

2–HYPERREFLEXIVITY AND HYPOREFLEXIVITY OF POWER PARTIAL ISOMETRIES

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Abstract. Power partial isometries are not always hyperreflexive neither reflexive. In the present paper it will be shown that power partial isometries are always hyporeflexive and 2-hyperreflexive.

Keywords: power partial isometry, reflexive subspace, hyperreflexive subspace, hyperreflexive operator, hyporeflexive algebra.

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1. INTRODUCTION

Reflexivity and hyperreflexivity of operator algebras on Hilbert spaces is connected with the problem of existence of a nontrivial invariant subspace. An algebra of operators is reflexive [24] if it has so many (common) invariant subspaces that they determine the algebra. It means that if any operator leaves invariant all subspaces which are invariant for all operators from the algebra, then it has to belong to the algebra. Equivalently, rank one operators contained in the preannihilator of an algebra generate the whole preannihilator. Hyporeflexivity [13] of an algebra of operators (weaker property than reflexivity) means that if any operator from the commutant of the given algebra leaves invariant all subspaces, which are invariant for all operators from the algebra, then it has to belong to the algebra. An algebra of operators is hyperreflexive [1] (much stronger property than reflexivity) if the usual distance from any operator to the algebra can be controlled by the distance given by rank one operators. Replacing rank one operators by operators of rank at most k in the corresponding conditions we obtain the concepts of k -reflexivity [3] and k -hyperreflexivity [16] as natural generalizations of reflexivity and hyperreflexivity.

A power partial isometry is an operator for which all its powers are partial isometries. In [5] full characterization of reflexivity of an algebra generated by completely non-unitary power partial isometries was given. In [21] it was shown that the same

conditions given in [5] characterize hyperreflexive algebras generated by power partial isometries. In the present paper we will show that algebras generated by power partial isometries are hyporeflexive, 2-reflexive and even 2-hyperreflexive.

2. PRELIMINARIES

Let $\mathbf{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on a complex separable Hilbert space \mathcal{H} . For a cardinal number d let $\mathcal{H}^{(d)}$ denote the orthogonal sum of \mathcal{H} with itself d times. If $T \in \mathbf{B}(\mathcal{H})$, then $T^{(d)}$ is the orthogonal sum of T with itself d times and for $\mathcal{S} \subset \mathbf{B}(\mathcal{H})$ we denote $\mathcal{S}^{(d)} = \{T^{(d)} : T \in \mathcal{S}\}$. Duality between trace class operators $\mathbf{B}_1(\mathcal{H})$ and the algebra $\mathbf{B}(\mathcal{H})$ is given by trace, i.e. $\langle T, f \rangle = \text{tr}(Tf)$ for $T \in \mathbf{B}(\mathcal{H})$, $f \in \mathbf{B}_1(\mathcal{H})$. By $\mathbf{F}_k(\mathcal{H})$ we denote the set of operators of rank at most k , $k \in \mathbb{N}$. For a subset $\mathcal{S} \subset \mathbf{B}(\mathcal{H})$ by \mathcal{S}_\perp we denote the preannihilator of \mathcal{S} , i.e.

$$\mathcal{S}_\perp = \{f \in \mathbf{B}_1(\mathcal{H}) : \langle T, f \rangle = 0 \text{ for all } T \in \mathcal{S}\}.$$

Let $\mathcal{S} \subset \mathbf{B}(\mathcal{H})$ be a subspace (i.e. a norm-closed linear manifold) and let $T \in \mathbf{B}(\mathcal{H})$. A subspace \mathcal{S} is *reflexive* ([18]) if

$$\mathcal{S} = \text{Ref } \mathcal{S} \stackrel{\text{df}}{=} \{A \in \mathbf{B}(\mathcal{H}) : Ax \in [\mathcal{S}x] \text{ for all } x \in \mathcal{H}\}.$$

(By $[\mathcal{M}]$ we denote the smallest closed subspace containing \mathcal{M} , in the appropriate space and topology.) Longstaff in [19] proved that a weak*-closed subspace $\mathcal{S} \subset \mathbf{B}(\mathcal{H})$ is reflexive if and only if $\mathcal{S}_\perp = [\mathcal{S}_\perp \cap \mathbf{F}_1(\mathcal{H})]$. A subspace \mathcal{S} is *k-reflexive* [3] if $\mathcal{S}^{(k)}$ is reflexive in $\mathbf{B}(\mathcal{H}^{(k)})$. Kraus and Larson in [17] gave equivalent condition, namely a weak*-closed subspace $\mathcal{S} \subset \mathbf{B}(\mathcal{H})$ is *k-reflexive* if and only if $\mathcal{S}_\perp = [\mathcal{S}_\perp \cap \mathbf{F}_k(\mathcal{H})]$.

