ such that } vw \in E(F)\} \). Note that

\[
\|F\| = \|F_l\| + \|B\| + |R|.
\]
Case 1. \( l = 8 \).

Observe that \(|N| = 0\), since otherwise a rainbow 3\(P_3\) is in \(K\).

We show that there is at most one edge in \(B\). Suppose that there are vertices \(x_1, x_2, x_3, x_4 \in V(B)\) such that \(x_1x_2, x_3x_4 \in E(F)\). If \(x_2 = x_3\), then \(x_1x_2x_4\) is rainbow. If \(x_2 \neq x_3\), then at least one of paths \(A^1 = x_1x_2x_3\) or \(A^2 = x_4x_3x_2\) is rainbow. Possibly deleting the edge with color \(c(x_2x_3)\) in \(C_8\) we obtain a rainbow subgraph of \(C_8\) which contains 2\(P_3\). It contradicts the assumption that there is no 3\(P_3\) in \(F\). Therefore, \(|B| \leq 1\).

Now, let \(e = xy\) be an edge in \(K\) such that \(x \in V(C_8)\) and \(y \in V(B)\). Obviously, \(e \notin E(F)\) and \(c(e)\) is one of colors from \(C(C_8)\). Then \(|F_b| \leq 23\).

Otherwise, deleting the edge with color \(c(e)\) in \(F_b\), by Theorem 2.2 we obtain a rainbow hamiltonian graph of order 8 without a color \(c(e)\) and joining \(e\) to the hamiltonian cycle we obtain a rainbow \(P_9\) in \(K\) and hence a rainbow 3\(P_3\) is in \(K\). Hence,

\[|F| = |F_b| + |B| \leq 23 + 1 = 24 < 2m - 1,\]

a contradiction.

Case 2. \( l = 7\).

Observe that \(|N| \leq 1\). Otherwise it is easy to obtain 3\(P_3\) in \(F\).

Analogously as in previous cases, we can show that \(|B| \leq 1\).

Suppose that \(|N| = 0\). So, \(|F_7| \leq 21\). Hence,

\[|F| \leq 21 + 1 = 22 < 2m - 1,\]

a contradiction.

Assume that \(|N| = 1\) and \(N = \{x\}\). Similarly as in a previous case, by Theorem 2.2, we obtain that \(F[V(C_7) \cup \{x\}]\) contains at most 23 edges. Hence,

\[|F| \leq 23 + 1 = 24 < 2m - 1,\]

a contradiction.

Case 3. \( l = 6\).

Analogously as in Case 1, we can show that \(|B| \leq 1\).

Denote the consecutive vertices in \(C_6\) by \(c_0, c_1, \ldots, c_5\) and by \(d_R(x)\) the number of edges in \(R\) incident with \(x\). Observe that since \(|B| \leq 1\), we have

\[|R| \geq 2m - 1 - (15 + 1) = 2m - 17 \geq m - 4\]

for every \(m > 12\). It implies that \(|R| > 0\) and there are at least two distinct vertices \(c_i, c_j\) such that \(d_R(c_i) \geq 1\) and \(d_R(c_j) \geq 1\). Without loss of generality, we can assume that \(d_R(c_0) \geq 2\).

The assumption that \(d_R(c_k) \geq 1\) for certain \(k \in \{1, 2, 4, 5\}\) leads us to a contradiction with the assumption that there is no 3\(P_3\) in \(F\). Therefore \(c_3\) is the other vertex with neighbors in \(N\) and moreover \(d_R(c_3) \geq 2\).

If \(|N| \leq 3\), then \(|R| \leq 6\), a contradiction. So \(|N| \geq 4\) which means that we can choose a rainbow 2\(P_3\) in \(F\) with middle vertices \(c_0\) and \(c_3\) and endpoints in \(N\).
The rainbow $3P_3$ in $K$ we can find as follows. If $c(c_1c_4) \notin C(2P_3)$, then we are done, since at least one of the paths $c_5c_4c_1$, $c_2c_1c_4$ is rainbow in $K$.

So suppose that $c(c_1c_4) \in C(2P_3)$. Without loss of generality let $xc_0y$ be one of above mentioned rainbow $2P_3$ and $c(c_1c_4) = c(xc_0)$. Then the other $P_3$ with middle vertex $c_3$, $c_2c_1c_4$ and $yc_0c_5$ form a rainbow $3P_3$ in $K$.

Hence we obtain a contradiction.

Case 4, $l = 5$.

Denote the consecutive vertices in $C_5$ by $\{c_0, c_1, \ldots, c_4\}$.

