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CHARACTERIZATIONS OF RECTANGULAR (PARA)-UNITARY RATIONAL FUNCTIONS

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Abstract. We here present three characterizations of not necessarily causal, rational functions which are (co)-isometric on the unit circle:
(i) through the realization matrix of Schur stable systems,
(ii) the Blaschke-Potapov product, which is then employed to introduce an easy-to-use description of all these functions with dimensions and McMillan degree as parameters,
(iii) through the (not necessarily reducible) Matrix Fraction Description (MFD).
In cases (ii) and (iii) the poles of the rational functions involved may be anywhere in the complex plane, but the unit circle (including both zero and infinity). A special attention is devoted to exploring the gap between the square and rectangular cases.

Keywords: isometry, coisometry, lossless, all-pass, realization, gramians, matrix fraction description, Blaschke-Potapov product.

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1. INTRODUCTION

This work is on the crossroads of Operator and Systems theory from the mathematical side and Control, Signal Processing and Communications theory from the engineering side. It addresses problems or employs tools from all these areas. Thus, it is meant to serve as a bridge between the corresponding communities. We start by formally laying out the set-up.

1.1. (PARA)-UNITARY SYMMETRY

Let $F(z)$ be $p \times m$-valued rational functions with poles everywhere in the complex plane $\mathbb{C}$ (including infinity), i.e. it can be written as

$$F(z) = C(zI - A)^{-1}B + D + \sum_{j=1}^{k} z^j E_j, \quad k \geq 0, \tag{1.1}$$

where the constant matrices A, B, C, and $D, E_1, \ldots, E_k$ are of dimensions $n \times n$, $m \times n$, $p \times n$ and $p \times m$, respectively. Whenever, $k \geq 1$, in system theory "dialect" $F(z)$ is said to have *poles at infinity* while in engineering "dialect" $F(z)$ is called an *improper* rational function. Furthermore, $F(z)$ may be viewed as the (two sided) Z-transform of an impulse response $\Phi(t)$, with t an integral variable. In particular, $k \geq 1$ means that $\Phi(t) \not\equiv 0$ for $t < 0$. Thus engineers call it *non-causal.*

Let $\mathbb{T}$ be the unit circle,

$$\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}.$$

In this work we focus on $\mathcal{U}$, the subclass of $p \times m$-valued rational functions in Eq. (1.1) having unitary symmetry on the unit circle, i.e.

$$\mathcal{U} := \left\{ F(z) : \begin{cases} (F(z))^* F(z) \equiv I_m & p \geq m \quad \text{isometry} \\ F(z)\,(F(z))^* \equiv I_p & m \geq p \quad \text{coisometry} \end{cases} \quad \forall z \in \mathbb{T} \right\}. \tag{1.2}$$

In signal processing "dialect" *unitary* is reserved to constant matrices while *para-unitary* means matrix-valued functions with some unitary symmetry as in $\mathcal{U}$, see Eq. (1.2). In mathematical literature, typically, both cases are referred to as *unitary.*

For a given $p \times m$-valued rational function $F(z)$, let $F^{\#}(z)$ be the $m \times p$-valued *conjugate* rational function, i.e.

$$F^{\#}(z) := \left(F\left(\tfrac{1}{z^*}\right)\right)^*.$$

Note that on the unit circle one has that

$$F^{\#}(z)_{|z \in \mathbb{T}} = \left(F(z)_{|z \in \mathbb{T}}\right)^*.$$

It is well known (see e.g. [1, Eq. (3.1)], [34, Eq. (1.9)]) that for rational functions condition Eq. (1.2) is equivalent to the following, for all $z \in \mathbb{C}$,

$$\begin{cases} F^{\#}(z)F(z) \equiv I_m & p \geq m \quad \text{isometry} \\ F(z)F^{\#}(z) \equiv I_p & m \geq p \quad \text{coisometry.} \end{cases}$$

The interest in the class $\mathcal{U}$ is from various aspects, see e.g. [1, 2, 9, 10, 13, 19, 21, 23, 33, 34, 36, 37, 39, 41, 44].

Clearly, whenever $F(z)$ is in $\mathcal{U}$ it must be analytic on $\mathbb{T}$. There are (at least) two common special cases:

(i) If $F(z)$ is analytic outside the closed unit disk (=Schur stable), then in engineering terminology it is called *lossless*[1)], see e.g. [21], [43, Section 14.2] or *all-pass*[2)].
(ii) If for $p \geq m$ ($m \geq p$) the matrix $I_m - (F(z))^*F(z)$ ($I_p - F(z)(F(z))^*$) is positive semi-definite, within the unit disk, $1 \geq |z|$, then $F(z)$ is anti Schur stable[3)], i.e. its conjugate $F^{\#}(z)$ is Schur stable.

The interest in rational functions within $\mathcal{U}$ is vast, see e.g. the books [14, 29], [32, Section 7.3], [40, Section 5.2], [43, Section 6.5] and the papers [3–5, 7, 13, 17, 20, 28, 30, 35, 38, 42, 46] and [47].

This work is aimed at three different communities: mathematicians interested in classical analysis, signal processing engineers and system and control engineers. Thus adopting the terminology familiar to one audience, may intimidate or even alienate the other. For example as we already mentioned, rational functions which are improper or have poles at infinity or non-causal, are virtually the same entity seen by a different community. Similarly, what is known to engineers as McMillan degree also arises in geometry of loop groups as an index.

Books like [12, 14, 40], and the theses [27, 34] have made an effort to be at least "bi-lingual". Lack of space prevents us from providing even a concise dictionary of relevant terms. Instead, we try to employ only basic concepts or indicate for references providing for the necessary background.

The differences between scientific communities go beyond terminology. Closely related problems are formulated not in the same framework. For example, in many of the engineering references in Eq. (1.1) $F(z)$ is assumed to be analytic outside the open unit disk (=Schur stable), i.e. $k = 0$ and the spectral radius of A is less than one. In other references $F(z)$ is a genuine matrix valued polynomial, i.e. in Eq. (1.1) B or C vanish or in Eq. (5.6) $q \geq N$. We here try to provide a simple, yet full, picture.

This work is organized as follows.

In Section 2 we show that a square rational $F(z)$ in $\mathcal{U}$ can always be truncated (by eliminating rows or columns) to a rectangular function in $\mathcal{U}$. Conversely, a rectangular rational function in $\mathcal{U}$, can always be embedded (by adding rows or columns) in a square function in $\mathcal{U}$.

On the one hand, in the special case where $F(z)$ is analytic outside the open unit disk, this result is well known. On the other hand if $\mathcal{U}$ is substituted by *indefinite* inner product, this result is not always true (see discussion below). This suggests that our result is not trivial.

In passing, we explore the controllability and observability gramians associated with rectangular Schur stable (co)-isometries on the unit circle.

In Section 3 we combine the classical Blaschke-Potapov product formula along with the main result of the preceding section, to introduce a characterization of rectangular

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1) Passive electrical circuits are either dissipative or lossless.

2) For example, in studying classical filters a "high-pass" could be viewed as an "all-pass" minus a "low-pass".

3) In control engineering circles a Schur stable functions in $\mathcal{U}$ is called "inner", see e.g. [48, Subsection 21.5.1], while in mathematical analysis the same term is attributed to the anti Schur stable case, see e.g. [10, Section 4].