If $T \in \mathbf{B}(\mathcal{H})$ by $\text{dist}(T, \mathcal{S})$ we denote the usual distance from T to \mathcal{S} , namely $\text{dist}(T, \mathcal{S}) = \inf\{\|T - S\| : S \in \mathcal{S}\}$. In what follows we will also consider the distances

$$\alpha_k(T, \mathcal{S}) = \sup\{|\langle T, f \rangle| : f \in \mathcal{S}_\perp \cap \mathbf{F}_k(\mathcal{H}), \|f\|_1 \leq 1\},$$

see [16]. Recall that $\alpha_k(T, \mathcal{S}) \leq \text{dist}(T, \mathcal{S})$, $k \in \mathbb{N}$. A subspace \mathcal{S} is called *k-hyperreflexive* ([1, 2, 16, 17]) if there is a constant $\kappa > 0$ such that

$$\text{dist}(T, \mathcal{S}) \leq \kappa \alpha_k(T, \mathcal{S}) \quad \text{for all } T \in \mathbf{B}(\mathcal{H}).$$

The infimum of all constants κ fulfilling this inequality is called the *k-hyperreflexivity constant* and denoted by $\kappa_k(\mathcal{S})$. We omit the letter k if $k = 1$ and say that \mathcal{S} is hyperreflexive. An operator $A \in \mathbf{B}(\mathcal{H})$ is called *k-reflexive* (*k-hyperreflexive*) if $\mathcal{W}(A)$ is *k-reflexive* (*k-hyperreflexive*) where $\mathcal{W}(A)$ denotes the smallest algebra containing polynomials in A and closed in the weak operator topology.

Recall that the *unilateral shift* is the operator $a_s \in \mathbf{B}(l_+^2)$ defined as

$$a_s(x_0, x_1, \dots) = (0, x_0, x_1, \dots).$$

The *backward shift* a_c is its adjoint

$$a_c(x_0, x_1, \dots) = (x_1, x_2, \dots).$$

For a k -dimensional Hilbert space H_k (isomorphically identified with \mathbb{C}^k) a *truncated shift* (Jordan block) $a_k \in \mathbf{B}(H_k)$ of order k , $1 \leq k < \infty$, is defined as

$$a_k(x_0, x_1, \dots, x_{k-1}) = (0, x_0, x_1, \dots, x_{k-2}).$$

Recall that $V \in \mathbf{B}(\mathcal{H})$ is a *power partial isometry* if all its powers V^n , $n \in \mathbb{N}$, are partial isometries. The operators a_s, a_c, a_k are examples of power partial isometries. Moreover, they appear in the model of a power partial isometry. Recall after [14]

Theorem 2.1. *Let $V \in \mathbf{B}(\mathcal{H})$ be a power partial isometry. There exist subspaces $\mathcal{H}_u(V), \mathcal{H}_s(V), \mathcal{H}_c(V), \mathcal{H}_t(V) \subset \mathcal{H}$ such that $\mathcal{H}_u(V), \mathcal{H}_s(V), \mathcal{H}_c(V), \mathcal{H}_t(V)$ reduce V and*

$V_u = V|_{\mathcal{H}_u(V)}$ *is a unitary operator,*

$V_s = V|_{\mathcal{H}_s(V)}$ *is a unilateral shift of arbitrary multiplicity,*

$V_c = V|_{\mathcal{H}_c(V)}$ *is a backward shift of arbitrary multiplicity,*

$V_t = V|_{\mathcal{H}_t(V)}$ *is possibly infinite orthogonal sum of truncated shifts and*

$$V = V_u \oplus V_s \oplus V_c \oplus V_t. \quad (2.1)$$

This decomposition is unique.

