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ON FIBONACCI NUMBERS IN EDGE COLOURED TREES

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Abstract. In this paper we show the applications of the Fibonacci numbers in edge coloured trees. We determine the second smallest number of all $(A, 2B)$-edge colourings in trees. We characterize the minimum tree achieving this second smallest value.

Keywords: edge colouring, tree, tripod, Fibonacci numbers.

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1. INTRODUCTION AND PRELIMINARY RESULTS

The n-th Fibonacci number F_n is defined recursively by the second order linear recurrence relation of the form $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with the initial conditions $F_0 = F_1 = 1$. Research on graph interpretations of the Fibonacci numbers were initiated in 1982 by H. Prodinger and R.F. Tichy in [4]. They showed that the number of all independent sets in n-vertex path is equal to F_n. This simple observation triggered counting problems in graphs related to the Fibonacci numbers. In this paper we consider a special parameter in edge coloured graph which allows to get another graph interpretation of the Fibonacci numbers.

For concepts not defined here see [2,3]. Let G be a finite, undirected, simple graph with the vertex set $V(G)$ and the edge set $E(G)$. The order (number of vertices) and size (number of edges) of G are denoted by n and m, respectively. By $P(m)$, $T(m)$ and $S(m)$ we denote a path, a tree and a star of size m, respectively. Recall that in a tree a vertex of degree at least 3 is a *branch vertex*, a vertex of degree 1 is a *leaf*. A *tripod* is a tree with exactly three leaves. In other words, every tripod has a unique branch vertex being the initial vertex of three elementary paths. Let $m \geq 3$, $p \geq 1$, $t \geq 1$ be integers. By $T(m, p, t)$ we mean a tripod of size m with paths of lengths p, t and $m - p - t$, starting from the branch vertex. These paths we will denote shortly: *p-path, t-path* and $(m - p - t)$*-path*, respectively.

We begin with a definition of $(A, 2B)$-*edge colouring*. Let G be a connected graph and let $\mathcal{C} = \{A, B\}$ be a set of two colours. A graph G is $(A, 2B)$-*edge coloured* if for every maximal (with respect to set inclusion) B-monochromatic subgraph H of G there exists a partition of H into edge disjoint paths of length 2. We have no restriction on the colour A, so $(A, 2B)$-edge colouring exists for an arbitrary graph G. It is worth mentioning that the concept of $(A, 2B)$-edge colouring is a special case of edge shade colouring of a graph (more details for edge shade colouring can be found in [1]).

Now, assume that $\mathcal{F}$ is a family of all distinct $(A, 2B)$-edge coloured graphs obtained by colouring of G, i.e. $\mathcal{F} = \{G^{(1)}, G^{(2)}, \dots, G^{(l)}\}$, where $l \geq 1$ and $G^{(p)}$ denotes a graph obtained by $(A, 2B)$-edge colouring of the graph G for $p = 1, 2, \dots l$. Let $\theta(G^{(p)})$, where $1 \leq p \leq l$, be the number of all partitions into edge disjoint paths of length 2 of all B-monochromatic subgraphs of $G^{(p)}$. If $G^{(p)}$ is A-monochromatic then we put $\theta(G^{(p)}) = 1$. Let us define a parameter $\sigma_{(A,2B)}(G)$ as follows:

$$\sigma_{(A,2B)}(G) = \sum_{p=1}^{l} \theta(G^{(p)}).$$

The parameter $\sigma_{(A,2B)}(G)$ was studied for different classes of graphs (see [1]). We recall the main result for trees.

Theorem 1.1 ([1])**.** *Let $T(m)$ be a tree of size $m, m \geq 1$. Then*

$$F_m \leq \sigma_{(A,2B)}(T(m)) \leq 1 + \sum_{j \geq 1} \binom{m}{2j} \prod_{p=0}^{j-1} [2j - (2p+1)].$$

Moreover, $\sigma_{(A,2B)}(P(m)) = F_m$ and

$$\sigma_{(A,2B)}(S(m)) = 1 + \sum_{j \geq 1} \binom{m}{2j} \prod_{p=0}^{j-1} [2j - (2p+1)].$$

From the above theorem follows that the graph $P(m)$ is the extremal tree achieving the minimum value of the parameter $\sigma_{(A,2B)}(T(m))$ and the graph $S(m)$ is the extremal tree achieving the maximum value of the parameter $\sigma_{(A,2B)}(T(m))$. It is natural to ask what are extremal trees achieving the second smallest and the second largest value of this parameter. In this paper we give the lower bound of the parameter $\sigma_{(A,2B)}(T(m))$ with the restriction that $T(m) \not\cong P(m)$ and we describe the extremal graph achieving the second smallest value of the parameter $\sigma_{(A,2B)}(T(m))$.