(co)-isometries on the unit circle, with poles everywhere (including infinity) excluding the unit circle.

In Section 4 we then exploit the above characterization to introduce in a compact, convex, easy-to-use, description of all rational functions in $\mathcal{U}$ parametrized by their McMillan degree and dimensions. Again, the poles may be everywhere (including infinity) excluding the unit circle. It is straightforward to restrict this parametrization to Schur stable functions.

This is in particular convenient if one wishes to:

(i) Design through optimization, a rational function (co)-isometric on the unit circle, see e.g. [17, 24, 38, 42] and [46];
(ii) Iteratively apply para-unitary similarity, see e.g. [27, Section 3.3], [33, 39]. In signal processing literature, this is associated with *channel equalization* and in communications literature with *decorrelation of signals*; or
(iii) Iteratively apply Q-R factorization in the framework of communications, see e.g. [15, 16].

In Section 5 we resort to the Matrix Fraction Description (MFD) of the $p \times m$-valued rational function $F(z)$, i.e.

$$F(z) = \begin{cases} N(z)\,(\Delta(z))^{-1} & N(z) \quad p \times m - \text{valuedpolynomial}, \\ & \Delta(z) \quad m \times m - \text{valuedpolynomial} \quad p \geq m, \\ \left(\tilde{\Delta}(z)\right)^{-1} \tilde{N}(z) & \tilde{N}(z) \quad p \times m - \text{valuedpolynomial}, \\ & \tilde{\Delta}(z) \quad p \times p - \text{valuedpolynomial} \quad m \geq p. \end{cases}$$

See e.g. [31, Chapter 6], [43, Section 13.3] or [45, Chapter 4]. In Theorem 5.1 we introduce an, MFD based, easy-to-check characterization of $F(z)$ in $\mathcal{U}$. Note that this test does not require any minimality of this representation.

In [5] we focus on the subclass rational functions: In mathematical terms $F(z)$ are $p \times m$-valued polynomials with powers of possibly mixed signs, i.e. where in Eq. (1.1) the matrix A is nilpotent (i.e. A^l vanishes for some natural l). In engineering "dialect" these are (not necessarily causal) *Finite Impulse Response* functions. We there present three characterizations of those functions within $\mathcal{U}$. Here, (in Theorem 5.2 below) we use Theorem 5.1 to offer an alternative proof of one of the main results in [5].

2. RECTANGULAR VS. SQUARE PARA-UNITARY RATIONAL FUNCTIONS

In this section we show that in the framework of (co)-isometric rational functions, the rectangular case is essentially equivalent (in a rigorous sense, see Theorem 2.3) to the square case.

We do it in two stages. First the easier Schur stable case and then extend it to rational functions with poles anywhere in the complex plane (including zero and infinity) but the unit circle.

2.1. MINIMAL STATE-SPACE REALIZATION OF SCHUR STABLE SYSTEMS

This subsection provides known background material used for the sequel.

Recall that if a $p \times m$-valued rational function $F(z)$ is so that

$$\text{there exists} \quad \lim_{z \to \infty} F(z),$$

i.e. in Eq. (1.1) $k = 0$, it is bounded at infinity[4)], then it admits a state space realization

$$F(z) = C(zI_n - A)^{-1}B + D. \tag{2.1}$$

Sometimes it is convenient to present $F(z)$ in Eq. (2.1) by its $(n+p) \times (n+m)$ realization matrix R, i.e.

$$R := \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right). \tag{2.2}$$

A realization is called *minimal* if n, the dimension of A, is the smallest possible.

Assuming that $F(z)$ in Eq. (2.1) is analytic outside the open unit disk, in Theorem 2.1 below we present a characterization, through the corresponding realization matrix R in Eq. (2.2), of Schur stable rectangular rational functions in $\mathcal{U}$.

We here mention some of the existing variants of this result: The basic case is where R in Eq. (2.2) is square and the associated inner-product is definite. An extension to indefinite inner product framework appeared in [1, Theorem 3.1], [2, Theorem 2.1] and [21, Lemma 2 and Theorem 3]. In [10, Theorem 4.5], the study was further generalized to the rectangular case, i.e. $F^*(z)J_pF(z) = J_m$ with J_p, J_m signature matrices, i.e. diagonal matrices satisfying $J_p^2 = I_p$ and $J_m^2 = I_m$, see [10, Theorem 3.1].

However, the result in [10] requires the introduction of a condition on the *defect* of $F(z)$, for definition see [18], [31, p. 460] and for detailed discussion in the context of rectangular isometries see [9, Section 2], [10, Section 2].

Restricting the discussion to the Schur stable case (spectrum within the open unit disk) enabled one to prove the above result by resorting to a more modest tool from Matrix Theory.

Theorem 2.1. *Let $F(z)$ be a $p \times m$-valued rational function with poles within the open unit disk* (*Schur stable*).

I. *Assume that $p \geq m$.*
   (i) *$F(z)$ is in $\mathcal{U}$ (=lossless) if and only if, it admits $(p+n) \times (m+n)$ minimal realization matrix Eq.* (2.2)

$$R := \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right).$$

satisfying

$$R^* \cdot \begin{pmatrix} I_n & 0 \\ 0 & I_p \end{pmatrix} \cdot R = \begin{pmatrix} I_n & 0 \\ 0 & I_p \end{pmatrix}. \tag{2.3}$$

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4) In engineering it is colloquially called *proper*. Note also that $F(z)$ is referred to as *causal*. This is since that when $F(z)$ is viewed as the (two-sided) Z-transform of a discrete-time sequence $\Phi(t)$ (t integral variable), then $\Phi(t) \equiv 0$ for all $t < 0$.

(ii) *If Eq.* (2.3) *holds, one can always find* $\tilde{B}\in\mathbb{C}^{n\times(p-m)}$ *and* $\tilde{D}\in\mathbb{C}^{p\times(p-m)}$ *so that the* $(n+p)\times(n+p)$ *augmented matrix*

$$R_{n+p} := \left(\begin{array}{c|cc} A & B & \tilde{B} \\ \hline C & D & \tilde{D} \end{array}\right), \tag{2.4}$$

is unitary, i.e.

$$R^*_{n+p}R_{n+p} = I_{n+p} = R_{n+p}R^*_{n+p}. \tag{2.5}$$

(iii) *If Eq.* (2.5) *holds, one can always find a constant isometry* U_{iso} *so that*

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = R = R_{n+p}\cdot\left(\begin{array}{c|c} I_n & 0_{n\times m} \\ \hline 0_{p\times n} & U_{\text{iso}} \end{array}\right) \quad U_{\text{iso}}\in\mathbb{C}^{p\times m}, \quad U^*_{\text{iso}}U_{\text{iso}} = I_m \tag{2.6}$$

II. *Assume that* $m\geq p$.

(i) $F(z)$ *is in* $\mathcal{U}$ *(=lossless) if and only if, it admits* $(p+n)\times(m+n)$ *minimal realization matrix Eq.* (2.2)

$$R := \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

satisfying

$$R\cdot\text{diag}\{I_n \quad I_m\}\cdot R^* = \text{diag}\{I_n \quad I_p\}. \tag{2.7}$$

