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ON CRITERIA FOR ALGEBRAIC INDEPENDENCE OF COLLECTIONS OF FUNCTIONS SATISFYING ALGEBRAIC DIFFERENCE RELATIONS

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Abstract. This paper gives conditions for algebraic independence of a collection of functions satisfying a certain kind of algebraic difference relations. As applications, we show algebraic independence of two collections of special functions: (1) Vignéras' multiple gamma functions and derivatives of the gamma function, (2) the logarithmic function, q-exponential functions and q-polylogarithm functions. In a similar way, we give a generalization of Ostrowski's theorem.

Keywords: difference algebra, systems of algebraic difference equations, algebraic independence, Vignéras' multiple gamma functions, q-polylogarithm functions.

Mathematics Subject Classification: 12H10, 39A10, 39A13.

1. INTRODUCTION

In [14], Ostrowski gave a criterion for algebraic independence of antiderivatives of given functions, which is stated as follows: Let $f_1(z), f_2(z), \ldots, f_n(z)$ be functions such that the derivative $\frac{d}{dz} f_i(z)$ is included in a differential field for each $i = 1, 2, \ldots, n$. If the $f_i(z)$'s are algebraically dependent over the field, then there exist constants $c_1, c_2, \ldots, c_n$ not all zero such that the linear combination $\sum_{i=1}^{n} c_i f_i(z)$ is included in the field. In [9], Kolchin gave an analogue of Ostrowski's theorem, which is a criterion for algebraic independence of exponentials of given functions. This analogue is stated as follows: Let $g_1(z), g_2(z), \ldots, g_m(z)$ be nontrivial functions such that the logarithmic derivative $\frac{d}{dz} \log g_j(z)$ is included in a differential field for each $j = 1, 2, \ldots m$. If the $g_j(z)$'s are algebraically dependent over the field, then there exist integers $r_1, r_2, \ldots, r_m$ not all zero such that $\prod_{j=1}^{m} g_j(z)^{r_j}$ is included in the field. We note that, in [9], Kolchin also gave a criterion for algebraic independence of the $f_i(z)$'s and the $g_j(z)$'s. In [6], by taking a Galoisian approach, Hardouin gave difference analogues of Ostrowski's theorem and Kolchin's theorem in order to study algebraic independence and differential independence of functions $f_1(z), f_2(z), \ldots, f_n(z)$ satisfying $\sigma(f_i(z)) = a_i(z)f_i(z)$ for the shift operator $\sigma : f(z) \mapsto f(z+\tau)$ and the q-shift operator $\sigma : f(z) \mapsto f(qz)$. As an application, the difference analogues give differential independence of compositions of the gamma functions with linear functions. On the other hand, by relying on algebraic structures of modules of differentials of a field extension, Ax [1] proved some conjecture for formal power series made by Schanuel. Inspired by this work, we gave another proof for each of Hardouin's analogues by using modules of differentials in difference algebra in [13].

In the present paper, by using the latter method, we extend the results of Hardouin to more general systems of difference equations. We consider the following system of functional equations with coefficients $a_{i,j}(z)$'s and $b_i(z)$'s:

$$\begin{cases} \sigma(y_1(z)) = a_{1,1}(z)y_1(z) & + b_1(z), \\ \sigma(y_2(z)) = a_{2,1}(z)y_1(z) + a_{2,2}(z)y_2(z) & + b_2(z), \\ \quad \vdots & \quad \vdots \\ \sigma(y_n(z)) = a_{n,1}(z)y_1(z) + \cdots + a_{n,n}(z)y_n(z) & + b_n(z), \end{cases} \tag{1.1}$$

where σ is called a transforming operator, a generalization of the shift operator and the q-shift operator as above. Hardouin's analogue of Ostrowski's theorem deals with (1.1) in case $(a_{i,j}(z))$ is the identity matrix. We also consider the following system of functional equations with coefficients $q_i(z)$'s :

$$\begin{cases} \sigma(x_1(z)) = q_1(z)x_1(z)^{p_{1,1}}, \\ \sigma(x_2(z)) = q_2(z)x_1(z)^{p_{2,1}}x_2(z)^{p_{2,2}}, \\ \quad \vdots \\ \sigma(x_m(z)) = q_m(z)x_1(z)^{p_{m,1}}x_2(z)^{p_{m,2}} \cdots x_m(z)^{p_{m,m}}, \end{cases} \tag{1.2}$$

where the $p_{i,j}$'s are integers. Hardouin's analogue of Kolchin's theorem deals with (1.2) in case $(p_{i,j})$ is the identity matrix.

By reformulating these equations in difference algebra, we give criteria for algebraic independence of solutions $y_1(z), y_2(z), \ldots, y_n(z)$ of (1.1) in Theorem 3.1 and solutions $x_1(z), x_2(z), \ldots, x_m(z)$ of (1.2) in Theorem 3.3, respectively. In Theorem 3.8, we give a criterion for algebraic independence of the $y_i(z)$'s and the $x_j(z)$'s when the both diagonal entries of $(a_{i,j})$ and $(p_{i,j})$ are unity. By these extended results, we can prove the algebraic independence of a collection of special functions such as

1. Vignéras' multiple gamma functions and derivatives of the gamma function,
2. the logarithmic function, q-exponential functions and q-polylogarithm functions.

Furthermore, by using a similar method to proofs for the extended results, we generalize Ostrowski's theorem in differential algebra.

In Section 2, we introduce notations to formulate main results and notions of difference algebra and module of differentials. In Section 3, we give main results by reformulating (1.1) and (1.2) in difference algebra. Then we show Theorems 3.1, 3.3

and 3.8. In Section 4, we apply Theorem 3.8 to two collections of special functions as above. Then we show Corollaries 4.2 and 4.4. In Section 5, as a differential analogue of Theorem 3.1, we give a generalization of Ostrowski's theorem.