In the sequel we will use the following notation. If $e \in E(G)$ is a fixed edge that is coloured by the colour A then we will write $c(e) = A$. If e is coloured by the colour B then we will write $c(e) = 2B$ to indicate that there is an edge e' adjacent to e and coloured by the colour B. Moreover, we will write $\sigma_{A(e)}(G)$ (resp. $\sigma_{2B(e)}(G)$) to denote the number of all $(A, 2B)$-edge colourings of G with $c(e) = A$ (resp. $c(e) = 2B$). The following lemmas give the basic rules for determining the parameter $\sigma_{(A,2B)}(G)$.

Lemma 1.2. *Let $e \in E(G)$ be a fixed edge. Then*

$$\sigma_{(A,2B)}(G) = \sigma_{A(e)}(G) + \sigma_{2B(e)}(G). \tag{1.1}$$

Lemma 1.3. *Let $G = H \cup T(l) \cup \{e\}$ be a connected graph, where H is a connected graph, $T(l)$ is a tree of size $l, l \geq 1$ and H and $T(l)$ are vertex disjoint. Assume that $e = uv$, where $u \in V(H)$, $v \in V(T(l))$ and e is a bridge in G. Then*

$$\sigma_{(A,2B)}(G) \geq \sigma_{(A,2B)}(H \cup P(l) \cup \{e\}).$$

Moreover, the equality holds if $T(l) \cong P(l)$.

Proof. Let $G = H \cup T(l) \cup \{e\}$ and $e = uv \in E(G)$ with $u \in V(H)$ and $v \in V(T(l))$. We distinguish the following cases:

Case 1. $c(e) = A$.
Then $\sigma_{A(e)}(G) = \sigma_{(A,2B)}(H)\sigma_{(A,2B)}(T(l))$.

Case 2. $c(e) = 2B$.
Then there exists an edge, say $e' \in E(G)$, adjacent to the edge e, such that $\{e, e'\}$ belongs to a partition of $2B$-monochromatic subgraph of G and $c(e') = 2B$. Clearly, either $e' \in E(H)$ or $e' \in E(T(l))$. Hence

$$\sigma_{2B(e)}(G) = \sigma_{2B(e)}(H \cup \{e\})\sigma_{(A,2B)}(T(l)) + \sigma_{(A,2B)}(H)\sigma_{2B(e)}(T(l) \cup \{e\}).$$

Consequently,

$$\begin{aligned}\sigma_{(A,2B)}(G) = {} & \sigma_{(A,2B)}(H)\sigma_{(A,2B)}(T(l)) + \sigma_{2B(e)}(H \cup \{e\})\sigma_{(A,2B)}(T(l)) \\ & + \sigma_{(A,2B)}(H)\sigma_{2B(e)}(T(l) \cup \{e\}).\end{aligned}$$

By Theorem 1.1, we have $\sigma_{(A,2B)}(T(l)) > \sigma_{(A,2B)}(P(l))$. Hence

$$\begin{aligned}\sigma_{(A,2B)}(G) \geq {} & \sigma_{(A,2B)}(H)\sigma_{(A,2B)}(P(l)) + \sigma_{2B(e)}(H \cup \{e\})\sigma_{(A,2B)}(P(l)) \\ & + \sigma_{(A,2B)}(H)\sigma_{2B(e)}(P(l+1)) = \sigma_{(A,2B)}(H \cup P(l) \cup \{e\}),\end{aligned}$$

which completes the proof. □

2. EXTREMAL TRIPODS WITH RESPECT TO $\sigma_{(A,2B)}(T(m,p,t))$

As it was mentioned earlier the path $P(m)$ is the extremal graph achieving the minimum value of the parameter $\sigma_{(A,2B)}(T(m))$ in the class of trees of size m. Looking for the second smallest value of the parameter $\sigma_{(A,2B)}$ in trees of size m let us consider the class of trees $T(m)$ such that $T(m) \ncong P(m)$. This means that there exists at least one branch vertex in $T(m)$. We start with the class of tripods because results obtained for this class will be crucial for the main result. Assume that $\mathcal{T} = \{T(m,p,t); m \geq 3, p \geq 1, t \geq 1\}$ is the family of tripods.