(ii) *If Eq.* (2.7) *holds, one can always find* $\tilde{C}\in\mathbb{C}^{(m-p)\times n}$ *and* $\tilde{D}\in\mathbb{C}^{(m-p)\times m}$ *so that the* $(n+m)\times(n+m)$ *augmented matrix*

$$R_{n+m} := \left(\begin{array}{c|c} A & B \\ \hline C & D \\ \tilde{C} & \tilde{D} \end{array}\right) \tag{2.8}$$

is unitary, i.e.

$$R^*_{n+m}R_{n+m} = I_{n+m} = R_{n+m}R^*_{n+m}. \tag{2.9}$$

(iii) *If Eq.* (2.9) *holds, one can always find, a constant coiometry* U_{coiso} *so that*

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = R = \left(\begin{array}{c|c} I_n & 0_{n\times m} \\ \hline 0_{p\times n} & U_{\text{coiso}} \end{array}\right)\cdot R_{n+m}, \quad U_{\text{cosio}}\in\mathbb{C}^{p\times m},\ U_{\text{coiso}}U^*_{\text{coiso}} = I_p. \tag{2.10}$$

Proof. Assume $p\geq m$.

Part (i) is an adaption of [43, Theorem 14.5.1].

Part (ii) appears in [48, Lemma 21.21].

Part (iii) follows from the fact that multiplying from the right a $(n+p)\times(n+p)$ unitary, by a $(n+p)\times(n+m)$ isometry yields another $(n+p)\times(n+m)$ isometry.

As the case $m\geq p$ is analogous, its proof is omitted. □

As already mentioned, the Schur stable case addressed in Theorem 2.1, will be extended to rational functions with poles anywhere in $\{\mathbb{C}\cup\infty\}\setminus\mathbb{T}$, in Theorem 2.3 in the next subsection.

Still in the Schur stable framework (the spectrum of A, the upper left block of R in Eq. (2.2) is within the open unit disk), we now recall the notion of Controllability and Observability Gramians (for the continuous-time case see e.g. [31, Subsections 9.2.1, 9.2.2], [48, Sections 3.8, 15.1]. We shall denote by $W_{\rm cont}$, $W_{\rm obs}$, the $n\times n$ Controllability and Observability Gramians, respectively, obtained from the solution to the corresponding Stein equations:

$$W_{\rm cont}-AW_{\rm cont}A^*=BB^*,\quad W_{\rm obs}-A^*W_{\rm obs}A=C^*C. \tag{2.11}$$

The following, is essentially known, for completeness a proof is provided.

Proposition 2.2. *Let $F(z)$ be a $p\times m$-valued rational function whose poles are within the open unit disk and denote by $W_{\rm cont}$, $W_{\rm obs}$ the associated controllability and observability gramians, respectively. Assume that $F(z)$ is in $\mathcal{U}$.*

I. *If $p\geq m$, $F(z)$ admits a state space realization R in Eq.* (2.3) *so that*

$$(I_n-W_{\rm cont})\ \textit{is positive semidefinite and}\ W_{\rm obs}=I_n.$$

II. *If $m\geq p$, $F(z)$ admits a state space realization R in Eq.* (2.7) *so that*

$$W_{\rm cont}=I_n\ \textit{and}\ (I_n-W_{\rm obs})\ \textit{is positive semidefinite.}$$

III. *If $p=m$, $F(z)$ admits a state space realization R in Eqs.* (2.3), (2.7) *so that*

$$W_{\rm cont}=I_n\quad\textit{and}\quad W_{\rm obs}=I_n.$$

Proof. Indeed, assume $p\geq m$. From the upper left block of Eq. (2.3), it follows that $W_{\rm obs}=I_n$. Consider now Eq. (2.4). The upper left block of the equation $R_{n+p}R_{n+p}^*={\rm diag}\{I_n\quad I_p\}$ reads

$$I_n-AA^*=BB^*+\tilde{B}\tilde{B}^*.$$

Now, from Eq. (2.11) we have that

$$W_{\rm cont}-AW_{\rm cont}A^*=BB^*.$$

Subtraction of the two equations yields

$$(I_n-W_{\rm cont})-A(I_n-W_{\rm cont})A^*=\tilde{B}\tilde{B}^*,$$

so the first part of the claim is established.

As the proof the second part is analogous, it is omitted. The third part follows from the first two. □

We conclude this subsection with a couple of brief comments.

(a) Part III of Proposition 2.2 is classical, see e.g. [1, Section 3], [21, Corollary 3] and later in [34, Proposition 1.2.1].
(b) The technique employed in Eqs. (2.4), (2.8) in the proof, is commonly used in system theory for the Hankel norm approximation and is known as *all-pass embedding.*

2.2. RECTANGULAR PARA-UNITARY RATIONAL FUNCTIONS

Theorem 2.3, our first main result, establishes a close connection between square and rectangular rational functions in $\mathcal{U}$, with poles at $\{\mathbb{C} \cup \infty\} \setminus \mathbb{T}$.

Theorem 2.3. *Let $F(z)$ be a $p \times m$-valued rational function.*

I. *Assume that $p \geq m$. $F(z)$ is in $\mathcal{U}$ if and only if, there exists in $\mathcal{U}$, a $p \times p$-valued rational function $F_p(z)$, so that*

$$F(z) = F_p(z)U_{\text{iso}}, \quad U_{\text{iso}} \in \mathbb{C}^{p\times m},\ U_{\text{iso}}^*U_{\text{iso}} = I_m.$$

II. *Assume that $m \geq p$. $F(z)$ is in $\mathcal{U}$ if and only if, there exists in $\mathcal{U}$ a $m \times m$-valued rational function $F_m(z)$, so that*

$$F(z) = U_{\text{coiso}}F_m(z), \quad U_{\text{coiso}} \in \mathbb{C}^{p\times m},\ U_{\text{coiso}}U_{\text{coiso}}^* = I_p.$$

The proof is relegated further down this subsection.

It should be pointed be pointed out that in [9, Proposition 2.1] a similar result is formulated for the case where on the imaginary axis (instead of the unit circle)

$$(F(z))^* J_p F(z) = J_m$$

with J_m, J_p signature matrices, i.e. diagonals satisfying $J_m^2 = I_m,\ J_p^2 = I_p$.

As already mentioned above, restricting the discussion here to $J_m = I_m,\ J_p = I_p$ enables us to prove the result through basic matrix theory tools and to avoid the introduction of the subtle notion of *defect* of $F(z)$.

In the sequel we shall use the fact that the scalar rational function (known as a Blaschke-Potapov factor)

$$\phi(z) = \frac{1 - \alpha^* z}{z - \alpha}, \quad \alpha \in \{\infty \cup \mathbb{C}\} \setminus \mathbb{T},$$

is well defined $\left(\phi(z)_{|\alpha=\infty} = z\right)$ and satisfies

$$|\phi(z)| = 1 \quad \text{for all} \quad z \in \mathbb{T}.$$

We start with an illustrative example.