2. PRELIMINARIES

In order to clarify the argument, we give notations of difference algebra according to [3, 10]. Throughout this paper, by a ring we always mean a commutative ring of characteristic 0. For a field K, we denote $K^\times = K\backslash\{0\}$. For a field K and an injective endomorphism σ of K, we call the pair (K,σ) a *difference field*. We call σ the *transforming operator* and K the *underlying field*. For example, for $f(z) \in \mathbb{C}(z)$, let $\sigma_1 : f(z) \mapsto f(z+\tau)$ be the shift operator with $\tau \in \mathbb{C}$, $\sigma_2 : f(z) \mapsto f(qz)$ the q-shift operator with $q \in \mathbb{C}^\times$ and $\sigma_3 : f(z) \mapsto f(z^n)$ the shift operator of Mahler type with $n \in \mathbb{Z}_{\geq 2}$. Then the pair $(\mathbb{C}(z),\sigma_i)$ is a difference field for each $i = 1,2,3$. For difference fields (L,σ_L) and (K,σ_K), we call $(L,\sigma_L)/(K,\sigma_K)$ a *difference field extension* if K is a subfield of L and $\sigma_L|_K = \sigma_K$. We frequently omit subscripts L and K from σ. If there is no confusion, we denote a difference field (K,σ) simply by K and a difference field extension $(L,\sigma)/(K,\sigma)$ simply by L/K. We define an intermediate difference field in a natural way. Let L/K be a difference field extension. For a subset $S \subset L$, we denote $K\langle S\rangle = K(\{\sigma^i(s) | s \in S, i \in \mathbb{Z}_{\geq 0}\})$. For a positive integer n, $y = (y_1, y_2, \ldots, y_n)^T \in L^n$ and an intermediate difference field M of L/K, we denote $M\langle y\rangle = M\langle y_1, y_2, \ldots, y_n\rangle$. We call $C_K = \{a \in K | \sigma(a) = a\}$ the *invariant field* of K. For a lower triangular matrix $A = (a_{i,j})$ in $M_n(K)$ and an intermediate difference field M of L/K, we define

$$V(M,A) = \left\{u \in M^n \,\middle|\, a_{n,n}u = A^T\sigma(u)\right\}.$$

For a vector $v = (v_1, v_2, \ldots, v_n)^T$, a matrix $A = (a_{i,j})_{1\leq i,j\leq n}$ and $r \leq n$, we denote $v_{(r)} = (v_1, v_2, \ldots, v_r)^T$ and $A_{(r)} = (a_{i,j})_{1\leq i,j\leq r}$.

Let L be an algebra over a ring K. A pair $(\Omega_{L/K}, d)$ of an L-module $\Omega_{L/K}$ and a K-linear derivation $d : L \to \Omega_{L/K}$ is called the *module of differentials* of L over K if for any L-module M and any K-linear derivation $D : L \to M$ there is a unique L-module homomorphism $f : \Omega_{L/K} \to M$ such that $D = f \circ d$. For any algebra L over any ring K, there exists its module of differentials (see [10, pp.91–92]). Furthermore, Johnson introduced the differential structure of the module of differentials of a differential field extension in [7]. For a vector $y = (y_1, y_2, \ldots, y_n)^T \in L^n$, we define $dy = (dy_1, dy_2, \ldots, dy_n)^T \in \Omega^n_{L/K}$. The following propositions are well-known.

Proposition 2.1 ([5]). *Let L/K be a field extension and $(\Omega_{L/K}, d)$ the module of differentials of L/K. A family $\{x_i\}_{i\in I}$ of elements of L is algebraically independent over K if and only if a family $\{dx_i\}_{i\in I}$ of elements of $\Omega_{L/K}$ is linearly independent over L. In particular, $x \in L$ is algebraic over K if and only if $dx = 0$.*

Proposition 2.2 ([15]). *Let L/K be a field extension and $(\Omega_{L/K}, d)$ the module of differentials of L/K. Suppose that $a_1, a_2, \dots, a_m \in K$ are linearly independent over $\mathbb{Q}$. If $y, x_1, x_2, \dots, x_m \in L$ satisfy*

$$dy + \sum_{i=1}^{m} a_i \frac{dx_i}{x_i} = 0,$$

then $dy = dx_1 = dx_2 = \cdots = dx_m = 0$.

Proposition 2.3 ([10]). *Let L/K be a difference field extension and $(\Omega_{L/K}, d)$ the module of differentials of L/K. Then there exists an additive mapping*

$$\sigma^* : \Omega_{L/K} \to \Omega_{L/K}$$

such that

$$\sigma^*(adb) = \sigma(a)d(\sigma(b)) \quad \text{for } a, b \in L.$$

3. MAIN RESULTS – CRITERIA FOR ALGEBRAIC INDEPENDENCE

In this section, we fix a difference field extension L/K with $C_L = C_K$ and positive integers n, m.

Theorem 3.1. *Let $y = (y_1, y_2, \dots, y_n)^T \in L^n$ satisfy*

$$\sigma(y) = Ay + b, \tag{3.1}$$

where $b \in K^n$ and $A = (a_{i,j})$ is a lower triangular matrix in $M_n(K)$. Suppose that for each $r = 1, 2, \dots, n$,

$$V(K\langle y_{(r)}\rangle, A_{(r)}) \subset K^r \tag{3.2}$$

holds. If $y_1, y_2, \dots, y_n$ are algebraically dependent over K, then there exist $r \in \mathbb{Z}$ with $1 \le r \le n$ and $u \in V(K\langle y_{(r)}\rangle, A_{(r)})\backslash\{0\}$ such that $t = u^T y_{(r)}$ is algebraic over K and t satisfies

$$\sigma(t) = a_{r,r}t + \sigma(u)^T b_{(r)}.$$

In particular, if $a_{r,r} \neq 0$, then there exists $f \in K$ such that

$$\sigma(f) = a_{r,r}f + \sigma(u)^T b_{(r)}.$$

Remark 3.2. Theorem 3.1 corresponds to the difference analogue of Ostrowski's theorem in case A is the identity matrix.

Proof of Theorem 3.1. Take a positive integer r with $1 \le r \le n$ such that $y_1, y_2, \dots, y_{r-1}$ are algebraically independent over K and $y_1, y_2, \dots, y_r$ are algebraically dependent over K. Let $(\Omega_{K\langle y_{(r)}\rangle/K}, d)$ be the module of differentials of $K\langle y_{(r)}\rangle/K$. By Proposition 2.1, there exists a column vector $u = (u_1, u_2, \dots, u_r)^T \in K\langle y_{(r)}\rangle^r\backslash\{0\}$ such that

$$u^T dy_{(r)} = 0.$$

Since $dy_1, dy_2, \dots, dy_{r-1}$ are linearly independent over $K\langle y_{(r)}\rangle$ by Proposition 2.1, we may take $u_r = 1$. Note that $\sigma(y_{(r)}) = A_{(r)}y_{(r)} + b_{(r)}$. By Proposition 2.3, we have

$$0 = \sigma(u)^T d\sigma(y_{(r)}) = \sigma(u)^T d(A_{(r)}y_{(r)} + b_{(r)}) = \sigma(u)^T A_{(r)} dy_{(r)}.$$