Theorem 2.1. *Let $m \geq 3, p \geq 1, t \geq 1$ be integers. Then for an arbitrary tripod $T(m,p,t) \in \mathcal{T}$ holds*

$$\sigma_{(A,2B)}(T(m,p,t)) = F_{p+t}F_{m-t-p} + F_{m-t-p-1}(F_{p-1}F_t + F_pF_{t-1}). \tag{2.1}$$

Proof. Let $T(m,p,t) \in \mathcal{T}$. If $m = 3$ then $p = t = 1$ and by the simple observation we have

$$\sigma_{(A,2B)}(T(3,1,1)) = 4 = F_2F_1 + F_0(F_0F_1 + F_1F_0).$$

Let $m \geq 4$ and let $x \in V(T(m,p,t))$ be the unique branch vertex of a tripod. Clearly $p \geq 2$ or $t \geq 2$ or $m-p-t \geq 2$. Without loss of generality suppose that $m-p-t \geq 2$. Let $e \in E(T(m,p,t))$ be the edge incident with the vertex x and e belongs to $(m-p-t)$-path of $T(m,p,t)$.

Consider the following cases:

Case 1. $c(e) = A$.
Then edges adjacent to e are coloured by A or $2B$. This means that we have exactly $F_{p+t}F_{m-t-p-1}$ distinct $(A,2B)$-edge colourings with $c(e) = A$.

Case 2. $c(e) = 2B$.
Then there exists an edge, say $e^{'} \in E(T(m,p,t))$, adjacent to e and coloured by $2B$ and $\{e,e'\}$ belongs to a partition of $2B$-monochromatic subgraph of $T(m,p,t)$. Clearly $e^{'}$ belongs to either p-path or t-path or $(m-t-p)$-path. Considering all these possibilities we obtain $F_{p-1}F_tF_{m-t-p-1} + F_pF_{t-1}F_{m-t-p-1} + F_{p+t}F_{m-t-p-2}$ $(A,2B)$-edge colourings with $c(e) = 2B$.

Finally, by the above, by Lemma 1.2 and by simple calculations we obtain

$$\begin{aligned}\sigma_{(A,2B)}(T(m,p,t)) &= F_{p+t}F_{m-t-p-1} + F_{p-1}F_tF_{m-t-p-1}\\ &\quad + F_{t-1}F_pF_{m-t-p-1} + F_{p+t}F_{m-t-p-2}\\ &= F_{p+t}F_{m-t-p} + F_{m-t-p-1}(F_{p-1}F_t + F_pF_{t-1}),\end{aligned}$$

which ends the proof. □

Corollary 2.2. *Let $m \geq 3, t \geq 1$ be integers. Then*
a) $\sigma_{(A,2B)}(T(m,1,t)) = F_{t+1}F_{m-t}$,
b) $\sigma_{(A,2B)}(T(m,1,1)) = 2F_{m-1}$.

Proof. a) Applying (2.1) for $p = 1$ and using the definition of Fibonacci numbers we have

$$\begin{aligned}\sigma_{(A,2B)}(T(m,1,t)) &= F_{t+1}F_{m-t-1} + F_{m-t-2}(F_t + F_{t-1})\\ &= F_{t+1}(F_{m-t-1} + F_{m-t-2}) = F_{t+1}F_{m-t}.\end{aligned}$$

b) Analogously by Theorem 2.1 for $p = t = 1$ we obtain

$$\sigma_{(A,2B)}(T(m,1,1)) = F_2F_{m-2} + 2F_{m-3} = 2F_{m-1}.$$ □

Using the above results we can give the maximum value of the parameter $\sigma_{(A,2B)}(T(m,p,t))$. We will need the following well-known identities:

$$F_{m-1} = F_{p+t}F_{m-p-t-1} + F_{p+t-1}F_{m-p-t-2}, \tag{2.2}$$

$$F_{m+n} = F_mF_n + F_{m-1}F_{n-1}. \tag{2.3}$$

Theorem 2.3. *Let $m \geq 4$, $p \geq 1, t \geq 1$ be integers. Then for an arbitrary tripod $T(m,p,t) \in \mathcal{T}$ holds*

$$\sigma_{(A,2B)}(T(m,p,t)) \leq 2F_{m-1}.$$

Moreover, $\sigma_{(A,2B)}(T(m,p,t)) = 2F_{m-1}$ iff $T(m,p,t) \cong T(m,1,1)$.