Example 2.4. In part II of Theorem 2.3 take $m = 2$,

$$F_m(z) := \frac{1}{\sqrt{2}} \begin{pmatrix} \phi(z) & \psi(z) \\ -\left(\psi(z)\right)^{\#} & \left(\phi(z)\right)^{\#} \end{pmatrix}, \tag{2.12}$$

where $\phi(z), \psi(z)$ are scalar rational functions. Then

$$\left(F_m(z)\right)^{\#} F_m(z) = \tfrac{1}{2} \left(\left(\phi(z)\right)^{\#} \phi(z) + \left(\psi(z)\right)^{\#} \psi(z) \right) I_2.$$

Construct from $F_m(z)$ in Eq. (2.12), the following 1×2-valued rational function

$$F(z) = U_{\text{coiso}}F_m(z), \quad U_{\text{coiso}} = (1 \quad 0), \tag{2.13}$$

i.e.

$$F(z) = \tfrac{1}{\sqrt{2}} \begin{pmatrix} \phi(z) & \psi(z) \end{pmatrix}.$$

Now, $F(z)$ in Eq. (2.13) is in $\mathcal{U}$, if and only if $F_m(z)$ in Eq. (2.12) is in $\mathcal{U}$.

This in turn, is equivalent to having $\phi(z)$, $\psi(z)$ of the form

$$\phi(z) = \prod_{j=1}^{\overline{j}} \frac{1 - {\alpha_j}^* z}{z - \alpha_j}, \quad \psi(z) = \prod_{k=1}^{\overline{k}} \frac{1 - {\beta_k}^* z}{z - \beta_k},$$

where $\overline{j}$ and $\overline{k}$ are non-negative integers, and $\alpha_j, \beta_k \in \{\infty \cup \mathbb{C}\} \setminus \mathbb{T}$. (Recall that $\prod_1^0 := 1$.)

To prove Theorem 2.3 we resort to the following.

Lemma 2.5. *Let $F(z)$ be a $p \times m$-valued rational function with poles at $\{\infty \cup \mathbb{C}\} \setminus \mathbb{T}$.*

I. *Assume $p \geq m$*
One can always find a $m \times m$-valued function $U_m(z)$ in $\mathcal{U}$, so that the poles of $F_o(z)$, i.e.

$$F_o(z) := F(z) U_m(z) \tag{2.14}$$

are all in the open unit disk (Schur stable).
Moreover, $F(z)$ is in $\mathcal{U}$, if and only if, $F_o(z)$ is in $\mathcal{U}$.

II. *Assume $m \geq p$*
One can always find a $p \times p$-valued function $U_p(z)$ in $\mathcal{U}$, so that the poles of $F_o(z)$, i.e.

$$F_o(z) := U_p(z) F(z)$$

are all in the open unit disk (Schur stable).

Moreover, $F(z)$ is in $\mathcal{U}$, if and only if, $F_o(z)$ is in $\mathcal{U}$.

Proof. I. Assume $p \geq m$

Clearly, for an arbitrary $m \times m$-valued $U_m(z)$ in $\mathcal{U}$, one has that in Eq. (2.14) $F(z)$ is in $\mathcal{U}$, if and only if, $F_o(z)$ is in $\mathcal{U}$.

Without loss of generality, we shall order the poles of $F(z)$ (including multiplicities) $\alpha_1, \ldots, \alpha_t, \alpha_{t+1}, \ldots, \alpha_l$ as

$$\infty \geq |\alpha_1| \geq \ldots \geq |\alpha_t| > 1 > |\alpha_{t+1}| \geq \ldots \geq |\alpha_l| \geq 0.$$

Take now in Eq. (2.14)

$$U_m(z) := \prod_{j=1}^{t} \frac{z - \alpha_j}{1 - \alpha_j^* z} I_m.$$

It is easy to verify that the poles of $F_o(z)$ in Eq. (2.14) are at

$$\frac{1}{{\alpha_1}^*}, \ldots, \frac{1}{{\alpha_t}^*}, \alpha_{t+1}, \ldots, \alpha_l$$

and in particular they are all in the open unit disk.

The proof of the case $m \geq p$ is analogous and thus omitted. □

There are numerous ways to construct $U_m(z)$ in Eq. (2.14) (or $U_p(z)$). The choice in the above proof was solely to simplify the presentation. It is by no means "good" in other senses.

We can now establish the main result of this section.

Proof of Theorem 2.3. If $F(z)$ is Schur stable (poles within the open unit disk), the claim is established by using $U_{\rm iso}$, $U_{\rm coiso}$ from Eqs. (2.6), (2.10), respectively.

If the poles of $F(z)$ are anywhere in $\{\infty \cup \mathbb{C}\} \setminus \mathbb{T}$, by employing Lemma 2.5 one may obtain a Schur stable $F_o(z)$. Now, by the first part, the claim is established. □

The following example illustrates some of the results of this section.

Example 2.6. From Example 2.4 we here consider the 1×2-valued $F(z)$ (see Eq. (2.13)) and the 2×2-valued $F_m(z)$ satisfying

$$F(z) = (1 \quad 0) F_m(z).$$

For simplicity take in Eq. (2.13) $\overline{j} = 1$, $\overline{k} = 0$ so that $F(z)$ and $F_m(z)$ are of the form

$$F(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1-\alpha^* z}{z-\alpha} & 1 \end{pmatrix}, \quad F_m(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1-\alpha^* z}{z-\alpha} & 1 \\ -1 & \frac{z-\alpha}{1-\alpha^* z} \end{pmatrix}, \quad \alpha \in \{\infty \cup \mathbb{C}\} \setminus \mathbb{T}. \tag{2.15}$$

Now, whenever α is restricted to be finite, $F(z)$ in Eq. (2.13) admits a (minimal) state space realization of the form Eq. (2.2) with,

$$R = \left(\begin{array}{c|cc} \alpha & \frac{1-|\alpha|^2}{\sqrt{2}} & 0 \\ \hline 1 & -\frac{\alpha^*}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{array}\right), \quad \alpha \in \{\mathbb{C} \setminus \mathbb{T}\}.$$

Furthermore, in accordance to part II of Theorem 2.1, it is only when $F(z)$ in Eq. (2.15) is lossless (i.e. $1 > |\alpha|$), that it admits an equivalent minimal realization,

$$\hat{R} = \left(\begin{array}{c|cc} \alpha & \sqrt{1-|\alpha|^2} & 0 \\ \hline \frac{\sqrt{1-|\alpha|^2}}{\sqrt{2}} & -\frac{\alpha^*}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{array}\right), \quad 1 > |\alpha|, \tag{2.16}$$

satisfying

$$\hat{R} \cdot \begin{pmatrix} 1 & 0 \\ 0 & I_2 \end{pmatrix} \cdot \hat{R}^* = \begin{pmatrix} 1 & 0 \\ 0 & I_2 \end{pmatrix}.$$

In fact, following part II of Proposition 2.2, here the observability gramian is $W_{\rm obs} = \frac{1}{2}$.