Since $A_{(r)}$ is lower triangular, the coefficient of dy_r in $(a_{r,r}u^T - \sigma(u)^T A_{(r)})dy_{(r)}$ is zero. From linear independence of $dy_1, dy_2, \dots, dy_{r-1}$, we obtain $a_{r,r}u^T - \sigma(u)^T A_{(r)} = 0$. By the assumption (3.2), we have $u \in K^r$. Putting $t = u^T y_{(r)}$, we see that t is algebraic over K and

$$\sigma(t) = a_{r,r}t + \sigma(u)^T b_{(r)}.$$

Assume that $a_{r,r} \neq 0$. Let

$$f(X) = \sum_{i=0}^{s} f_i X^i, \quad f_i \in K, f_s = 1,$$

be the minimal polynomial of t. Then we have

$$\sum_{i=0}^{s} f_i t^i = 0, \tag{3.3}$$

$$\sum_{i=0}^{s} \sigma(f_i)(a_{r,r}t + \sigma(u)^T b_{(r)})^i = 0. \tag{3.4}$$

Since $f(X)$ is the minimal polynomial of t satisfying (3.3) and (3.4), the polynomial $a_{r,r}^s f(X) - \sum_{i=0}^{s} \sigma(f_i)(a_{r,r}X + \sigma(u)^T b_{(r)})^i$ must be trivial. Then f_{s-1} satisfies

$$\sigma(f_{s-1}) = a_{r,r}f_{s-1} - s\sigma(u)^T b_{(r)}.$$

Putting $\tilde{f} = -f_{s-1}/s$, we obtain $\sigma(\tilde{f}) = a_{r,r}\tilde{f} + \sigma(u)^T b_{(r)}$. $\square$

Theorem 3.3. *Let $x = (x_1, x_2, \dots, x_m)^T \in L^{\times m}$ satisfy*

$$\begin{cases} \sigma(x_1) = q_1 x_1^{p_{1,1}}, \\ \sigma(x_2) = q_2 x_1^{p_{2,1}} x_2^{p_{2,2}}, \\ \quad \vdots \\ \sigma(x_m) = q_m x_1^{p_{m,1}} x_2^{p_{m,2}} \cdots x_m^{p_{m,m}}, \end{cases} \tag{3.5}$$

where $(q_1, q_2, \dots, q_m)^T \in K^{\times m}$ and $P = (p_{i,j})$ is a lower triangular matrix in $M_m(\mathbb{Z})$. Suppose that for each $s = 1, 2, \dots, m$,

$$V(K\langle x_{(s)}\rangle, P_{(s)}) \subset K^s \tag{3.6}$$

holds. If $x_1, x_2, \dots, x_m$ are algebraically dependent over K, then there exist $s \in \mathbb{Z}$ with $1 \le s \le m$ and $l = (l_1, l_2, \dots, l_s)^T \in (\mathbb{Z}^s \cap V(K\langle x_{(s)}\rangle, P_{(s)}))\backslash\{0\}$ such that $\tilde{x} = \prod_{i=1}^{s} x_i^{l_i}$ is algebraic over K and $\sigma(\tilde{x}) = q\tilde{x}^{p_{s,s}}$ holds for $q = \prod_{i=1}^{s} q_i^{l_i}$. In particular, if $p_{s,s} = 1$, then there exist $f \in K^\times$ and $r \in \mathbb{Z}_{\ge 1}$ such that

$$\sigma(f) = q^r f.$$

Remark 3.4. Theorem 3.3 corresponds to the difference analogue of Kolchin's theorem in case P is the identity matrix.

Proof of Theorem 3.3. Take a positive integer s with $1 \leq s \leq m$ such that $x_1, x_2, \ldots, x_{s-1}$ are algebraically independent over K and $x_1, x_2, \ldots, x_s$ are algebraically dependent over K. Let $(\Omega_{K\langle x_{(s)}\rangle/K}, d)$ be the module of differentials of $K\langle x_{(s)}\rangle/K$. By Proposition 2.1, there exists a column vector $u = (u_1, u_2, \ldots, u_s)^T \in K\langle x_{(s)}\rangle^s \backslash \{0\}$ such that

$$\sum_{i=1}^{s} u_i \frac{dx_i}{x_i} = 0.$$

Since $dx_1, dx_2, \ldots, dx_{s-1}$ are linearly independent over $K\langle x_{(s)}\rangle$ by Proposition 2.1, we may take $u_s = 1$. By Proposition 2.3, we have

$$0 = \sum_{i=1}^{s} \sigma(u_i) \frac{d\sigma(x_i)}{\sigma(x_i)} = \sum_{j=1}^{s} \Big(\sum_{i=j}^{s} p_{i,j} \sigma(u_i) \Big) \frac{dx_j}{x_j}.$$

Put $w = (dx_1/x_1, dx_2/x_2, \ldots, dx_s/x_s)^T$. Since $P_{(s)}$ is lower triangular, the coefficient of dx_s/x_s in $(p_{s,s}u^T - \sigma(u)^T P_{(s)})w$ is zero. From linear independence of $dx_1/x_1, dx_2/x_2, \ldots, dx_{s-1}/x_{s-1}$, we obtain $p_{s,s}u^T - \sigma(u)^T P_{(s)} = 0$. By the assumption (3.6), we have $u \in K^s$. There exist $\alpha_1, \alpha_2, \ldots, \alpha_t \in K$ such that they are linearly independent over $\mathbb{Q}$ and $u_i = \sum_{j=1}^{t} l_{i,j}\alpha_j$ with $l_{i,j} \in \mathbb{Z}$. Putting $\tilde{x}_j = \prod_{i=1}^{s} x_i^{l_{i,j}}$, we obtain

$$\sum_{j=1}^{t} \alpha_j \frac{d\tilde{x}_j}{\tilde{x}_j} = 0.$$

We may assume that $(l_{1,1}, l_{2,1}, \ldots, l_{s,1}) \neq 0$. By Proposition 2.2, $\tilde{x}_1$ is algebraic over K. Putting $q = \prod_{i=1}^{s} q_i^{l_{i,1}}$, we have

$$\sigma(\tilde{x}_1) = q \Bigg(\prod_{j=1}^{s-1} x_j^{\left(\sum_{i=j}^{s} p_{i,j} l_{i,1}\right) - p_{s,s} l_{j,1}} \Bigg) \tilde{x}_1^{p_{s,s}}.$$

Then $\sigma(\tilde{x}_1)/\tilde{x}_1^{p_{s,s}}$ is algebraic over K and so is $\prod_{j=1}^{s-1} x_j^{(\sum_{i=j}^{s} p_{i,j} l_{i,1}) - p_{s,s} l_{j,1}}$. Then $\sum_{i=j}^{s} p_{i,j} l_{i,1} = p_{s,s} l_{j,1}$ holds for each $j = 1, 2, \ldots, s$. Hence $\sigma(\tilde{x}_1) = q\tilde{x}_1^{p_{s,s}}$ holds. Putting $l = (l_{1,1}, l_{2,1}, \ldots, l_{s,1})^T$, we have $P_{(s)}^T l = p_{s,s} l$.