In the following paper the theorem above will be the starting point in the main proofs. As we can realize “the proper” behaviour of reflexivity and hyperreflexivity as to orthogonal sums and heredity to subspaces will be needed.

Remark 2.2. In [16–18] may be found theorems, which deal with heredity of hyperreflexivity for subspaces and property $\mathbb{A}_1(r)$. This results were presented in our context in [21, Proposition 2.2].

Remark 2.3. Combining [12, Theorem 6.16], [16, Theorem 5.1], [13, Theorems 3.8, 4.1] we get theorem, which deals with orthogonal sums of algebras and operators in the context of hyperreflexivity and property $\mathbb{A}_1(r)$. The combined version can be found in [21, Proposition 2.3].

In both Remarks above property $\mathbb{A}_1(r)$ were used. Recall after [7] that linear manifold $\mathcal{S} \subset \mathbf{B}(\mathcal{H})$ has *property \mathbb{A}_1* if for any weak*-continuous functional ϕ on \mathcal{S} there are $x, y \in \mathcal{H}$ such that $\phi(S) = \text{tr}(S(x \otimes y))$ for all $S \in \mathcal{S}$. (By $(x \otimes y)$ we denote rank one operator defined as $(x \otimes y)z = (z, y)x$ for $z \in \mathcal{H}$.) It is said that \mathcal{S} has *property $\mathbb{A}_1(r)$* , $r \geq 1$, if \mathcal{S} has property \mathbb{A}_1 and for any $\varepsilon > 0$ vectors x, y can be chosen such that $\|x \otimes y\|_1 \leq (r + \varepsilon)\|\phi\|$.

3. POWER PARTIAL ISOMETRIES ARE 2-HYPERREFLEXIVE

Theorem 3.1. *If $V \in \mathbf{B}(\mathcal{H})$ is a power partial isometry, then V is 2-hyperreflexive.*

Proof. Recall that $V = V_u \oplus V_s \oplus V_c \oplus V_t$ (see (2.1)). An algebra $\mathcal{W}(V_u)$ is hyperreflexive (see [23, Lemma 3.1]) and has property $\mathbb{A}_1(1)$ (see [9, Proposition 60.1]). Thus $\mathcal{W}(V_u^{(2)})$ is hyperreflexive and has property $\mathbb{A}_1(1)$ ([21, Proposition 2.3]). Similarly, since $\mathcal{W}(V_s)$

is hyperreflexive, $\kappa(\mathcal{W}(a_s)) < 11, 4$ (see [10,15]) and has property $\mathbb{A}_1(1)$ ([9, Proposition 60.5]), thus $\mathcal{W}(V_s^{(2)})$ is hyperreflexive and has property $\mathbb{A}_1(1)$ ([21, Proposition 2.3]). Recall also that both hyperreflexivity (with the same hyperreflexivity constant) and property $\mathbb{A}_1(1)$ are kept, when we take the adjoint, hence $\mathcal{W}(V_c^{(2)})$ is hyperreflexive and has property $\mathbb{A}_1(1)$ (see also [21, Proposition 3.1]). By [6, Proposition III.1.21] we get that $\mathcal{W}(V_t^{(2)})$ has property $\mathbb{A}_1(1)$. If $V_t = \oplus_{i=1}^m a_{k_i}$, then $\mathcal{W}(V_t^{(2)})$ is reflexive, since the largest block a_{k_m} appears at least twice in the decomposition (2.1) (see [11, Theorem 2]). Thus $\mathcal{W}(V_t^{(2)})$ is hyperreflexive, since, in such a case, underlying Hilbert space $\mathcal{H}(V)^{(2)}$ is finite dimensional, see [17,20]. If $V_t = \oplus_{i=1}^{\infty} a_{k_i}$ then $\mathcal{W}(V_t^{(2)})$ is also hyperreflexive (see [21, Theorem 3.3, Proposition 3.1]). Let us note that