Moreover, following Eq. (2.8), $\hat{R}$ in Eq. (2.16) may be extended to (here $n = 1$, $m = 2$),

$$R_{n+m} = \left(\begin{array}{c|cc} \alpha & \sqrt{1-|\alpha|^2} & 0 \\ \hline \frac{\sqrt{1-|\alpha|^2}}{\sqrt{2}} & -\frac{\alpha^*}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{\sqrt{1-|\alpha|^2}}{\sqrt{2}} & -\frac{\alpha^*}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{array}\right),$$

satisfying

$$R_{n+m} R_{n+m}^* = I_{m+n} = R_{n+m}^* R_{n+m}.$$

3. A CHARACTERIZATION THROUGH THE BLASCHKE-POTAPOV PRODUCT

We first recall Potapov's classical characterization of the set of rational functions in $\mathcal{U}$. Here is a brief perspective. The Fundamental Theorem, see [37, p. 133], was formulated in the following framework,

$$\begin{array}{llll} J - F(z)JF^*(z) & \text{positive semidefinite} & 1 \geq |z| & J \quad \text{diagonal} \\ J = F(z)JF^*(z) & & 1 = |z| & J^2 = I. \end{array} \tag{3.1}$$

A similar result, independently appeared in [19, Theorem 17] and yet another independent (and more general) version in [21, Theorem 9].

A special case of this result where $J = I$, was advertized in the Signal Processing community in [43, Section 14.9.1], see also [20]. In all these cases it was assumed that $F(z)$ is analytic outside the open unit disk (Schur stable).

In [1, Theorem 3.11], Potapov's Fundamental Theorem was extended to the case where $F(z)$ is analytic on the circle only (with poles possibly at infinity as well).

We shall denote by P a rank one orthogonal projection, i.e.

$$P^* = P = P^2, \quad \mathrm{rank}(P) = 1.$$

Recall that if P is $k \times k$ it can always be written as

$$P = vv^*, \quad v^*v = 1, \quad v \in \mathbb{C}^k. \tag{3.2}$$

Recall also that a rank $k-1$ orthogonal projection Q, i.e.

$$Q^* = Q^2 = Q, \quad \mathrm{rank}(Q) = k-1,$$

can always be written as

$$Q := I_k - vv^*, \quad v^*v = 1, \quad v \in \mathbb{C}^k, \tag{3.3}$$

as in Eq. (3.2).

Theorem 3.1. *Let $F(z)$ be a $p \times m$-valued rational function of McMillan degree d. $F(z)$ is in $\mathcal{U}$, Eq. (1.2), if and only if it can be written as*

$$\begin{array}{l} p \geq m \quad F(z) = \left(\prod_{j=1}^{d} \left(I_p + \left(\frac{1-\alpha_j^* z}{z-\alpha_j} - 1 \right) v_j v_j^* \right) \right) U_{\mathrm{iso}}, \\ v_j \in \mathbb{C}^p, \quad v_j^* v_j = 1, \quad U_{\mathrm{iso}} \in \mathbb{C}^{p \times m}, \quad U_{\mathrm{iso}}^* U_{\mathrm{iso}} = I_m, \quad \alpha_j \in \{\infty \cup \mathbb{C}\} \setminus \mathbb{T}, \\ m \geq p \quad F(z) = U_{\mathrm{coiso}} \left(\prod_{j=1}^{d} \left(I_m + \left(\frac{1-\alpha_j^* z}{z-\alpha_j} - 1 \right) v_j v_j^* \right) \right) \\ v_j \in \mathbb{C}^m \quad v_j^* v_j = 1, \quad U_{\mathrm{coiso}} \in \mathbb{C}^{p \times m}, \quad U_{\mathrm{coiso}} U_{\mathrm{coiso}}^* = I_p. \end{array} \tag{3.4}$$

Recall $\prod_{j=1}^{0} := I$

Proof. Substituting in [1, Theorem 3.11] the special case $J = I$ (definite inner product), yields the following:

An $m \times m$-valued rational function $F(z)$, of McMillan degree d, is in $\mathcal{U}$ (see Eq. (1.2)) if and only if (up to multiplication by a constant $m \times m$ unitary matrix from the left or from the right) it can be written as

$$F(z) = \prod_{j=1}^{d} \left(I_m + \left(\tfrac{1-\alpha_j^* z}{z-\alpha_j} - 1 \right) v_j v_j^* \right), \quad \alpha_j \in \{\infty \cup \mathbb{C}\} \setminus \mathbb{T}. \tag{3.5}$$

Using, Eqs. (3.2) and (3.3), establishes Eq. (3.4) for $m = p$.

To obtain the rectangular case, apply Theorem 2.3. □

Three remarks are now in order.

(a) It is tempting to combine [1, Theorem 3.11] along with the above Theorem 3.1, to formulate a rectangular version of Blaschke-Potapov product result with poles in $\{\infty \cup \mathbb{C}\} \setminus \mathbb{T}$ for *indefinite* inner product, see Eq. (3.1). However, this requires some caution as then, the notion of the *defect* of $F(z)$ needs to be addressed. For definition see [18], [31, p. 460] and for detailed discussion in the context of rectangular isometries see [9, Section 2].

(b) Theorem 3.1 asserts that whenever $F \in \mathcal{U}$ is of McMillan degree d, *there exist* rank one orthogonal projections $v_1 v_1^*, \ldots, v_d v_d^*$, satisfying Eq. (3.4). In general, the McMillan degree of the product in the right hand side of Eq. (3.5) is *at most d*. For example,

$$\begin{aligned} &(I + (\phi_1(z) - 1)v_1 v_1^*)\,(I + (\phi_2(z) - 1)v_2 v_2^*)_{|_{v_1 v_1^* = v_2 v_2^*}} \\ &= (I + (\phi_1(z)\phi_2(z) - 1)v_1 v_1^*)_{|_{\phi_1(z)\phi_2(z) \equiv 1}} = I, \end{aligned}$$

which is a zero degree rational function.

(c) Note that products of the form

$$v_1 v_1^* v_2 v_2^* \cdots v_k v_k^* = \left(\prod_{j=1}^{k-1} v_j^* v_{j+1} \right) v_1 v_k^*, \quad k \geq 2,$$

which appear in Eq. (3.4), always produce a rank one matrix. In the special case where $v_1 v_1^* = \ldots = v_d v_d^*$ this is an orthogonal projection, else it is a strict contraction.

4. PARAMETRIZATION OF ALL PARA-UNITARY RATIONAL FUNCTIONS

We next exploit the above Theorem 3.1 to describe all rational function in $\mathcal{U}$, parametrized by dimensions and the McMillan degree.

To this end, we introduce the following matrix theory notation

$$\begin{aligned} \mathbb{U}_{\text{Iso}} &:= \{U \in \mathbb{C}^{p \times m} \quad p \geq m : U^* U = I_m\}, \\ \mathbb{U}_{\text{Coiso}} &:= \{U \in \mathbb{C}^{p \times m} \quad m \geq p : U U^* = I_p\}. \end{aligned} \tag{4.1}$$

Lemma 4.1. *The set* $\mathbb{U}_{\mathrm{Iso}}$ *in Eq.* (4.1) *may be completely parametrized by*

$$[0, 2\pi)^{m(2p-m)}. \tag{4.2}$$

Similarly, the set $\mathbb{U}_{\mathrm{Coiso}}$ *in Eq.* (4.1) *may be completely parametrized by*

$$[0, 2\pi)^{p(2m-p)}.$$

Indeed, due to symmetry, one address only the case of $p \geq m$. Now, the set of all $v \in \mathbb{C}^p$ with $v^*v = 1$, i.e. the $\|\cdot\|_2$ unit sphere in $\in \mathbb{C}^p$ may be identified with with

$$[0, 2\pi)^{2p-1}.$$

For example, for $p = 3$ this v is of the form

$$v = \begin{pmatrix} \cos(\alpha)e^{i\eta} \\ \cos(\beta)\sin(\alpha)e^{i\gamma} \\ \sin(\beta)\sin(\alpha)e^{i\delta} \end{pmatrix}, \quad \alpha, \beta, \gamma, \delta, \eta \in [0, 2\pi).$$

To obtain all m-dimensional orthonormal bases of such vectors, one resorts to Eq. (4.2), so the claim is established.