Assume that $p_{s,s} = 1$. Let

$$f(X) = \sum_{i=0}^{r} f_i X^i, \quad f_i \in K, f_r = 1, f_0 \neq 0,$$

be the minimal polynomial of $\tilde{x}_1$. Then we have

$$\sum_{i=0}^{r} f_i \tilde{x}_1^i = 0, \tag{3.7}$$

$$\sum_{i=0}^{r} \sigma(f_i) q^i \tilde{x}_1^i = 0. \tag{3.8}$$

Since $f(X)$ is the minimal polynomial of $\tilde{x}_1$ satisfying (3.7) and (3.8), the polynomial $q^r f(X) - \sum_{i=0}^{r} \sigma(f_i) q^i X^i$ must be trivial. Then we obtain $\sigma(f_0) = q^r f_0$. $\square$

Proposition 3.5. *Let $y = (y_1, y_2, \dots, y_n)^T \in L^n$ and $x = (x_1, x_2, \dots, x_m)^T \in L^{\times m}$ satisfy* (3.1) *and* (3.5)*, respectively. Suppose that for each $r = 1, 2, \dots, n$ and each $s = 1, 2, \dots, m$,*

$$V(K\langle x\rangle\langle y\rangle, A_{(r)}) \subset K^r,$$
$$V(K\langle x\rangle\langle y\rangle, P_{(s)}) \subset K^s,$$
$$\left\{ w \in K\langle x\rangle\langle y\rangle^n \,\middle|\, p_{s,s} w = A^T \sigma(w) \right\} \subset K^n$$

hold. Suppose that $x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n$ are algebraically dependent over K. Then at least one of the following statements holds:

1. *There exist $r \in \mathbb{Z}$ with $1 \le r \le n$ and $u \in V(K\langle y_{(r)}\rangle, A_{(r)}) \backslash \{0\}$ such that $t = u^T y_{(r)}$ is algebraic over K and t satisfies $\sigma(t) = a_{r,r} t + \sigma(u)^T b_{(r)}$. In particular, if $a_{r,r} \ne 0$, then there exists $f \in K$ such that $\sigma(f) = a_{r,r} f + \sigma(u)^T b_{(r)}$.*
2. *There exist $s \in \mathbb{Z}$ with $1 \le s \le m$ and $l = (l_1, l_2, \dots, l_s)^T \in (\mathbb{Z}^s \cap V(K\langle x\rangle\langle y\rangle, P_{(s)})) \backslash \{0\}$ such that $\tilde{x} = \prod_{i=1}^{s} x_i^{l_i}$ is algebraic over K and $\sigma(\tilde{x}) = q \tilde{x}^{p_{s,s}}$ holds for $q = \prod_{i=1}^{s} q_i^{l_i}$. In particular, if $p_{s,s} = 1$, then there exist $f \in K^\times$ and $r \in \mathbb{Z}_{\ge 1}$ such that $\sigma(f) = q^r f$.*

Proof. By Theorem 3.1, we may assume that $y_1, y_2, \dots, y_n$ are algebraically independent over K. Take a positive integer s with $1 \le s \le m$ such that $y_1, y_2, \dots, y_n, x_1, x_2, \dots, x_{s-1}$ are algebraically independent over K and $y_1, y_2, \dots, y_n, x_1, x_2, \dots, x_s$ are algebraically dependent over K.

Let $(\Omega_{K\langle x\rangle\langle y\rangle/K}, d)$ be the module of differentials of $K\langle x\rangle\langle y\rangle/K$. There exist column vectors $u = (u_1, u_2, \dots, u_n)^T \in K\langle x\rangle\langle y\rangle^n$ and $v = (v_1, v_2, \dots, v_s)^T \in K\langle x\rangle\langle y\rangle^s \backslash \{0\}$ such that

$$u^T dy + \sum_{i=1}^{s} v_i \frac{dx_i}{x_i} = 0.$$

Since $dy_1, dy_2, \dots, dy_n, dx_1, dx_2, \dots, dx_{s-1}$ are linearly independent over $K\langle x\rangle\langle y\rangle$, we may take $v_s = 1$. Then we have

$$\sigma(u)^T A dy + \sum_{j=1}^{s} \Big(\sum_{i=j}^{s} p_{i,j} \sigma(v_i) \Big) \frac{dx_j}{x_j} = 0.$$

It follows that $p_{s,s}v = P_{(s)}^T\sigma(v)$ and $p_{s,s}u = A^T\sigma(u)$, so that $u \in K^n$ and $v \in K^s$ by the assumption. There exist $\alpha_1, \alpha_2, \ldots, \alpha_t \in K$ such that they are linearly independent over $\mathbb{Q}$ and $v_i = \sum_{j=1}^t l_{i,j}\alpha_j$ with $l_{i,j} \in \mathbb{Z}$. Putting $\tilde{x}_j = \prod_{i=1}^s x_i^{l_{i,j}}$, we obtain

$$d(u^T y) + \sum_{j=1}^t \alpha_j \frac{d\tilde{x}_j}{\tilde{x}_j} = 0.$$

By Proposition 2.2 and by the same argument as in the proof of Theorem 3.3, we see that the assertion holds true. □

Lemma 3.6. *Let* $y = (y_1, y_2, \ldots, y_n)^T \in L^n$ *and* $x = (x_1, x_2, \ldots, x_m)^T \in L^{\times m}$ *satisfy* (3.1) *and* (3.5)*, respectively. Suppose that there exists* $r \in \mathbb{Z}$ *with* $1 \le r \le n$ *such that*

$$V(K\langle x\rangle\langle y\rangle, A_{(r)}) \subset K^r,$$
$$\{w \in L \mid \sigma(w) = a_{r,r}w\} \subset \overline{K},$$

where $\overline{K}$ *is the algebraic closure of* K *in* L*. If there exist* $f \in K$ *and* $u \in V(K\langle x\rangle\langle y\rangle, A_{(r)})\setminus\{0\}$ *such that* $\sigma(f) = a_{r,r}f + \sigma(u)^T b_{(r)}$*, then* $y_1, y_2, \ldots, y_r$ *are algebraically dependent over* K*.*