Proof. It suffices to prove that

$$F_{p+t}F_{m-t-p} + F_{m-t-p-1}(F_{p-1}F_t + F_pF_{t-1}) - 2F_{m-1} \leq 0.$$

Applying (2.2) and the definition of Fibonacci numbers we have

$$\begin{aligned}
&F_{p+t}F_{m-t-p} + F_{m-t-p-1}(F_{p-1}F_t + F_pF_{t-1}) - 2F_{p+t}F_{m-t-p-1} - 2F_{t+p-1}F_{m-t-p-2}\\
&= F_{p+t}F_{m-t-p-1} + F_{p+t}F_{m-t-p-2} + F_{m-t-p-1}(F_{p-1}F_t + F_pF_{t-1})\\
&\quad - 2F_{p+t}F_{m-t-p-1} - 2F_{t+p-1}F_{m-t-p-2}\\
&= F_{m-t-p-1}(F_{p+t} + F_{p-1}F_t + F_pF_{t-1} - 2F_{p+t}) + F_{m-t-p-2}(F_{p+t} - 2F_{p+t-1}).
\end{aligned}$$

By (2.3), we obtain

$$\begin{aligned}
&F_{m-t-p-1}(F_{p-1}F_t + F_pF_{t-1} - F_pF_t - F_{p-1}F_{t-1})\\
&\quad + F_{m-t-p-2}(F_{p+t-1} + F_{p+t-2} - 2F_{p+t-1})\\
&= F_{m-t-p-1}(F_t(F_{p-1} - F_p) - F_{t-1}(F_{p-1} - F_p)) + F_{m-t-p-2}(F_{p+t-2} - F_{p+t-1})\\
&= F_{m-t-p-1}(F_{p-1} - F_p)(F_t - F_{t-1}) + F_{m-t-p-2}(F_{p+t-2} - F_{p+t-1}) \leq 0
\end{aligned}$$

by $F_{p-1} \leq F_p$ and $F_{p+t-2} \leq F_{p+t-1}$.

Moreover, the equality holds if and only if $F_{p+t-2} = F_{p+t-1}$ and $F_{p-1} = F_p$ or $F_{p+t-2} = F_{p+t-1}$ and $F_t = F_{t-1}$. Clearly, $F_{p+t-2} = F_{p+t-1}$ only for $p+t-2 = 0$ and $p+t-1 = 1$. Hence $p+t = 2$, so $p = t = 1$. Consequently, $F_{p-1} = F_p$ and $F_t = F_{t-1}$, which immediately gives that the extremal tripod achieving the maximum value of the parameter $\sigma_{(A,2B)}(T(m,p,t))$ is only the tripod $T(m,1,1)$. $\square$

Now we give the recurrence rule for determining the parameter $\sigma_{(A,2B)}(T(m,p,t))$.

Theorem 2.4. *Let $m \geq 3, p \geq 1, t \geq 1$ be integers. Then for an arbitrary tripod $T(m,p,t) \in \mathcal{T}$ and $m-p-t \geq 3$ holds*

$$\sigma_{(A,2B)}(T(m,p,t)) = \sigma_{(A,2B)}(T(m-1,p,t)) + \sigma_{(A,2B)}(T(m-2,p,t)) \tag{2.4}$$

with initial conditions
$\sigma_{(A,2B)}(T(p+t+1,p,t)) = F_{p+1}F_{t+1}$ and $\sigma_{(A,2B)}(T(p+t+2,p,t)) = F_{p+1}F_{t+1}+F_{p+t}$.

Proof. Let $T(m,p,t) \in \mathcal{T}$. If $m = p+t+1$ then $(m-p-t)$-path has length 1. Let $e \in E(T(p+t+1,p,t))$ be the unique edge of the $(m-p-t)$-path. We distinguish the following cases.

Case 1. $c(e) = A$.
Then edges adjacent to e are coloured by A or $2B$. This means that $T(p+t+1,p,t)\backslash e \cong P(p+t)$ and, by Theorem 1.1, $\sigma_{A(e)}(T(p+t+1,p,t)) = F_{p+t}$.

Case 2. $c(e) = 2B$.
Then there exists an edge $e^{'} \in E(T(p+t+1, p, t) \setminus e)$ adjacent to e such that $c(e^{'}) = 2B$. Clearly $e^{'}$ belongs to either p-path or t-path, by $m = p + t + 1$. Considering these two possibilities we obtain that $\sigma_{2B(e)}(T(p + t + 1, p, t)) = F_{p-1}F_t + F_pF_{t-1}$.