A word of caution. Consider for simplicity the case of unitary matrices where $p = m$ are prescribed. One can ask the two following questions:

(i) How many parameters are required to completely describe the whole set?
(ii) How many parameters are required to completely describe all unitary similarity transformations?

The above lemma addresses the first question. The following example illustrates the gap between these two.

Example 4.2. Consider for simplicity the case of $p = m = 2$.

Every unitary matrix U may be written as

$$U = \begin{pmatrix} e^{i(\gamma-\beta)}\cos(\alpha) & e^{i\delta}\sin(\alpha) \\ -e^{-i\beta}\sin(\alpha) & e^{i(\delta-\gamma)}\cos(\alpha) \end{pmatrix}, \quad \alpha, \beta, \gamma, \delta \in [0, 2\pi).$$

Namely, this set may be identified with $[0, 2\pi)^4$.

However, if for a given 2×2 matrix M, one is interested in all unitary similarity transformations of the form U^*MU, without loss of generality, one can assume that in the above U,

$$\beta = \gamma = \delta.$$

Namely, two of the angles are redundant, so all 2×2 unitary similarity transformations may be identified with $[0, 2\pi)^2$.

In this case the complex version of the Givens (sometimes named after Jacobi), rotations is obtained (for the real version see e.g. [22, Section 3.4], [25, Example 2.2.3] [43, Section 14.6.1]). Thus, it is parametrized by two (and not four) angles.

In the literature these two problems were treated in numerous places (in some cases, with a slight confusion between them), see e.g. [22, Section 3.4], [27, Propriété 41], [33], [38, Eq. (19)], [39, Section 3], [41] and [43, Section 14.6.1].

Theorem 3.1 along with Lemma 4.1 enable us to introduce the following easy-to-use description of all rational functions in $\mathcal{U}$ of prescribed McMillan degree d and dimensions p and m, as real set which is virtually d copies of real polytopes.

Proposition 4.3. *All $p \times m$-valued rational functions of McMillan degree d in $\mathcal{U}$ may be parametrized by*

$$\begin{aligned} &(\{0\} \cup \{\infty\} \cup (((0,\infty) \setminus \{1\}) \cdot [0,2\pi)))^d \cdot [0,2\pi)^{2d(p-1)+m(2p-m)}, \quad p \geq m, \\ &(\{0\} \cup \{\infty\} \cup (((0,\infty) \setminus \{1\}) \cdot [0,2\pi)))^d \cdot [0,2\pi)^{2d(m-1)+p(2m-p)}, \quad m \geq p. \end{aligned} \tag{4.3}$$

The Schur stable subset is parametrized by

$$\begin{aligned} &(\{0\} \cup ((0,1) \cdot [0,2\pi)))^d \cdot [0,2\pi)^{2d(p-1)+m(2p-m)}, \quad p \geq m. \\ &(\{0\} \cup ((0,1) \cdot [0,2\pi)))^d \cdot [0,2\pi)^{2d(m-1)+p(2m-p)}, \quad m \geq p. \end{aligned}$$

Proof. Assume that $p \geq m$. As in Lemma 4.1, the set of all $v \in \mathbb{C}^p$ with $v^*v = 1$, i.e. the $\|\cdot\|_2$ unit sphere in $\mathbb{C}^p$, may be identified with

$$[0,2\pi)^{2p-1}.$$

As v and $e^{i\eta}v$ produce the same vv^*, to parametrize all $p \times p$ rank one orthogonal projections in Eq. (3.2), one angle is redundant, so one can use

$$[0,2\pi)^{2(p-1)}.$$

We next address the poles $\alpha_1, \ldots, \alpha_d$ in Eq. (3.4). If a pole α_j is in the complex plane, excluding zero, infinity and the unit circle, it may be parametrized by the usual polar representation

$$((0,\infty) \setminus \{1\}) \cdot [0,2\pi). \tag{4.4}$$

Thus, to parametrize a single Blaschke-Potapov factor in Eq. (3.4), one needs

$$(\{0\} \cup \{\infty\} \cup ((0,\infty) \setminus \{1\}) \cdot [0,2\pi)) \cdot [0,2\pi)^{2(p-1)}.$$

Note that this set is nearly a real polytope. Now taking d copies, yields

$$(\{0\} \cup \{\infty\} \cup ((0,\infty) \setminus \{1\}) \cdot [0,2\pi))^d \cdot [0,2\pi)^{2d(p-1)}.$$

Along with Eq. (4.2) from Lemma 4.1 the first part of Eq. (4.3) is obtained.

Due to symmetry, we omit the case $m \geq p$, so the construction is complete. □

The above parameterization is in particular convenient if one wishes to design through optimization, a rational function (co)-isometric on the unit circle. For example, given a $p \times m$-valued function $G(z)$ which is not necessarily rational, not necessarily (co)-isometric on the unit circle, and not necessarily Schur stable, find $F(z)$ its best Schur stable approximation in $\mathcal{U}$ of a prescribed McMillan degree d, i.e.

$$\min_{(\{0\}\cup((0,1)\cdot[0,2\pi)))^d\cdot[0,2\pi)^{2d(p-1)+m(2p-m)}} \|F(z) - G(z)\|, \quad p \geq m.$$

For other type optimization problems see e.g. [17, 24, 38, 42] and [46].

5. MATRIX-FRACTION DESCRIPTION

So far, we confined the discussion to rational functions $F(z)$ presented in their *minimal realization.* We next relax this restriction.

Following e.g. [31, Chapter 6], [43, Section 13.3] or [45, Chapter 4], a $p\times m$-valued rational function of the form Eq. (1.1), can always be written as

$$F(z) = \begin{cases} N(z)\left(\Delta(z)\right)^{-1} = \left(N_0 + zN_1 + \ldots + z^{\nu}N_{\nu}\right)\left(\Delta_0 + z\Delta_1 + \ldots + z^{\delta}\Delta_{\delta}\right)^{-1} & \text{RMFD},\\ \left(\tilde{\Delta}(z)\right)^{-1}\tilde{N}(z) = \left(\tilde{\Delta}_0 + z\tilde{\Delta}_1 + \ldots + z^{\tilde{\delta}}\tilde{\Delta}_{\tilde{\delta}}\right)^{-1}\left(\tilde{N}_0 + z\tilde{N}_1 + \ldots + z^{\tilde{\nu}}\tilde{N}_{\tilde{\nu}}\right) & \text{LMFD}, \end{cases} \tag{5.1}$$

where $\Delta(z)$ and $\tilde{\Delta}(z)$ are $m\times m$-valued and $p\times p$-valued polynomials, respectively, each of a full normal rank, while both $N(z)$ and $\tilde{N}(z)$ are $p\times m$-valued polynomials. $N(z)\left(\Delta(z)\right)^{-1}$ is called a *right matrix fraction description* (RMFD) of $F(z)$ while $\left(\tilde{\Delta}(z)\right)^{-1}\tilde{N}(z)$ is a *left matrix fraction description* (LMFD) of $F(z)$.