Proof. Putting $t = u^T y_{(r)}$, we have $\sigma(t) = a_{r,r}t + \sigma(u)^T b_{(r)}$. Since $\sigma(f-t) = a_{r,r}(f-t)$, it follows from the assumption that $f - t \in \overline{K}$. Therefore $t \in \overline{K}$ holds. □

Lemma 3.7. *Let* $y = (y_1, y_2, \ldots, y_n)^T \in L^n$ *and* $x = (x_1, x_2, \ldots, x_m)^T \in L^{\times m}$ *satisfy* (3.1) *and* (3.5)*, respectively. Suppose that there exists* $s \in \mathbb{Z}$ *with* $1 \le s \le m$ *such that*

$$\{w \in L \mid \sigma(w) = w^{p_{s,s}}\} \subset \overline{K},$$

where $\overline{K}$ *is the algebraic closure of* K *in* L*. If there exist* $f \in K^\times$ *and* $l = (l_1, l_2, \ldots, l_s)^T \in (\mathbb{Z}^s \cap V(K\langle x\rangle\langle y\rangle, P_{(s)}))\setminus\{0\}$ *such that* $\sigma(f) = qf^{p_{s,s}}$ *for* $q = \prod_{i=1}^s q_i^{l_i}$*, then* $x_1, x_2, \ldots, x_s$ *are algebraically dependent over* K*.*

Proof. Putting $\tilde{x} = \prod_{i=1}^s x_i^{l_i}$, we have $\sigma(\tilde{x}) = q\tilde{x}^{p_{s,s}}$, so that $\sigma(\tilde{x}/f) = (\tilde{x}/f)^{p_{s,s}}$ holds. From the assumption, we obtain $\tilde{x} \in \overline{K}$. □

Theorem 3.8. *Let* $y = (y_1, y_2, \ldots, y_n)^T \in L^n$ *and* $x = (x_1, x_2, \ldots, x_m)^T \in L^{\times m}$ *satisfy* (3.1) *and* (3.5)*, respectively. Suppose that* $a_{r,r} = 1$ *for each* $r = 1, 2, \ldots, n$ *in* (3.1) *and* $p_{s,s} = 1$ *for each* $s = 1, 2, \ldots, m$ *in* (3.5)*. Suppose that for each* $r = 1, 2, \ldots, n$ *and each* $s = 1, 2, \ldots, m$*,*

$$V(K\langle x\rangle\langle y\rangle, A_{(r)}) \subset K^r,$$
$$V(K\langle x\rangle\langle y\rangle, P_{(s)}) \subset K^s$$

hold. Then $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$ *are algebraically dependent over* K *if and only if at least one of the following statements holds:*

1. *There exist* $f \in K, r \in \mathbb{Z}$ *with* $1 \le r \le n$ *and* $u \in V(K\langle x\rangle\langle y\rangle, A_{(r)})\setminus\{0\}$ *such that* $\sigma(f) = f + \sigma(u)^T b_{(r)}$*.*

2. *There exist* $f \in K^{\times}, s \in \mathbb{Z}$ *with* $1 \leq s \leq m$ *and* $l = (l_1, l_2, \dots, l_s)^T \in (\mathbb{Z}^s \cap V(K\langle x\rangle\langle y\rangle, P_{(s)}))\backslash\{0\}$ *such that* $\sigma(f) = qf$ *for* $q = \prod_{i=1}^{s} q_i^{l_i}$.

Proof. Sufficiency. By the assumption, we have

$$\{w \in L \mid \sigma(w) = a_{r,r}w\} = \{w \in L \mid \sigma(w) = w^{p_{s,s}}\} = C_L.$$

It follows from Lemma 3.6 and Lemma 3.7 that the x_i's or the y_j's are algebraically dependent over K.

Necessity. We have

$$\left\{w \in K\langle x\rangle\langle y\rangle^n \,\middle|\, p_{s,s}w = A^T\sigma(w)\right\} = V(K\langle x\rangle\langle y\rangle, A) \subset K^n.$$

Hence the assertion follows from Proposition 3.5. □

4. APPLICATIONS

4.1. ALGEBRAIC INDEPENDENCE OF VIGNÉRAS' MULTIPLE GAMMA FUNCTIONS AND DERIVATIVES OF THE GAMMA FUNCTION

Vignéras' multiple gamma functions $G_1(z), G_2(z), \dots, G_m(z), \dots$ are meromorphic functions satisfying the following relations [16]:

1. $G_0(z) = z,\ G_m(z+1) = G_{m-1}(z)G_m(z)$ for $m \geq 1$,
2. $G_m(1) = 1$,
3. $\dfrac{d^{m+1}}{dz^{m+1}} \log G_m(z+1) \geq 0 \quad$ for $z \geq 0$.

Then $G_1(z)$ is equal to the gamma function $\Gamma(z)$. In this section, we denote the derivative of a differentiable function $f(z)$ by $f'(z)$. In [12], Nishizawa showed that $G_m(z)$ does not satisfy any nontrivial algebraic differential equation over $\mathbb{C}(z)$ and gave the following relation between logarithmic derivatives:

$$\begin{aligned}\frac{G_{m+1}(z+1)'}{G_{m+1}(z+1)} &= \frac{z-m+1}{m}\frac{G_m(z+1)'}{G_m(z+1)} + p_m(z)\\ &= \binom{z}{m}\frac{\Gamma(z+1)'}{\Gamma(z+1)} + P_m(z),\end{aligned}$$

where $p_m(z), P_m(z)$ are polynomials of degree less than or equal to m. We show that $G_1(z) = \Gamma(z), G_2(z), \dots, G_m(z), \dots$ and $\Gamma'(z), \Gamma''(z), \dots, \Gamma^{(n)}(z), \dots$ are algebraically independent over $\mathbb{C}(z)$; namely, $\Gamma(z)$ does not satisfy any nontrivial algebraic differential equation over the field $\mathbb{C}(z)(\{G_i(z)\}_{i\geq 2})$.

Proposition 4.1. *Define a transforming operator σ by $\sigma(f(z)) = f(z+1)$ for each $f \in \mathbb{C}(z)$. Let $(L,\sigma)/(\mathbb{C}(z),\sigma)$ be a difference field extension. If $\{x_i\}_{i\in\mathbb{Z}_{\geq 1}}, \{y_i\}_{i\in\mathbb{Z}_{\geq 1}} \subset L^\times$ satisfy*

$$\begin{cases} \sigma(x_1) = zx_1, & \\ \sigma(x_i) = x_{i-1}x_i & \text{for } i \in \mathbb{Z}_{\geq 2}, \\ \sigma(y_i) = y_i + (-1)^{i-1}(i-1)!z^{-i} & \text{for } i \in \mathbb{Z}_{\geq 1}, \end{cases} \tag{4.1}$$

then they are algebraically independent over $C_L(z)$.