$$\begin{aligned} \mathcal{W}(V^{(2)}) &= \mathcal{W}(V_u^{(2)} \oplus V_s^{(2)} \oplus V_c^{(2)} \oplus V_t^{(2)}) \\ &\subset \mathcal{W}(V_u^{(2)}) \oplus \mathcal{W}(V_s^{(2)}) \oplus \mathcal{W}(V_c^{(2)}) \oplus \mathcal{W}(V_t^{(2)}) \end{aligned}$$

and $\mathcal{W}(V_u^{(2)}) \oplus \mathcal{W}(V_s^{(2)}) \oplus \mathcal{W}(V_c^{(2)}) \oplus \mathcal{W}(V_t^{(2)})$ is hyperreflexive and has property $\mathbb{A}_1(1)$ ([21, Proposition 2.3]). Thus using [21, Proposition 2.2] we get hyperreflexivity of $\mathcal{W}(V^{(2)})$. By [16, Theorem 3.5] we obtain 2-hyperreflexivity of $\mathcal{W}(V)$. \square

Remark 3.2. It is worth to note that even if in the proof above we have shown hyperreflexivity we have not got an estimation of $\kappa_2(\mathcal{W}(V))$. The main reason is that it is not known whether $\kappa(\mathcal{W}(a_k \oplus a_k))$ is bounded independently on k .

Remark 3.3. 2-reflexivity of a power partial isometry is a weaker version of Theorem 3.1. On the other hand, it can be proved directly using the similar technique as in the proof of Theorem 3.1.

4. POWER PARTIAL ISOMETRIES ARE HYPOREFLEXIVE

Hyporeflexivity was introduced in [13]. We say that a commutative algebra $\mathcal{W} \subset \mathbf{B}(\mathcal{H})$ is *hyporeflexive* if

$$\mathcal{W} = \mathcal{W}' \cap \text{Alg Lat } \mathcal{W},$$

where \mathcal{W}' denotes a commutant of \mathcal{W} . Let us note that if a commutative algebra is reflexive, then it is hyporeflexive. An operator $T \in \mathbf{B}(\mathcal{H})$ is *hyporeflexive* if $\mathcal{W}(T)$ is hyporeflexive. Note that every reflexive operator $B \in \mathbf{B}(\mathcal{H})$ is hyporeflexive since an algebra $\mathcal{W}(B)$ is commutative. Let us recall that an operator acting on a finite dimensional Hilbert space is always hyporeflexive (see [8, Theorem 10]).

Now we prove two technical lemmas.

Lemma 4.1. *Let $\mathcal{H} = \oplus_{n \in \mathbb{N}} \mathcal{H}_n$ be an orthogonal sum of Hilbert spaces. Let us consider an operator $\oplus_{n \in \mathbb{N}} A_n \in \oplus_{n \in \mathbb{N}} \mathbf{B}(\mathcal{H}_n) \subset \mathbf{B}(\mathcal{H})$. Then*

$$(\mathcal{W}(\oplus_{n \in \mathbb{N}} A_n))' \cap (\oplus_{n \in \mathbb{N}} \mathbf{B}(\mathcal{H}_n)) = (\oplus_{n \in \mathbb{N}} \mathcal{W}(A_n))' \cap (\oplus_{n \in \mathbb{N}} \mathbf{B}(\mathcal{H}_n)).$$

Proof. Clearly $\mathcal{W}(\oplus_{n \in \mathbb{N}} A_n) \subset \oplus_{n \in \mathbb{N}} \mathcal{W}(A_n)$, thus by [9, Proposition 12.2] we get

$$(\mathcal{W}(\oplus_{n \in \mathbb{N}} A_n))' \supset (\oplus_{n \in \mathbb{N}} \mathcal{W}(A_n))',$$

so

$$(\mathcal{W}(\oplus_{n \in \mathbb{N}} A_n))' \cap (\oplus_{n \in \mathbb{N}} \mathbf{B}(\mathcal{H}_n)) \supset (\oplus_{n \in \mathbb{N}} \mathcal{W}(A_n))' \cap (\oplus_{n \in \mathbb{N}} \mathbf{B}(\mathcal{H}_n)).$$

Let us take $T = \oplus_{n \in \mathbb{N}} T_n \in (\mathcal{W}(\oplus_{n \in \mathbb{N}} A_n))' \cap (\oplus_{n \in \mathbb{N}} \mathbf{B}(\mathcal{H}_n))$. To prove the converse inclusion we should check that

$$(\oplus_{n \in \mathbb{N}} T_n)(\oplus_{n \in \mathbb{N}} B_n) = (\oplus_{n \in \mathbb{N}} B_n)(\oplus_{n \in \mathbb{N}} T_n) \tag{4.1}$$

for $\oplus_{n \in \mathbb{N}} B_n \in \oplus_{n \in \mathbb{N}} \mathcal{W}(A_n)$.