Consequently, by (2.3), we have

$$\begin{aligned}\sigma_{(A,2B)}(T(p + t + 1, p, t)) &= F_{p+t} + F_{p-1}F_t + F_pF_{t-1}\\ &= F_pF_t + F_{p-1}F_{t-1} + F_{p-1}F_t + F_pF_{t-1}\\ &= F_t(F_p + F_{p-1}) + F_{t-1}(F_p + F_{p-1})\\ &= F_{t+1}F_{p+1}.\end{aligned}$$

If $m = p + t + 2$ then, with respect to an edge $e \in E(T(m + p + t + 2, p, t))$ belonging to the $(m - p - t)$-path and incident with a leaf, we obtain

$$\begin{aligned}\sigma_{(A,2B)}(T(p + t + 2, p, t)) &= F_{p+t} + \sigma_{(A,2B)}(T(p + t + 1, p, t))\\ &= F_{p+t} + F_{t+1}F_{p+1}.\end{aligned}$$

Assume now $m - p - t \geq 3$. Let $e \in E(T(m, p, t))$ be an edge of the $(m - p - t)$-path incident with a leaf. By analogy we obtain

$$\sigma_{(A,2B)}(T(m, p, t)) = \sigma_{(A,2B)}(T(m - 1, p, t)) + \sigma_{(A,2B)}(T(m - 2, p, t)),$$

which completes the proof. □

Solving the recurrence relation (2.4) we obtain the Binet formulas for the parameters $\sigma_{(A,2B)}(T(m, 1, 1))$ and $\sigma_{(A,2B)}(T(m, 2, 2))$.

$$\begin{aligned}\sigma_{(A,2B)}(T(m, 1, 1)) &= \frac{2\sqrt{5}}{5}\left[\left(\frac{1+\sqrt{5}}{2}\right)^m - \left(\frac{1-\sqrt{5}}{2}\right)^m\right] \quad \text{for } m \geq 3,\\ \sigma_{(A,2B)}(T(m, 2, 2)) &= \left(\frac{4\sqrt{5}}{5} - 1\right)\left(\frac{1+\sqrt{5}}{2}\right)^m\\ &\quad - \left(\frac{4\sqrt{5}}{5} + 1\right)\left(\frac{1-\sqrt{5}}{2}\right)^m \quad \text{for } m \geq 5.\end{aligned}$$

We shall show that $T(m, 2, 2)$ is the extremal tripod achieving the minimum value of the parameter $\sigma_{(A,2B)}(T(m, p, t))$ in the class $\mathcal{T}$. Consider non-isomorphic tripods of size $m = 5, 6$ (Figs 1 and 2).

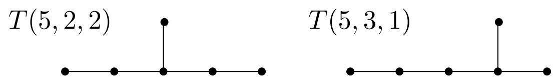

Fig 1. All non-isomorphic tripods of size 5

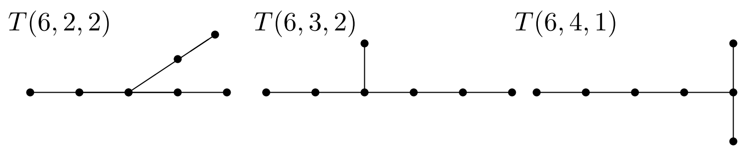

Fig 2. All non-isomorphic tripods of size 6

For these tripods values of the parameter $\sigma_{(A,2B)}(T(m,p,t))$ are given in Table 1.

Table 1.

$T(m,p,t)$	$T(5,2,2)$	$T(5,3,1)$	$T(6,2,2)$	$T(6,3,2)$	$T(6,4,1)$
$\sigma_{(A,2B)}(T(m,p,t))$	9	10	14	15	16

Theorem 2.5. *Let $m \geq 5$, $p \geq 1, t \geq 1$ be integers. Then for an arbitrary tripod $T(m,p,t) \in \mathcal{T}$ holds*

$$\sigma_{(A,2B)}(T(m,p,t)) \geq F_{m-1} + 2F_{m-3}.$$

Moreover, the equality holds if $T(m,p,t) \cong T(m,2,2)$.

Proof (*by induction on m*). If $m = 5, 6$ then the result follows immediately from Table 1, Figures 1 and 2 and the definition of Fibonacci numbers.

Let $m \geq 7$. Assume that for all $n < m$ holds $\sigma_{(A,2B)}(T(n,p,t)) \geq F_{n-1} + 2F_{n-3}$. We shall show that the theorem is true for m. Since $m \geq 7$, we have that at least one path of tripod $T(m,p,t)$ has length at least 3. Without loss of generality we can assume that $m - p - t \geq 3$. Using Theorem 2.4 and the induction hypothesis we have

$$\begin{aligned} \sigma_{(A,2B)}(T(m,p,t)) &= \sigma_{(A,2B)}(T(m-1,p,t)) + \sigma_{(A,2B)}(T(m-2,p,t)) \\ &\geq F_{m-2} + 2F_{m-4} + F_{m-3} + 2F_{m-5} = F_{m-1} + 2F_{m-3} \end{aligned}$$

and the theorem follows.