Proof. Put $K = C_L(z)$. Then we have $C_L = C_K$. Put $x = (x_1, x_2, \ldots, x_m)^T$ and $y = (y_1, y_2, \ldots, y_n)^T$ for $n + m > 0$. We put matrices $A = I_n$ and

$$P = (p_{i,j})_{1\leq i,j\leq m}, \quad p_{i,j} = \begin{cases} 1 & \text{for } i = j \text{ or } i = j+1, \\ 0 & \text{otherwise.} \end{cases}$$

Since $A_{(r)}$ is the identity matrix, we have $V(K\langle x\rangle\langle y\rangle, A_{(r)}) \subset C_L^r \subset K^r$ for each $r = 1, 2, \ldots, n$. Next we show $V(K\langle x\rangle\langle y\rangle, P_{(s)}) \subset K^s$ for each $s = 1, 2, \ldots, m$. Let $w \in K\langle x\rangle\langle y\rangle$ satisfy $\sigma(w) = w + \alpha$ for $\alpha \in C_L[z]$. By induction on the degree of α in z, we show $w \in C_L[z]$. If $\alpha \in C_L$, then $\sigma(w - \alpha z) = w - \alpha z$ holds, which implies $w = \alpha z + c$ for some $c \in C_L$. Assume that the degree of α in z is equal to $r \in \mathbb{Z}_{\geq 1}$. Let α_r be the leading coefficient of α. Putting $\tilde{w} = w - \alpha_r z^{r+1}/(r+1)$, we have $\sigma(\tilde{w}) = \tilde{w} + \tilde{\alpha}$ for some polynomial $\tilde{\alpha} \in C_L[z]$ of degree less than r. From the induction hypothesis, we obtain $\tilde{w} \in C_L[z]$, and hence $w \in C_L[z]$. By the definition, any element $u = (u_1, u_2, \ldots, u_s)^T \in V(K\langle x\rangle\langle y\rangle, P_{(s)})$ satisfies

$$\begin{aligned} &\sigma(u_s) = u_s, \\ &\sigma(u_i) + \sigma(u_{i+1}) = u_i \quad \text{for } i = 1, 2, \ldots, s-1. \end{aligned}$$

Then we find $u_s, u_{s-1}, \ldots, u_1 \in C_L[z]$ by the above argument. Hence we have $V(K\langle x\rangle\langle y\rangle, P_{(s)}) \subset C_L[z]^s \subset K^s$ for each $s = 1, 2, \ldots, m$. Therefore, in order to prove this proposition, we have only to check the conditions 1 and 2 in Theorem 3.8.

Suppose that there exist $f \in K^\times, s \in \mathbb{Z}$ with $1 \leq s \leq m$ and $l = (l_1, l_2, \ldots, l_s)^T \in (\mathbb{Z}^s \cap V(K\langle x\rangle\langle y\rangle, P_{(s)}))\backslash\{0\}$ such that $\sigma(f) = z^{l_1}f$. It follows from $P_{(s)}^T l = l$ that $l_2 = l_3 = \cdots = l_s = 0$ holds. Hence we have $l_1 \neq 0$. Put $f = p/q$, where $p, q \in C_L[z]$ are relatively prime. Then we have $\sigma(p)q = z^{l_1}p\sigma(q)$. Comparing the degrees, we obtain $l_1 = 0$, a contradiction.

Next, suppose that there exist $f \in K, r \in \mathbb{Z}$ with $1 \leq r \leq n$ and $u = (u_1, u_2, \ldots, u_r)^T \in C_L^r\backslash\{0\}$ such that

$$\sigma(f) = f + \sum_{i=1}^{r} \frac{(-1)^{i-1}(i-1)!u_i}{z^i}. \tag{4.2}$$

For $\lambda \in C_L$, we denote by $v_{z-\lambda}$ the discrete valuation associated to $z - \lambda$ of the rational function field $C_L(z)/C_L$. Note that $v_{z+\lambda}(f) = v_{z+\lambda+1}(\sigma(f))$. It follows from

(4.2) that $v_z(f) < 0$ or $v_z(\sigma(f)) < 0$. Suppose that $v_z(f) < 0$. We show $v_{z+i}(f) < 0$ for each $i \in \mathbb{Z}_{\geq 0}$. We have $v_{z+1}(\sigma(f)) < 0$ by $v_z(f) < 0$. By (4.2), the equality $v_{z+1}(\sigma(f) - f) \geq 0$ holds, which implies $v_{z+1}(f) < 0$. Hence we have $v_{z+2}(\sigma(f)) < 0$. Repeating the argument, we obtain $v_{z+i}(f) < 0$ for each $i \in \mathbb{Z}_{\geq 0}$. Next, suppose that $v_z(\sigma(f)) < 0$. In the same way, $v_{z-i-1}(f) < 0$ holds for each $i \in \mathbb{Z}_{\geq 0}$. This implies that f has infinitely many poles, a contradiction.

It follows from Theorem 3.8 that $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$ are algebraically independent over K. □

We put

$$L = \mathbb{C}(z)(\{G_k(z)\}_{k \in \mathbb{Z}_{\geq 1}}, \{\Gamma^{(k)}(z)\}_{k \in \mathbb{Z}_{\geq 1}}),$$

$x_i = G_i(z)$ and $y_i = (\log \Gamma(z))^{(i)}$ for each $i \in \mathbb{Z}_{\geq 1}$. Then we obtain the following corollary:

Corollary 4.2. *$G_2(z)$, $G_3(z)$, $\ldots$, $G_m(z)$, $\ldots$ and $\Gamma(z)$, $\Gamma'(z)$, $\ldots$, $\Gamma^{(n)}(z)$, $\ldots$ are algebraically independent over $\mathbb{C}(z)$.*