Let p be a polynomial. Then

$$\begin{aligned} (\oplus_{n \in \mathbb{N}} T_n) (p(A_1) \oplus (\oplus_{n \neq 1} 0)) &= (\oplus_{n \in \mathbb{N}} T_n) p(\oplus_{n \in \mathbb{N}} A_n) (I \oplus (\oplus_{n \neq 1} 0)) \\ &= p(\oplus_{n \in \mathbb{N}} A_n) (\oplus_{n \in \mathbb{N}} T_n) (I \oplus (\oplus_{n \neq 1} 0)) \\ &= p(\oplus_{n \in \mathbb{N}} A_n) (I \oplus (\oplus_{n \neq 1} 0)) (\oplus_{n \in \mathbb{N}} T_n) \\ &= (p(A_1) \oplus (\oplus_{n \neq 1} 0)) (\oplus_{n \in \mathbb{N}} T_n). \end{aligned}$$

Let us take $B_1 \in \mathcal{W}(A_1)$. There is a net of polynomials p_η such that $p_\eta(A_1)$ converges in the weak operator topology to B_1 . Then for $\oplus_{n \in \mathbb{N}} h_n, \oplus_{n \in \mathbb{N}} g_n \in \mathcal{H}$ we have

$$\begin{aligned} &\langle (\oplus_{n \in \mathbb{N}} T_n)(B_1 \oplus (\oplus_{n \neq 1} 0)) (\oplus_{n \in \mathbb{N}} h_n), (\oplus_{n \in \mathbb{N}} g_n) \rangle \\ &= \langle (B_1 \oplus (\oplus_{n \neq 1} 0)) (\oplus_{n \in \mathbb{N}} h_n), (\oplus_{n \in \mathbb{N}} T_n^*) (\oplus_{n \in \mathbb{N}} g_n) \rangle \\ &= \langle B_1 h_1, T_1^* g_1 \rangle = \lim \langle p_\eta(A_1) h_1, T_1^* g_1 \rangle \\ &= \lim \langle (\oplus_{n \in \mathbb{N}} T_n) (p_\eta(A_1) \oplus (\oplus_{n \neq 1} 0)) (\oplus_{n \in \mathbb{N}} h_n), (\oplus_{n \in \mathbb{N}} g_n) \rangle \\ &= \lim \langle (p_\eta(A_1) \oplus (\oplus_{n \neq 1} 0)) (\oplus_{n \in \mathbb{N}} T_n) (\oplus_{n \in \mathbb{N}} h_n), (\oplus_{n \in \mathbb{N}} g_n) \rangle \\ &= \lim \langle p_\eta(A_1) T_1 h_1, g_1 \rangle = \langle B_1 T_1 h_1, g_1 \rangle \\ &= \langle (B_1 \oplus (\oplus_{n \neq 1} 0)) (\oplus_{n \in \mathbb{N}} T_n) (\oplus_{n \in \mathbb{N}} h_n), (\oplus_{n \in \mathbb{N}} g_n) \rangle. \end{aligned}$$

Thus, (4.1) holds for $B_1 \oplus (\oplus_{n \neq 1} 0)$. Hence, it is also fulfilled for $\oplus_{n=1}^k B_n \oplus (\oplus_{n > k} 0)$ for any k . Since $\oplus_{n \in \mathbb{N}} B_n$ is the limit of $\oplus_{n=1}^k B_n \oplus (\oplus_{n > k} 0)$ ($k \rightarrow \infty$) in the weak and strong operator topology thus, as above, we get (4.1) for $\oplus_{n \in \mathbb{N}} B_n$. \square