Now we shall show that $\sigma_{(A,2B)}(T(m,2,2)) = F_{m-1} + 2F_{m-3}$. By Theorem 2.1 and by the definition of Fibonacci numbers, we obtain

$$\begin{aligned} \sigma_{(A,2B)}(T(m,2,2)) &= 5F_{m-4} + 4F_{m-5} = 4F_{m-3} + F_{m-4} \\ &= F_{m-2} + 3F_{m-3} = F_{m-1} + 2F_{m-3}, \end{aligned}$$

which completes the proof. $\square$

3. MAIN RESULTS

In this section we determine the second smallest value of the parameter $\sigma_{(A,2B)}(T(m))$. We show that the tripod $T(m,2,2)$ realizes this second minimum value of $\sigma_{(A,2B)}(T(m))$.

Let $r \geq 1, \Delta \geq 3$ be integers. For $m \geq 3$ by a tree $S_r(m,\Delta)$ we mean a graph with a unique branch vertex obtained from the star with maximum degree Δ by inserting new vertices of degree 2 into some edges of the star such that in the resulting tree $S_r(m,\Delta)$ the longest path starting from the branch vertex has length r. In particular, $S_1(m,\Delta)$ is isomorphic to a star $S(m)$ and $S_r(m,3)$ is isomorphic to a tripod $T(m,r,t)$, for some $t \geq 1$.

Theorem 3.1. *Let $m \geq 4$, $\Delta \geq 3$ be integers. Then*

$$\sigma_{(A,2B)}\left(S_2(m,\Delta)\right) = \begin{cases} \sigma_{(A,2B)}(S(m-1)) + \sigma_{(A,2B)}(S(m-2)), \\ \quad \textit{if } S_2(m,\Delta) \textit{ has the unique } 2-\textit{path}, \\ \sigma_{(A,2B)}(S_2(m-1,\Delta)) + \sigma_{(A,2B)}(S_2(m-2,\Delta-1)), \\ \quad \textit{otherwise}. \end{cases}$$

Proof. Let $m \geq 4$ and $\Delta \geq 3$ be integers. Consider two cases.

Case 1. There exists a unique 2-path in the tree $S_2(m,\Delta)$.
Let $e \in E(S_2(m,\Delta))$ be an edge which belongs to the 2-path and e is incident with a leaf. We have two possibilities.

Case 1.1. $c(e) = A$.
Then $\sigma_{A(e)}\left(S_2(m,\Delta)\right) = \sigma_{(A,2B)}(S(m-1))$.

Case 1.2. $c(e) = 2B$.
Then $\sigma_{2B(e)}\left(S_2(m,\Delta)\right) = \sigma_{(A,2B)}(S(m-2))$. Hence

$$\sigma_{(A,2B)}\left(S_2(m,\Delta)\right) = \sigma_{(A,2B)}(S(m-1)) + \sigma_{(A,2B)}(S(m-2)).$$

Case 2. There exist at least two 2-paths in the tree $S_2(m,\Delta)$.
Let $e \in E(S_2(m,\Delta))$ be an edge which belongs to any 2-path and e is incident with a leaf. We have two possibilities.

Case 2.1. $c(e) = A$.
Then $\sigma_{A(e)}\left(S_2(m,\Delta)\right) = \sigma_{(A,2B)}(S_2(m-1,\Delta))$.

Case 2.2. $c(e) = 2B$.
Then $\sigma_{2B(e)}\left(S_2(m,\Delta)\right) = \sigma_{(A,2B)}(S_2(m-2,\Delta-1))$. Hence

$$\sigma_{(A,2B)}\left(S_2(m,\Delta)\right) = \sigma_{(A,2B)}(S_2(m-1,\Delta)) + \sigma_{(A,2B)}(S_2(m-2,\Delta-1)),$$

which completes the proof. □

Theorem 3.2. *Let $m \geq 4$, $\Delta \geq 4$, $r \geq 1$ be integers. Then for an arbitrary tripod $T(m,p,t) \in \mathcal{T}$ holds*

$$\sigma_{(A,2B)}\left(S_r(m,\Delta)\right) > \sigma_{(A,2B)}\left(T(m,p,t)\right). \tag{3.1}$$

Proof. Let m, Δ, r be as in the statement of the theorem. We consider the following cases:

Case 1. There exists a unique r-path in the tree $S_r(m,\Delta)$.
Clearly $r \geq 2$ and $m \geq 5$. We use induction on m and r. If $m = 5$ then $r = 2$ and the result is obvious. Let $m \geq 6$ and $r \geq 2$ and assume that the inequality (3.1) holds for all $n < m$ and $k < r$. Let $e \in E(S_r(m,\Delta))$ belongs to the r-path and e is incident with a leaf of $S_r(m,\Delta)$. We need to consider two cases.