4.2. ALGEBRAIC INDEPENDENCE OF THE LOGARITHMIC FUNCTION, q-EXPONENTIAL FUNCTIONS AND q-POLYLOGARITHM FUNCTIONS

Let q be a nonzero complex number being not a root of unity. In [2, 11, 17], q-multiple polylogarithm functions $\mathrm{Li}_{q;n_1,n_2,\ldots,n_l}(z_1, z_2, \ldots, z_l)$ are defined by

$$\mathrm{Li}_{q;n_1,n_2,\ldots,n_l}(z_1, z_2, \ldots, z_l) = \sum_{0<k_1<k_2<\cdots<k_l} \frac{z_1^{k_1} z_2^{k_2} \cdots z_l^{k_l}}{[k_1]_q^{n_1} [k_2]_q^{n_2} \cdots [k_l]_q^{n_l}},$$

where $[k]_q = (1 - q^k)/(1 - q)$ is the q-analogue of the number k. When $l = 1$, these functions are the q-analogues of the polylogarithm functions,

$$\mathrm{Li}_{q;n}(z) = \sum_{k=1}^{\infty} \frac{z^k}{[k]_q^n}.$$

By simple calculations, we have

$$\begin{aligned} \mathrm{Li}_{q;1}(qz) &= \mathrm{Li}_{q;1}(z) - (1-q)\frac{z}{1-z}, \\ \mathrm{Li}_{q;n}(qz) &= \mathrm{Li}_{q;n}(z) - (1-q)\mathrm{Li}_{q;n-1}(z) \quad \text{for } n \in \mathbb{Z}_{\geq 2}. \end{aligned}$$

We define two q-analogues of the exponential function

$$e_q^z = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}, \quad E_q^z = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{z^n}{[n]_q!},$$

where $[n]_q!$ is the q-analogue of $n!$, that is,

$$[n]_q! = \begin{cases} 1 & \text{for } n = 0, \\ \prod_{k=1}^{n} [k]_q & \text{for } n \in \mathbb{Z}_{\geq 1}. \end{cases}$$

These q-exponential functions satisfy

$$e_q^{qz} = (1 - (1-q)z)e_q^z,$$
$$E_q^{qz} = (1 + (1-q)z)^{-1}E_q^z.$$

For other properties and relations, the reader may refer to [4, 8]. We show that $\log z, e_q^z, E_q^z, \mathrm{Li}_{q;1}(z), \mathrm{Li}_{q;2}(z), \ldots, \mathrm{Li}_{q;n}(z), \ldots$ are algebraically independent over $\mathbb{C}(z)$.

Proposition 4.3. *Let q be a nonzero complex number being not a root of unity. Define a transforming operator σ by the q-shift operator $\sigma(f(z)) = f(qz)$ for each $f \in \mathbb{C}(z)$. Let $(L,\sigma)/(\mathbb{C}(z),\sigma)$ be a difference field extension. Put $\hat{q} = 1 - q$. If $w, x_1, x_2 \in L^\times$ and $\{y_i\}_{i\in\mathbb{Z}_{\geq 1}} \subset L^\times$ satisfy*

$$\begin{cases} \sigma(w) = w + 1, \\ \sigma(x_1) = (1 - \hat{q}z)x_1, \\ \sigma(x_2) = (1 + \hat{q}z)^{-1}x_2, \\ \sigma(y_1) = y_1 - \hat{q}z/(1-z), \\ \sigma(y_n) = y_n - \hat{q}y_{n-1} \quad \text{for } n \in \mathbb{Z}_{\geq 2}, \end{cases} \tag{4.3}$$

then they are algebraically independent over $C_L(z)$.

Proof. Put $M = C_L(w)$ and $K = M(z) = C_L(w,z)$. Then we have $L \supset K \supset M$ and $C_L = C_K = C_M$. We see that w is transcendental over $C_L(z)$. Indeed, assume that w is algebraic over $C_L(z)$. Let

$$f(X) = \sum_{i=0}^{r} f_i X^i, \qquad r \in \mathbb{Z}_{\geq 1}, f_i \in C_L(z), f_r = 1,$$

be the minimal polynomial of w. Then we have

$$\sum_{i=0}^{r} f_i w^i = 0, \quad \sum_{i=0}^{r} \sigma(f_i)(w+1)^i = 0.$$

Comparing the coefficients, we obtain the equality

$$\sigma(f_{r-1}) = f_{r-1} - r. \tag{4.4}$$

Defining a transforming operator σ of the field of formal Laurent series $C_L((z))$ by $\sigma(\sum_k a_k z^k) = \sum_k a_k q^k z^k$, we have the difference field extension $(C_L((z)),\sigma)/(C_L(z),\sigma)$. From the Laurent series expansion $f_{r-1} = \sum_k \hat{f}_k z^k$ and the equality (4.4), we obtain $\hat{f}_0 = \hat{f}_0 - r$, a contradiction. For $\lambda \in M$, we denote by $v_{z-\lambda}$ the discrete valuation associated to $z - \lambda$ of the rational function field $M(z)/M$. Since M is inversive, i.e., $\sigma(M) = M$, for each $f \in M(z)$ we obtain the equality

$$v_{z-\lambda}(f) = v_{z-q^{-1}\sigma(\lambda)}(\sigma(f)). \tag{4.5}$$

Put $y=(y_1,y_2,\dots,y_n)^T$ for $n\geq 0$. We put matrices $P=I_2$ and

$$A=(a_{i,j})_{1\leq i,j\leq n},\quad a_{i,j}=\begin{cases}1 & \text{for } i=j,\\ -\hat{q} & \text{for } i=j+1,\\ 0 & \text{otherwise.}\end{cases}$$

Since $P_{(s)}$ is the identity matrix, we have $V(K\langle x_1,x_2\rangle\langle y\rangle,P_{(s)})\subset C_L^s\subset K^s$ for each $s=1,2$. In the same way as in the proof of Proposition 4.1, if $u\in K\langle x_1,x_2\rangle\langle y\rangle$ satisfies $\sigma(u)=u+\alpha$ for $\alpha\in C_L[w]$, then we have $u\in C_L[w]$. From the same argument as in the proof Proposition 4.1, we obtain $V(K\langle x_1,x_2\rangle\langle y\rangle,A_{(r)})\subset C_L[w]^r\subset K^r$ for each $r=1,2,\dots,n$.

Suppose that there exist $f\in K^\times$ and $l=(l_1,l_2)\in\mathbb{Z}^2\backslash\{0\}$ such that

$$\sigma(f)=\frac{(1-\hat{q}z)^{l_1}}{(1+\hat{q}z)^{l_2}}f. \tag{4.6}$$

Put $f=h/g$, where $h,g\in M[z]$ are relatively prime. Then we have

$$\sigma(h)g(1+\hat{q}z)^{l_2}=h(1-\hat{q}z)^{l_1}\sigma(g).$$

By comparing the degrees with respect to z, we have $l_1=l_2\neq 0$. Hence the equality (4.6) implies that $v_{z-\hat{q}^{-1}}(\sigma(f))=v_{z-\hat{q}^{-1}}(f)+l_1$. Using (4.5) and (4.6), we obtain $v_{z-q^k\hat{q}^{-1}}(f)=v_{z-\hat{q}^{-1}}(f)+l_1$ for each $k\in\mathbb{Z}_{\geq 1}$. Then f must satisfy $v_{z-\hat{q}^{-1}}(f)=-l_1$. From the same argument, we obtain $v_{z-q^{-k}\hat{q}^{-1}}(f)=-l_1$ for each $k\in\mathbb{Z}_{\geq 0}$. This yields a contradiction.