Lemma 4.2. *Let $\mathcal{H} = \oplus_{n \in \mathbb{N}} \mathcal{H}_n$ be an orthogonal sum of Hilbert spaces. Let us consider an operator $\oplus_{n \in \mathbb{N}} A_n \in \oplus_{n \in \mathbb{N}} \mathbf{B}(\mathcal{H}_n) \subset \mathbf{B}(\mathcal{H})$. Then*

$$\text{Alg Lat } \mathcal{W}(\oplus_{n \in \mathbb{N}} A_n) \subset \oplus_{n \in \mathbb{N}} \mathbf{B}(\mathcal{H}_n).$$

Proof. Since $\mathcal{W}(\oplus_{n \in \mathbb{N}} A_n) \subset \oplus_{n \in \mathbb{N}} \mathcal{W}(A_n)$, thus by [4, Proposition 5.6] we get

$$\begin{aligned} \text{Alg Lat } \mathcal{W}(\oplus_{n \in \mathbb{N}} A_n) &\subset \text{Alg Lat}(\oplus_{n \in \mathbb{N}} \mathcal{W}(A_n)) \\ &= \oplus_{n \in \mathbb{N}} \text{Alg Lat } \mathcal{W}(A_n) \subset \oplus_{n \in \mathbb{N}} \mathbf{B}(\mathcal{H}_n). \end{aligned} \quad \square$$

Now we prove two main theorems of this section.

Theorem 4.3. *Let $\mathcal{H} = \bigoplus_{n \in \mathbb{N}} \mathcal{H}_n$ be an orthogonal sum of Hilbert spaces. Let $A_n \in \mathbf{B}(\mathcal{H}_n)$ be a hyporeflexive operator and let $\mathcal{W}(A_n)$ have property $\mathbb{A}_1(1)$ for any $n \in \mathbb{N}$. Then $\mathcal{W}(\bigoplus_{n \in \mathbb{N}} A_n)$ is hyporeflexive.*

Proof. Let us note that

$$\begin{aligned} & (\mathcal{W}(\bigoplus_{n \in \mathbb{N}} A_n))' \cap (\bigoplus_{n \in \mathbb{N}} \mathbf{B}(\mathcal{H}_n)) \cap \text{Alg Lat } \mathcal{W}(\bigoplus_{n \in \mathbb{N}} A_n) \\ & \subset (\mathcal{W}(\bigoplus_{n \in \mathbb{N}} A_n))' \cap (\bigoplus_{n \in \mathbb{N}} \mathbf{B}(\mathcal{H}_n)) \cap \text{Alg Lat}(\bigoplus_{n \in \mathbb{N}} \mathcal{W}(A_n)). \end{aligned}$$

By Lemma 4.2, we get that

$$\text{Alg Lat } \mathcal{W}(\bigoplus_{n \in \mathbb{N}} A_n) \subset \bigoplus_{n \in \mathbb{N}} \mathbf{B}(\mathcal{H}_n)$$

and from [4, Proposition 5.6] it follows that

$$\text{Alg Lat}(\bigoplus_{n \in \mathbb{N}} \mathcal{W}(A_n)) = \bigoplus_{n \in \mathbb{N}} \text{Alg Lat } \mathcal{W}(A_n) \subset \bigoplus_{n \in \mathbb{N}} \mathbf{B}(\mathcal{H}_n),$$

thus

$$\begin{aligned} & (\mathcal{W}(\bigoplus_{n \in \mathbb{N}} A_n))' \cap \text{Alg Lat } \mathcal{W}(\bigoplus_{n \in \mathbb{N}} A_n) \\ & \subset (\mathcal{W}(\bigoplus_{n \in \mathbb{N}} A_n))' \cap \text{Alg Lat}(\bigoplus_{n \in \mathbb{N}} \mathcal{W}(A_n)). \end{aligned}$$