Specifically, ν, δ, $\tilde{\nu}$ and $\tilde{\delta}$ in Eq. (5.1) are non-negative integers. If they are the smallest possible[5)] the matrix fraction description of $F(z)$ in Eq. (5.1) is said to be *irreducible*, see e.g. [31, subsection 6.5]. Then, the polynomials $N(z)$ and $\Delta(z)$ are right coprime or the polynomials $\tilde{N}(z)$ and $\tilde{\Delta}(z)$ are left coprime, for details, see e.g. [31, Subsection 6.5] or [45, Chapter 4].

For a given $F(z)$, finding an *irreducible* MFD, may be challenging. However, here we look for *some* MFD. Specifically, let,

$$\alpha \geq \max(\nu,\delta), \quad \beta \geq \max(\tilde{\nu},\tilde{\delta}),$$

and by formally adding zero matrices to Eq. (5.1), we shall hereafter use the following MFD, where the numerator and denominator polynomials have the same power,

$$F(z) = \begin{cases} N(z)\left(\Delta(z)\right)^{-1} = \left(N_0 + zN_1 + \ldots + z^{\alpha}N_{\alpha}\right)\left(\Delta_0 + z\Delta_1 + \ldots + z^{\alpha}\Delta_{\alpha}\right)^{-1},\\ \left(\tilde{\Delta}(z)\right)^{-1}\tilde{N}(z) = \left(\tilde{\Delta}_0 + z\tilde{\Delta}_1 + \ldots + z^{\beta}\tilde{\Delta}_{\beta}\right)^{-1}\left(\tilde{N}_0 + z\tilde{N}_1 + \ldots + z^{\beta}\tilde{N}_{\beta}\right). \end{cases} \tag{5.2}$$

Recall also that with the polynomials in the RMFD in Eq. (5.2) one can associate the following ($p(\alpha+1)\times m(\alpha+1)$ and $m(\alpha+1)\times m(\alpha+1)$, respectively) Hankel matrices

$$\mathbf{H}_N := \begin{pmatrix} N_0 & N_1 & & N_{\alpha-1} & N_{\alpha}\\ N_1 & & \iddots & \iddots & \\ & \iddots & \iddots & & \\ N_{\alpha-1} & \iddots & & & \\ N_{\alpha} & & & & \end{pmatrix}, \quad \mathbf{H}_{\Delta} := \begin{pmatrix} \Delta_0 & \Delta_1 & & \Delta_{\alpha-1} & \Delta_{\alpha}\\ \Delta_1 & & \iddots & \iddots & \\ & \iddots & \iddots & & \\ \Delta_{\alpha-1} & \iddots & & & \\ \Delta_{\alpha} & & & & \end{pmatrix}. \tag{5.3}$$

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5) In principle, for arbitrary $m\times m$-valued polynomial $R(z)$, another RMFD is $F(z) = N(z)R(z)\left(\Delta(z)R(z)\right)^{-1}$.

By construction, both $\mathbf{H}_N^*\mathbf{H}_N$ and $\mathbf{H}_\Delta^*\mathbf{H}_\Delta$ are of the same dimensions $m(\alpha+1)\times m(\alpha+1)$.

Similarly, with the polynomials in the LMFD in Eq. (5.2), one can associate the following ($p(\beta+1)\times m(\beta+1)$ and $p(\beta+1)\times p(\beta+1)$, respectively) Hankel matrices,

$$\mathbf{H}_{\tilde{N}} := \begin{pmatrix} \tilde{N}_0 & \tilde{N}_1 & & \tilde{N}_{\beta-1} & \tilde{N}_\beta \\ \tilde{N}_1 & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & \\ \tilde{N}_{\beta-1} & \cdot^{\cdot^{\cdot}} & & & \\ \tilde{N}_\beta & & & & \end{pmatrix}, \quad \mathbf{H}_{\tilde{\Delta}} := \begin{pmatrix} \tilde{\Delta}_0 & \tilde{\Delta}_1 & & \tilde{\Delta}_{\beta-1} & \tilde{\Delta}_\beta \\ \tilde{\Delta}_1 & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & \\ \tilde{\Delta}_{\beta-1} & \cdot^{\cdot^{\cdot}} & & & \\ \tilde{\Delta}_\beta & & & & \end{pmatrix}. \tag{5.4}$$

By construction, both $\mathbf{H}_{\tilde{N}}\mathbf{H}_{\tilde{N}}^*$ and $\mathbf{H}_{\tilde{\Delta}}\mathbf{H}_{\tilde{\Delta}}^*$ are of the same dimensions $p(\beta+1)\times p(\beta+1)$.

We can now state the main result of this section.

Theorem 5.1. *Let $F(z)$ be a $p\times m$-valued rational function with a* (*not necessarily reducible*) *Matrix Fraction Description in Eq.* (5.2)*.*

I. *For $p\geq m$ let $\mathbf{H}_N$ and $\mathbf{H}_\Delta$ in Eq.* (5.3) *be the Hankel matrices associated with $N(z)$ and $\Delta(z)$, respectively. $F(z)$ is in $\mathcal{U}$, if and only if,*

$$(\mathbf{H}_\Delta^*\mathbf{H}_\Delta - \mathbf{H}_N^*\mathbf{H}_N)\begin{pmatrix} I_m \\ 0_{m\alpha\times m}\end{pmatrix} = 0_{m(\alpha+1)\times m}. \tag{5.5}$$

II. *For $m\geq p$ let $\mathbf{H}_{\tilde{N}}$ and $\mathbf{H}_{\tilde{\Delta}}$ in Eq.* (5.4) *be the Hankel matrices associated with $\tilde{N}(z)$ and $\tilde{\Delta}(z)$, respectively. $F(z)$ is in $\mathcal{U}$, if and only if,*

$$\left(\mathbf{H}_{\tilde{\Delta}}\mathbf{H}_{\tilde{\Delta}}^* - \mathbf{H}_{\tilde{N}}\mathbf{H}_{\tilde{N}}^*\right)\begin{pmatrix} I_p \\ 0_{p\beta\times p}\end{pmatrix} = 0_{p(\beta+1)\times p}.$$

Proof. Assume that $p\geq m$. Take the right RMFD of $F(z)$ and consider the following (where to simplify the presentation we omit the explicit dependence on the variable z)

$$F^\# F = \left(N\Delta^{-1}\right)^\# N\Delta^{-1} = \left(\Delta^{-1}\right)^\# N^\# N\Delta^{-1}$$

Now, having $F(z)$ is in $\mathcal{U}$ is equivalent to

$$I_m = F^\# F = \left(\Delta^{-1}\right)^\# N^\# N\Delta^{-1}.$$

Multiplying by $\Delta^\#$ from the left and Δ from the right yields

$$\Delta^\#\Delta = N^\# N.$$

Substituting now Eq. (5.2) in the above reads

$$\begin{aligned}&(\Delta_0 + z\Delta_1 + \ldots + z^\alpha\Delta_\alpha)^\# (\Delta_0 + z\Delta_1 + \ldots + z^\alpha\Delta_\alpha)\\ &= (N_0 + zN_1 + \ldots + z^\alpha N_\alpha)^\# (N_0 + zN_1 + \ldots + z^\alpha N_\alpha),\end{aligned}$$

which is equal to

$$\left(\Delta_0^* + \frac{1}{z}\Delta_1^* + \ldots + \frac{1}{z^\alpha}\Delta_\alpha^*\right)(\Delta_0 + z\Delta_1 + \ldots + z^\alpha \Delta_\alpha)$$
$$= \left(N_0^* + \frac{1}{z}N_1^* + \ldots + \frac{1}{z^\alpha}N_\alpha^*\right)(N_0 + zN_1 + \ldots + z^\alpha N_\alpha).$$