Next, suppose that there exist $f\in K, r\in\mathbb{Z}$ with $1\leq r\leq n$ and $u=(u_1,u_2,\dots,u_r)^T\in V(K\langle x_1,x_2\rangle\langle y\rangle,A_{(r)})\backslash\{0\}$ such that

$$\sigma(f)=f-\frac{\hat{q}\sigma(u_1)z}{1-z}. \tag{4.7}$$

It follows from $u\neq 0$ that $\sigma(u_1)\neq 0$. Then the inequality $v_{z-1}(\sigma(f)-f)<0$ implies that $v_{z-1}(\sigma(f))<0$ or $v_{z-1}(f)<0$. In a similar way to the proof of Proposition 4.1, we show that if $v_{z-1}(f)<0$ then $v_{z-q^{-k}}(f)<0$ holds for each $k\in\mathbb{Z}_{\geq 0}$. Suppose that $v_{z-1}(f)<0$. We have $v_{z-q^{-1}}(\sigma(f))<0$ by (4.5). By (4.7), the equality $v_{z-q^{-1}}(\sigma(f)-f)=0$ holds, which implies $v_{z-q^{-1}}(f)<0$. Hence we have $v_{z-q^{-2}}(\sigma(f))<0$. Repeating the argument, we obtain $v_{z-q^{-k}}(f)<0$ for each $k\in\mathbb{Z}_{\geq 0}$. In the same way, $v_{z-q^k}(f)<0$ holds for each $k\in\mathbb{Z}_{\geq 1}$ if $v_{z-1}(\sigma(f))<0$. This yields a contradiction.

It follows from Theorem 3.8 that $x_1,x_2,y_1,y_2,\dots,y_n$ are algebraically independent over K. □

Note that the argument of q is fixed. Then, in the Riemann surface of $\log z$, we have the identity

$$\log(qz)=\log z+\log q.$$

We put $L = \mathbb{C}(z)(\log z, e_q^z, E_q^z, \{\mathrm{Li}_{q;i}(z)\}_{i\in\mathbb{Z}_{\geq 1}}), w = \log z/\log q, x_1 = e_q^z, x_2 = E_q^z$ and $y_i = \mathrm{Li}_{q;i}(z)$ for each $i \in \mathbb{Z}_{\geq 1}$. Then we obtain the following corollary:

Corollary 4.4. *Let q be a nonzero complex number being not a root of unity. Then $\log z, e_q^z, E_q^z, Li_{q;1}(z),\ Li_{q;2}(z), \ldots, Li_{q;n}(z), \ldots$ are algebraically independent over $\mathbb{C}(z)$.*

5. GENERALIZATION OF OSTROWSKI'S THEOREM

We have discussed the algebraic independence of a collection of special functions having algebraic difference relations as above. By using differential algebra, a method similar to the proof of Theorem 3.1 gives a generalization of Ostrowski's theorem.

For a field K and a derivation operator D from K to itself, we call the pair (K, D) a *differential field*. We define a differential field extension in a natural way. If there is no confusion, we denote a differential field (K, D) simply by K and a differential field extension $(L, D)/(K, D)$ simply by L/K. Let L/K be a differential field extension. Denote by $a' = Da$, the derivative of a, and by $C_K = \{a \in K \mid a' = 0\}$ the constant field of K. For a subset $S \subset L$, we denote by $K\langle S\rangle$ the differential field generated by S over K. For $y = (y_1, y_2, \ldots, y_n)^T \in L^n$, we put $K\langle y\rangle = K\langle y_1, y_2, \ldots, y_n\rangle$. For a lower triangular matrix $A = (a_{i,j})$ in $M_n(K)$ and an intermediate differential field M of L/K, we define

$$V_D(M, A) = \left\{u \in M^n \,\middle|\, u' = (a_{n,n} - A^T)u\right\}.$$

Theorem 5.1. *Let L/K be a differential field extension with $C_L = C_K$. Let n be a positive integer and $y = (y_1, y_2, \ldots, y_n)^T \in L^n$ satisfy*

$$y' = Ay + b, \tag{5.1}$$

where $b \in K^n$ and $A = (a_{i,j})$ is a lower triangular matrix in $M_n(K)$. Suppose that for each $r = 1, 2, \ldots, n$,

$$V_D(K\langle y_{(r)}\rangle, A_{(r)}) \subset K^r \tag{5.2}$$

holds. If $y_1, y_2, \ldots, y_n$ are algebraically dependent over K, then there exist $f \in K$, $r \in \mathbb{Z}$ with $1 \leq r \leq n$ and $u \in V_D(K\langle y_{(r)}\rangle, A_{(r)})\backslash\{0\}$ such that

$$f' = a_{r,r}f + u^T b_{(r)}.$$

Remark 5.2. Theorem 5.1 corresponds to the original Ostrowski's theorem in case A is the zero matrix.

Proof of Theorem 5.1. Take a positive integer r with $1 \leq r \leq n$ such that $y_1, y_2, \ldots, y_{r-1}$ are algebraically independent over K and $y_1, y_2, \ldots, y_r$ are algebraically dependent over K. Let $(\Omega_{K\langle y_{(r)}\rangle/K}, d)$ be the module of differentials of $K\langle y_{(r)}\rangle/K$. There exists a column vector $u = (u_1, u_2, \ldots, u_r)^T \in K\langle y_{(r)}\rangle^r$ such that $u_r = 1$ and $u^T dy_{(r)} = 0$. Note that $y'_{(r)} = A_{(r)}y_{(r)} + b_{(r)}$. By the Lie derivative $adb \mapsto a'db + adb'$ for $a, b \in L$ (see [1]), we have

$$0 = u'^T dy_{(r)} + u^T dy'_{(r)} = (u'^T + u^T A_{(r)})dy_{(r)}.$$

Hence we have $u'^T + u^T A_{(r)} = a_{r,r} u^T$, that is, $u' = (a_{r,r} - A_{(r)}^T)u$. By the assumption of (5.2), $u \in K^r$ holds. Putting $t = u^T y_{(r)}$, it follows that t is algebraic over K and

$$t' = a_{r,r} t + u^T b_{(r)}.$$

By considering the minimal polynomial of t, there exists $f \in K$ such that $f' = a_{r,r} f + u^T b_{(r)}$. $\square$

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