Hence, using Lemma 4.1 we get

$$\begin{aligned} & (\mathcal{W}(\bigoplus_{n \in \mathbb{N}} A_n))' \cap \text{Alg Lat } \mathcal{W}(\bigoplus_{n \in \mathbb{N}} A_n) \\ & \subset (\mathcal{W}(\bigoplus_{n \in \mathbb{N}} A_n))' \cap \text{Alg Lat}(\bigoplus_{n \in \mathbb{N}} \mathcal{W}(A_n)) \\ & = (\mathcal{W}(\bigoplus_{n \in \mathbb{N}} A_n))' \cap (\bigoplus_{n \in \mathbb{N}} \mathbf{B}(\mathcal{H}_n)) \cap \text{Alg Lat}(\bigoplus_{n \in \mathbb{N}} \mathcal{W}(A_n)) \\ & = (\bigoplus_{n \in \mathbb{N}} \mathcal{W}(A_n))' \cap (\bigoplus_{n \in \mathbb{N}} \mathbf{B}(\mathcal{H}_n)) \cap \text{Alg Lat}(\bigoplus_{n \in \mathbb{N}} \mathcal{W}(A_n)) \\ & = (\bigoplus_{n \in \mathbb{N}} \mathcal{W}(A_n))' \cap \text{Alg Lat}(\bigoplus_{n \in \mathbb{N}} \mathcal{W}(A_n)) = \bigoplus_{n \in \mathbb{N}} \mathcal{W}(A_n). \end{aligned}$$

The last equality follows from the fact that since all operators A_n are hyporeflexive, for any $n \in \mathbb{N}$, thus the algebra $\bigoplus_{n \in \mathbb{N}} \mathcal{W}(A_n)$ is also hyporeflexive (see [13, Theorem 6.2(1)]). Now each algebra $\mathcal{W}(A_n)$ has property $\mathbb{A}_1(1)$ (for any $n \in \mathbb{N}$), thus the sum $\bigoplus_{n \in \mathbb{N}} \mathcal{W}(A_n)$ has property $\mathbb{A}_1(1)$ ([21, Proposition 2.3]). Property $\mathbb{A}_1(1)$ is hereditary ([21, Proposition 2.2]), thus

$$(\mathcal{W}(\bigoplus_{n \in \mathbb{N}} A_n))' \cap \text{Alg Lat } \mathcal{W}(\bigoplus_{n \in \mathbb{N}} A_n)$$

has property $\mathbb{A}_1(1)$. Using [13, Theorem 6.2(3)] we get hyporeflexivity of $\mathcal{W}(\bigoplus_{n \in \mathbb{N}} A_n)$. \square

Theorem 4.4. *Let $V \in \mathbf{B}(\mathcal{H})$ be a power partial isometry. Then V is hyporeflexive.*

Proof. Let $V = V_u \oplus V_s \oplus V_c \oplus V_t$ (see (2.1)). Since V_u is a unitary operator, hence $\mathcal{W}(V_u)$ is reflexive (see [22, Theorem 9.21]), thus hyporeflexive and has property $\mathbb{A}_1(1)$ by [9, Proposition 60.1].

The unilateral shift $a_s \in \mathbf{B}(l_+^2)$ is reflexive (see [24]), thus also hyporeflexive. Recall that $\mathcal{W}(a_s)$ has property $\mathbb{A}_1(1)$ (see [9, Proposition 60.5]), thus using Theorem 4.3 we get hyporeflexivity of $\mathcal{W}(V_s)$. It also has property $\mathbb{A}_1(1)$ by [21, Proposition 2.3]. The backward shift $a_c \in \mathbf{B}(l_+^2)$ is reflexive and has property $\mathbb{A}_1(1)$ (since both properties are preserved by taking the adjoint of operator), thus in the same way as above we get hyporeflexivity and property $\mathbb{A}_1(1)$ of $\mathcal{W}(V_c)$.

A truncated shift $a_k \in \mathbf{B}(H_k)$ is hyporeflexive, since every operator acting on finite dimensional space is hyporeflexive, see [8, Theorem 10]). Moreover, a_k has property $\mathbb{A}_1(1)$ ([6, Proposition III.1.21]). Hyporeflexivity and property $\mathbb{A}_1(1)$ of $\mathcal{W}(V_t)$ we get from Theorem 4.3.

Since $V = V_u \oplus V_s \oplus V_c \oplus V_t$, the hyporeflexivity of $\mathcal{W}(V)$ follows once again from Theorem 4.3. \square

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