Suppose that $P_3 = P$ is contained in $B$. Note that either one can find rainbow $2P_3$ with one vertex in $V(B) \setminus V(P)$ and five vertices in $V(C_5)$ or for each $u \in (V(B) \setminus V(P))$ we have $c(u) = c(c_{i+2} \mod 5c_{i+3} \mod 5)$. In the latter case we have a rainbow $2P_3$; $c_0u$ $c_1$, $c_2u$c_3$, where $u_1, u_2 \in (V(B) \setminus V(P))$. The rainbow $2P_1$ forms a rainbow $3P_3$ with $P$. A contradiction.

Therefore we can assume that $B = sK_3 \cup (m - 5 - 2s)K_1$.

Note that each vertex $u \in V(B)$ is adjacent to at most two vertices on the cycle $(c_i, c_{i+2} \mod 5)$ otherwise $F$ contains a longer cycle.

Moreover, if at least one $u \in V(B)$ has two neighbors on the cycle, then $\|F_5\| \leq 8$ $(c_{i+1} \mod 5c_{i+4} \mod 5 \notin E(F), c_{i+1} \mod 5c_{i+3} \mod 5 \notin E(F))$ otherwise $F$ contains a longer cycle.

Finally, note that if $u_1u_2 \in E(F)$, then there are at most two edges between $\{u_1, u_2\}$ and $V(C_5)$ otherwise $F$ contains a longer cycle.

Hence if at least one $u \in V(B)$ has two neighbors on the cycle, then

$2m - 1 = \|F\| = \|F_5\| + \|B\| + |R| \leq 8 + s + 2s + 2(m - 5) - 2s = 2m - s - 2.\)

A contradiction.

If all $u \in V(B)$ have at most one neighbor on the cycle, then

$2m - 1 = \|F\| = \|F_5\| + \|B\| + |R| \leq 10 + s + (m - 5) = m + s + 5.$

Since $s \leq \lfloor \frac{m - 5}{2} \rfloor$, we have a contradiction. $\square$

Next we consider two copies of a star with three rays.

**Theorem 3.5.** Let $m \geq 69$. Then $ar(m, 2K_{1,3}) = \lfloor \frac{m - 1}{2} \rfloor + m$.

**Proof.** The lower bound results from Theorems 3.1 and 2.4. To show the upper bound we color the edges of a complete graph $K_m = K$ with $\lfloor \frac{m - 1}{2} \rfloor + m + 1$ colors arbitrarily and assume that there is no rainbow $2K_{1,3}$.

Let $F$ be a selective subgraph of $K$ chosen in such a way that the maximal degree $\Delta(F)$ is as big as possible and let $x_0$ be the vertex of $K$ such that $d(x_0) = \Delta(F) = d$. Note that, since $\lfloor \frac{m - 1}{2} \rfloor + m + 1 > m + 1 = ar(m, K_{1,4})$ (see Theorem 2.5), we can assume that $d \geq 4$. Obviously $d \leq m - 1$. Let $N_F(x_0) = \{x_1, x_2, \ldots, x_d\}$ and $V(F) \setminus N_F[x_0] = \{x_{d+1}, x_{d+2}, \ldots, x_m\}$. The latter set is empty if $d = m - 1$.

Let us consider the case $d \geq 8$ firstly. Let $F^- = F - x_0$. Note that

$$\|F^-\| \geq \|F\| - (m - 1) = \left\lfloor \frac{m - 1}{2} \right\rfloor + m + 1 - (m - 1) = \left\lfloor \frac{m - 1}{2} \right\rfloor + 2. \quad (3.1)$$
Since $|F^-| = m - 1$, we obtain that $K_{1,2} \subset F^-$. Without loss of generality let $x_1 x_2 x_3$ be this star with the center $x_1$. Note that there is no other edges with the end
$x_1$ in $F^-$ otherwise there would be rainbow $2K_{1,3}$ in $F$ (one star with center $x_1$ and
one with $x_0$).