Case 1.1. $c(e) = A$.
Then $\sigma_{A(e)}\left(S_r(m,\Delta)\right) = \sigma_{(A,2B)}\left(S_{r-1}(m-1,\Delta)\right)$.

Case 1.2. $c(e) = 2B$.
If $r = 2$ then the unique 2-path P is coloured by $2B$ and the graph $S_2\left(m,\Delta\right) \setminus P$ is isomorphic to a star $S(m-2)$. Hence $\sigma_{2B(e)}\left(S_2(m,\Delta)\right) = \sigma_{(A,2B)}\left(S(m-2)\right)$.
If $r \geq 3$ then the graph $S_r\left(m,\Delta\right) \setminus P$ is isomorphic to a graph $S_{k<r}\left(m-2,\Delta\right)$.
If $r = 2$ then by the above and using the induction hypothesis we obtain

$$\begin{aligned}\sigma_{(A,2B)}\left(S_2(m,\Delta)\right) &= \sigma_{(A,2B)}\left(S_1(m-1,\Delta)\right) + \sigma_{(A,2B)}\left(S(m-2)\right)\\ &> \sigma_{(A,2B)}\left(T(m-1,p,t)\right) + \sigma_{(A,2B)}\left(S(m-2)\right).\end{aligned}$$

We shall show that for $T(m,p,t) \in \mathcal{T}$

$$\sigma_{(A,2B)}\left(T(m-1,p,t)\right) + \sigma_{(A,2B)}\left(S(m-2)\right) > \sigma_{(A,2B)}\left(T(m,p,t)\right).$$

It suffices to prove the following inequality

$$\sigma_{(A,2B)}\left(T(m-1,p,t)\right) + \sigma_{(A,2B)}\left(S(m-2)\right) - \sigma_{(A,2B)}\left(T(m,p,t)\right) > 0.$$

By Theorem 2.4, we have

$$\begin{aligned}&\sigma_{(A,2B)}\left(T(m-1,p,t)\right) + \sigma_{(A,2B)}\left(S(m-2)\right) - \sigma_{(A,2B)}\left(T(m-1,p,t)\right)\\ &- \sigma_{(A,2B)}\left(T(m-2,p,t)\right) > 0\end{aligned}$$

because the star maximizes this parameter in trees.

If $r \geq 3$ then using the induction hypothesis we obtain

$$\begin{aligned}\sigma_{(A,2B)}\left(S_r(m,\Delta)\right) &= \sigma_{(A,2B)}\left(S_{r-1}(m-1,\Delta)\right) + \sigma_{(A,2B)}\left(S_{k<r}(m-2,\Delta)\right)\\ &> \sigma_{(A,2B)}\left(T(m-1,p,t)\right) + \sigma_{(A,2B)}\left(T(m-2,p,t)\right)\\ &= \sigma_{(A,2B)}\left(T(m,p,t)\right),\end{aligned}$$

which completes the proof of this case.

Case 2. There exist at least two r-paths in $S_r\left(m,\Delta\right)$.

For $r = 1$ the result is obvious since $S_1\left(m,\Delta\right)$ is isomorphic to the star. Let $r \geq 2$. Then $m \geq 6$. We now proceed by induction on m. If $m = 6$ then $r = 2$ and the result is obvious.

Let $m \geq 7$ and assume that for all $n < m$ the inequality holds. We distinguish two possibilities.

Case 2.1. $c(e) = A$.
Then $\sigma_{A(e)}(S_r(m,\Delta)) = \sigma_{(A,2B)}(S_r(m-1,\Delta))$.
Case 2.2. $c(e) = 2B$.
If $r = 2$ then $S_2(m,\Delta) \setminus P$ is isomorphic to $S_2(m-2,\Delta-1)$. If $r \geq 3$ then $S_2(m,\Delta) \setminus P$ is isomorphic to $S_2(m-2,\Delta)$. Let $r = 2$. Then from these possibilities and by the induction hypothesis we obtain

$$\begin{aligned}\sigma_{(A,2B)}(S_2(m,\Delta)) &= \sigma_{(A,2B)}(S_2(m-1,\Delta)) + \sigma_{(A,2B)}(S_2(m-2,\Delta-1))\\ &> \sigma_{(A,2B)}(T(m-1,p,t)) + \sigma_{(A,2B)}(S_2(m-2,\Delta-1)).\end{aligned}$$