Note that in both, the numerator and the denominator, for each $k \in [1, \alpha]$, the coefficient of $\frac{1}{z^k}$, is the complex conjugate transpose, $()^*$, of the coefficient of z^k. Thus, without loss of generality, one can equate only the coefficients of z^k for $k \in [0, \alpha]$. This means that

$$\mathbf{H}_\Delta^* \begin{pmatrix} \Delta_o \\ \vdots \\ \Delta_\alpha \end{pmatrix} = \mathbf{H}_N^* \begin{pmatrix} N_o \\ \vdots \\ N_\alpha \end{pmatrix},$$

with the Hankel matrices from Eq. (5.3). This in turn may be equivalently written as

$$\mathbf{H}_\Delta^* \mathbf{H}_\Delta \begin{pmatrix} I_m \\ 0_{m\delta \times m} \end{pmatrix} = \mathbf{H}_N^* \mathbf{H}_N \begin{pmatrix} I_m \\ 0_{m\delta \times m} \end{pmatrix},$$

so Eq. (5.5) is established.

Due to symmetry, establishing the case $m \geq p$, is analogous and thus omitted. □

This work is devoted to $p \times m$-valued *rational functions* within $\mathcal{U}$. In [5] we focused on the subset of (possibly Laurent) polynomials (within $\mathcal{U}$), i.e.

$$F(z) = z^q(B_o + zB_1 + \ldots + z^\gamma B_\gamma), \quad \gamma \text{ is natural}, q \text{ is integral parameter}, \tag{5.6}$$

and $B_o, B_1, \ldots, B_\gamma$ constant matrices[6]. Note that for $-1 \geq q$ this is no longer a genuine polynomial. Although modest is size, there is a vast literature on this family, see e.g. [5] and references therein.

In Theorem 5.2 below we show how to use Hankel matrices to characterize this subset. In fact, this is a citation of [5, Theorem 4.1]. However, as the original proof is somewhat different. Using the above Theorem 5.1, we next establish the same result independently.

Here are the details: Substituting $q = 0$ in Eq. (5.6) one obtains,

$$F_0(z) := F(z)_{|q=0} = B_o + zB_1 + \ldots + z^\gamma B_\gamma.$$

With $F_0(z)$ one can associate the following $p(\gamma+1) \times m(\gamma+1)$ Hankel matrix,

$$\mathbf{H}_0 := \begin{pmatrix} B_0 & B_1 & & B_{\gamma-1} & B_\gamma \\ B_1 & & \iddots & \iddots & \\ & \iddots & \iddots & & \\ B_{\gamma-1} & \iddots & & & \\ B_\gamma & & & & \end{pmatrix}. \tag{5.7}$$

[6] Strictly speaking, the notation in [5] is slightly different, but equivalent.

Theorem 5.2. *Let $F(z)$ be a $p \times m$ polynomial in Eq.* (5.6) *and let $\mathbf{H}_0$ be the associated Hankel matrix as in Eq.* (5.7). *The polynomial $F(z)$ is in $\mathcal{U}$, if and only if,*

$$\begin{aligned} &\left(I_{m(\gamma+1)} - \mathbf{H}_0^*\mathbf{H}_0\right)\begin{pmatrix} I_m \\ 0_{m\gamma\times m}\end{pmatrix} = 0_{m(\gamma+1)\times m}, \quad p \geq m, \\ &\begin{pmatrix} I_p & 0_{p\times p\gamma}\end{pmatrix}\left(I_{p(\gamma+1)} - \mathbf{H}_0\mathbf{H}_0^*\right) = 0_{p\times p(\gamma+1)}, \qquad m \geq p. \end{aligned} \tag{5.8}$$

Proof. First, note that if $F(z)$ in Eq. (5.6) is in $\mathcal{U}$ for *some* q, it is in $\mathcal{U}$ for *all* q. Thus, without loss of generality, we characterize $F_0(z)$ in $\mathcal{U}$.

First, note that as a rational function $F_o(z)$ can be written as a RMFD in Eq.(5.2) with $\Delta_0 = I_m$, $\Delta_1 = \ldots = \Delta_\gamma = 0$ and $N_j = B_j$ for $j = 0, \ldots, \gamma$. Thus, using $\mathbf{H}_0$ from Eq. (5.7), here Eq. (5.3) takes the form

$$\mathbf{H}_N = \mathbf{H}_0 \quad \mathbf{H}_\Delta = \begin{pmatrix} I_m & 0 \\ 0 & 0_{m\gamma\times m\gamma}\end{pmatrix}.$$

Thus, for $p \geq m$ using Eq. (5.5) one has that

$$\begin{aligned} &\left(I_{m(\gamma+1)} - \mathbf{H}_0^*\mathbf{H}_0\right)\begin{pmatrix} I_m \\ 0_{m\gamma\times m}\end{pmatrix} \\ &= \left(I_{m(\gamma+1)} - \mathbf{H}_N^*\mathbf{H}_N\right)\begin{pmatrix} I_m \\ 0_{m\gamma\times m}\end{pmatrix} \\ &= \left(I_{m(\gamma+1)} - \mathbf{H}_N^*\mathbf{H}_N + \mathbf{H}_\Delta^*\mathbf{H}_\Delta - \mathbf{H}_\Delta^*\mathbf{H}_\Delta\right)\begin{pmatrix} I_m \\ 0_{m\gamma\times m}\end{pmatrix} \\ &= \left(I_{m(\gamma+1)} - \mathbf{H}_\Delta^*\mathbf{H}_\Delta\right)\begin{pmatrix} I_m \\ 0_{m\gamma\times m}\end{pmatrix} + \left(\mathbf{H}_\Delta^*\mathbf{H}_\Delta - \mathbf{H}_N^*\mathbf{H}_N\right)\begin{pmatrix} I_m \\ 0_{m\gamma\times m}\end{pmatrix} \\ &= \left(I_{m(\gamma+1)} - \mathbf{H}_\Delta^*\mathbf{H}_\Delta\right)\begin{pmatrix} I_m \\ 0_{m\gamma\times m}\end{pmatrix} \\ &= \left(\begin{pmatrix} I_m & 0 \\ 0 & I_{m\gamma}\end{pmatrix} - \begin{pmatrix} I_m & 0 \\ 0 & 0_{m\gamma\times m\gamma}\end{pmatrix}\right)\begin{pmatrix} I_m \\ 0_{m\gamma\times m}\end{pmatrix} \\ &= 0_{m(\gamma+1)\times m}. \end{aligned}$$

Thus, the first part of Eq. (5.8) is obtained.

Due to symmetry, establishing the case $m \geq p$ is analogous and thus omitted. □

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