Moreover, note that $G^-$ does not contain two edge-disjoint stars $K_{1,2}$. If $x_1 x_2 x_3$
and $x_2 x_3 x_1$ be such a stars with centers $x_1$ and $x_1$, respectively, then at least one of
the stars $x_1 x_2 x_3$, or $x_2 x_3 x_1$ would be rainbow in $K$ and form a rainbow $2K_{1,3}$
together with a certain star with a center $x_0$, even if $c(x_1 x_i) \in C(x_0)$. Therefore $G^-$
is a subset of (i) $K_{1,2} \cup \left\lfloor \frac{m-4}{2} \right\rfloor K_2$ or (ii) $P_4 \cup \left\lfloor \frac{m-5}{2} \right\rfloor K_2$ or (iii) $K_{3,3} \cup \left\lfloor \frac{m-4}{2} \right\rfloor K_2$. In case
(i) we have $\|F^-\| \leq 2 + \left\lfloor \frac{m-4}{2} \right\rfloor = \left\lfloor \frac{m}{2} \right\rfloor$ and in case (ii) $\|F^-\| \leq 3 + \left\lfloor \frac{m-5}{2} \right\rfloor = \left\lfloor \frac{m+1}{2} \right\rfloor$, which is a contradiction to (3.1). There is no similar contradiction in case (iii) only
in case $F^- \simeq K_3 \cup \left\lfloor \frac{m-4}{2} \right\rfloor K_2$. In that case let $x_1, x_2, x_3$ be the vertices of the
triangle and $x_i x_j$ be an edge of $F^-$, $i, j \notin \{x_1, x_2, x_3\}$. Now we look at colors of the edges in $K$. If $c(x_1 x_i) \notin \{c(x_1 x_2), c(x_1 x_3)\}$ or $c(x_2 x_i) \notin \{c(x_2 x_1), c(x_2 x_3)\}$ or $c(x_3 x_i) \notin \{c(x_3 x_1), c(x_3 x_2)\}$, then we have a rainbow star $K_{1,3}$ with a centrum $x_0$ for a certain $s \in \{1, 2, 3\}$ which forms a rainbow $2K_{1,3}$ with the rainbow $K_{1,3}$
with a centrum $x_0$. Therefore we can assume that $c(x_1 x_i) \in \{c(x_1 x_2), c(x_1 x_3)\}$ and $c(x_2 x_i) \in \{c(x_2 x_1), c(x_2 x_3)\}$ and $c(x_3 x_i) \in \{c(x_3 x_1), c(x_3 x_2)\}$. So there is a rainbow
star $K_{1,3} x_i x_i x_j$ with a centrum $x_i$ for a certain $s, t \in \{1, 2, 3\}$. So again we have
the rainbow $2K_{1,3}$ with the rainbow $K_{1,3}$ with a centrum $x_0$. A contradiction.

Now consider the case $4 \leq d \leq 7$. Note that each $x_i$, $i = 1, 2, \ldots, d$ can
have at most two neighbors in $\{x_{d+1}, x_{d+2}, \ldots, x_{m-1}\}$ otherwise we can easily find
$2K_{1,3}$ in $F$. So there is at most $d + 2d + \binom{d}{2}$ edges with at least one endpoint in
$\{x_0, x_1, x_2, \ldots, x_d\}$. Hence at least $\left\lfloor \frac{m-1}{2} \right\rfloor + m + 1 - 3d - \binom{d}{2}$ edges have both end-
points in $\{x_{d+1}, x_{d+2}, \ldots, x_{m-1}\}$. Note that at least one of these vertices has three
neighbors in this set, since

$$\left\lfloor \frac{m-1}{2} \right\rfloor + m + 1 - 3d - \binom{d}{2} > 2(m - d - 1)/2$$

for $m \geq 69$. So again we have $2K_{1,3}$ in $F$. A contradiction.

3.2. DISJOINT TRIANGLES

It is unlikely that the lower bound we discuss is sharp in any case. By the results
of Erdős et al. [3, 4], it follows that if $G$ is a graph which is not bipartite and does
not become bipartite after deleting a single edge, then $ar(m, G)$ and $ex(m, G^-)$ are
asymptotically equal, where $ex(m, H)$ denotes well known Turán number for a family
$H$ and $G^-$ is the family of all graphs obtained from $G$ by deleting one edge. Moreover,
recently Schiermeyer and Sötk [18] showed that for a graph $G$ with cyclomatic number
at least 2 the anti-Ramsey number $ar(m, G)$ cannot be bounded above by a function
which is linear in $m$.

As an example we present the following theorem.

Theorem 3.6. Let $m \geq 6$. Then $ar(m, 2K_3) \geq \left\lfloor \frac{m^2}{4} \right\rfloor + 1$. 


Proof. To construct an appropriate coloring of the edges of $K_m$ we proceed as follows. We choose a triangle-free subgraph $H$ with maximum possible number of edges (Turán graph) and assign to each edge a different color. Then we put one extra color to all remaining edges. Certainly, by the Turán theorem, $|E(H)| = \left\lfloor \frac{m^2}{4} \right\rfloor$. Obviously, there is no rainbow $2K_3$ in such a coloring, hence the proof is completed.

REFERENCES


Izolda Gorgol
gorgol@pollub.pl

Lublin University of Technology
Department of Applied Mathematics
Nadbystrzycka 38D, 20-618 Lublin, Poland

Agnieszka Görlich
gorlich@agh.edu.pl

AGH University of Science and Technology
Faculty of Applied Mathematics
al. A. Mickiewicza 30, 30-059 Krakow, Poland

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