We shall show that

$$\sigma_{(A,2B)}(T(m-1,p,t)) + \sigma_{(A,2B)}(S_2(m-2,\Delta-1)) > \sigma_{(A,2B)}(T(m,p,t))$$

for all $T(m,p,t) \in \mathcal{T}$. It suffices to prove that

$$\sigma_{(A,2B)}(T(m-1,p,t)) + \sigma_{(A,2B)}(S_2(m-2,\Delta-1)) - \sigma_{(A,2B)}(T(m,p,t)) > 0.$$

Suppose that there exist l ($l \geq 2$) 2-paths in $S_2(m,\Delta)$. Then by (2.4) and applying the induction hypothesis in l steps we obtain

$$\begin{aligned}&\sigma_{(A,2B)}(T(m-1,p,t)) + \sigma_{(A,2B)}(S_2(m-2,\Delta-1))\\ &\quad - \sigma_{(A,2B)}(T(m-1,p,t)) - \sigma_{(A,2B)}(T(m-2,p,t))\\ &= \sigma_{(A,2B)}(S_2(m-2,\Delta-1)) - \sigma_{(A,2B)}(T(m-2,p,t)) > 0\end{aligned}$$

in the first step. Consequently in the lth step

$$\sigma_{(A,2B)}(S_2(m-l-1,\Delta-l)) - \sigma_{(A,2B)}(T(m-l-1,p,t)) > 0$$

since $S_2(m-l-1,\Delta-l)$ is isomorphic to the star $S(m-l-1)$ and the result immediately follows.

Let $r \geq 3$. Then from Cases 2.1 and 2.2 and using the induction hypothesis we obtain

$$\begin{aligned}\sigma_{(A,2B)}(S_r(m,\Delta)) &= \sigma_{(A,2B)}(S_r(m-1,\Delta)) + \sigma_{(A,2B)}(S_r(m-2,\Delta))\\ &> \sigma_{(A,2B)}(T(m-1,p,t)) + \sigma_{(A,2B)}(T(m-2,p,t))\\ &= \sigma_{(A,2B)}(T(m,p,t)),\end{aligned}$$

and the proof is complete. □

Corollary 3.3. *Let $m \geq 4$, $\Delta \geq 4$, $r \geq 1$ be integers. Then*

$$\sigma_{(A,2B)}(S_r(m,\Delta)) > F_{m-1} + 2F_{m-3}.$$

Proof. By Theorems 3.2 and 2.5, we immediately obtain

$$\sigma_{(A,2B)}(S_r(m,\Delta)) > \sigma_{(A,2B)}(T(m,p,t)) \geq F_{m-1} + 2F_{m-3}. \qquad \square$$

Theorem 3.4. *Let $T(m) \ncong P(m)$ be a tree of the size m. Then*

$$\sigma_{(A,2B)}\left(T(m)\right) \geq F_{m-1} + 2F_{m-3}. \tag{3.2}$$

Moreover, $\sigma_{(A,2B)}\left(T(m)\right) = F_{m-1} + 2F_{m-3}$ *if* $T(m) \cong T(m,2,2)$.

Proof. Assume that $T(m)$ is a tree of size m non-isomorphic to the path $P(m)$. Since $T(m) \ncong P(m)$, there exists in $T(m)$ at least one branch vertex, say x. If $T(m)$ has a unique branch vertex then the result follows by Theorem 3.2. Suppose that $T(m)$ has at least two branch vertices and let $u, v \in V(T(m))$ be such vertices. Let $e \in E(T(m))$ be an edge belonging to the path $u - v$ in $T(m)$. Then $T(m) = T_1(m_1) \cup T_2(m_2) \cup \{e\}$, where $T_i(m_i)$ for $i = 1, 2$ are trees of the size m_i, $m_i \geq 2$. Applying Lemma 1.3 we obtain

$$\begin{aligned} \sigma_{(A,2B)}\left(T(m)\right) &= \sigma_{(A,2B)}(T_1(m_1) \cup T_2(m_2) \cup \{e\}) \\ &\geq \sigma_{(A,2B)}(T_1(m_1) \cup P(m_2) \cup \{e\}). \end{aligned}$$

If $T_1(m_1) \cup P(m_2) \cup \{e\}$ is $S_r(m, \Delta)$, then by Theorem 3.2 the result follows. Otherwise, it has at least two branch vertices and we repeat the above procedure until we get a tree T^* of the same size m. By Theorem 3.2 we have $\sigma_{(A,2B)}(T^*) > \sigma_{(A,2B)}(T(m,p,t))$. In the class $\mathcal{T}$ the minimum tripod $T(m,2,2)$ has the parameter $\sigma_{(A,2B)}\left(T(m,2,2)\right) = F_{m-1} + 2F_{m-3}$, which completes the proof. □

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