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ON A PROBLEM OF GEVORKYAN FOR THE FRANKLIN SYSTEM

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Abstract. In 1870 G. Cantor proved that if $\lim_{N\to\infty}\sum_{n=-N}^{N} c_n e^{inx} = 0$ for every real x, where $\bar{c}_n = c_n$ ($n \in \mathbb{Z}$), then all coefficients c_n are equal to zero. Later, in 1950 V.Ya. Kozlov proved that there exists a trigonometric series for which a subsequence of its partial sums converges to zero, where not all coefficients of the series are zero. In 2004 G. Gevorkyan raised the issue that if Cantor's result extends to the Franklin system. The conjecture remains open until now. In the present paper we show however that Kozlov's version remains true for Franklin's system.

Keywords: Franklin system, orthonormal spline system, trigonometric system, uniqueness of series.

Mathematics Subject Classification: 42C10, 42C25, 41A15.

1. INTRODUCTION

In 1870 G. Cantor proved in [5] the following theorem.

Theorem 1.1. *If* $\lim_{N\to\infty}\sum_{n=-N}^{N} c_n e^{inx} = 0$ *for every real number* x, *where* $\bar{c}_n = c_n$, *then* $c_n = 0$ *for* $n \in \mathbb{Z}$.

In 1950 V.Ya. Kozlov [13] proved that there exists a trigonometric series $\sum_{n=0}^{\infty} a_n \cos nx + b_n \sin nx$ such that a subsequence of its partial sums is convergent to zero for $x \in \mathbb{R}$ and not all the coefficients are equal to zero. By the Gram-Schmidt process to the Schauder basis Ph. Franklin constructed an orthonormal system of continuous piecewise linear functions with dyadic knots. It is an orthonormal Schauder basis in the space $C[0,1]$, and also in the space $L^2[0,1]$. In 1963 Z. Ciesielski [6] proved exponential type estimates for Franklin functions. Since then, it has been studied by many authors from different points of view. The Franklin system is an unconditional basis in $L^p[0,1]$, $1 < p < \infty$ (S.V. Bochkarev [4]) and $H^1[0,1]$ (P. Wojtaszczyk [16]).

It has been used to prove the existence of a basis in the space $C^1([0,1]\times[0,1])$ (independently by Z. Ciesielski [7] and by S. Schonefeld [15]).

In the case of Haar and Walsh systems the uniqueness problem was solved by F.G. Aratunyan and A.A. Talalyan in [2]. In 2004 G. Gevorkyan [12] raised the issue if Cantor's result extends to the Franklin system. This conjecture remains open until now, and also for the periodic Franklin system which is obtained similarly to the nonperiodic case. The periodic Franklin system has been used to construct a basis in a disc algebra (S.V. Bochkarev [4]).

In 1938 J. Marcinkiewicz [14] obtained the following result.

Theorem 1.2. *For any complete in $L^2[0,1]$ an orthonormal system $\{\varphi_n\}_{n=1}^{\infty}$ there exists a nonzero series $\sum_{n=1}^{\infty} c_n\varphi_n(x)$ with a subsequence of its partial sums converging to zero almost everywhere in the interval $[0,1]$.*

The purpose of the paper is to prove the ensuing result.

Theorem 1.3. *There is a nonzero Franklin series $\sum_{n=0}^{\infty} a_n f_n(x)$ such that a subsequence $\{s_{n_k}(x)\}_{k=1}^{\infty}$ of its partial sums is convergent to zero in the interval $[0,1]$.*

2. PROOF OF THEOREM 1.3

It suffices to prove the theorem for the Franklin system for the interval $I=[-1,1]$. We use odd functions in the proof. Because of it and the simplicity of calculations, we shall consider the Franklin system for this interval. Consider the following sequence $\{\Delta_n\}_{n=1}^{\infty}$ of dyadic partitions of the interval I: $\Delta_n=\{s_{n,i}\}_{i=0}^{n}$, $s_{1,0}=-1$, $s_{1,1}=1$,

$$s_{n,i}=\begin{cases}\frac{i}{2^{\mu}}-1 & \text{for } i=0,1,\ldots,2\nu,\\ \frac{i-\nu}{2^{\mu-1}}-1 & \text{for } i=2\nu+1,\ldots,n\end{cases}\tag{2.1}$$

for $n=2^{\mu}+\nu$, $\mu=0,1,\ldots$, $\nu=1,2,\ldots,2^{\mu}$.

We can obtain the Franklin system by means of cubic splines. We put

$$f_0=\frac{1}{\sqrt{2}},\quad f_1=\sqrt{\frac{3}{2}}\,x.$$

Let g_n be a cubic spline with respect to the partition Δ_n, i.e. $g_n\in C^2(I)$ and it is a polynomial of degree at most 3 in each interval $[s_{n,i-1},s_{n,i}]$. We assume that $g_n(s_{n-1,j})=0$ for $j=0,1,\ldots,n-1$ and $g_n(s_{n,k})=1$ for $s_{n,k}=\Delta_n\setminus\Delta_{n-1}$ with $g_n'(\pm1)=0$. The spline g_n is unique. For the proof we refer to [1]. It is similar to that of the uniqueness of a spline S_n below. Because of its crucial role in the proof of the main result, we shall give it in detail. Integrating by parts, we check that the system $\{f_n\}_{n=0}^{\infty}$, where

$$f_n=\frac{g_n''}{\|g_n''\|},\quad \|g_n''\|^2=\int\limits_{-1}^{1}[g_n''(x)]^2\,dx,\quad n=2,3,\ldots,$$

is orthonormal in the interval I (see [1,17,18]).

Let

$$F(x) = \begin{cases} -1 & \text{for } x \in [-1,0), \\ 0 & \text{for } x = 0, \\ 1 & \text{for } x \in (0,1]. \end{cases} \tag{2.2}$$

We define the following sequence of functions $\{S_n(x)\}_{n=0}^{\infty}$. S_n is a cubic spline interpolating the function F on Δ_n, i.e. $S_n(s_{n,i}) = F(s_{n,i})$, $S_n'(\pm 1) = 0$. Let

$$M_i = S_n''(s_{n,i}),\ h_i = s_{n,i} - s_{n,i-1},\ y_i = F(s_{n,i}), \quad i = 0,1,\dots,n,$$

$$y_0' = F'(-1),\ y_n' = F'(1),\ d_0 = \frac{6}{h_1}\left(\frac{y_1 - y_0}{h_1} - y_0'\right),\ d_n = \frac{6}{h_n}\left(y_n' - \frac{y_n - y_{n-1}}{h_n}\right),$$

$$d_j = \frac{6}{h_j + h_{j+1}}\left(\frac{y_{j+1} - y_j}{h_{j+1}} - \frac{y_j - y_{j-1}}{h_j}\right),\ \lambda_j = \frac{h_{j+1}}{h_j + h_{j+1}},$$

$$\mu_j = 1 - \lambda_j,\ j = 1,\dots,n-1, \text{ and } \lambda_0 = \mu_n = 1,\ \lambda_n = \mu_0 = 0.$$

S_n is piecewise linear. Hence for $x \in [s_{n,i-1}, s_{n,i}]$, $i = 1,\dots,n$, we get

$$\begin{aligned} S_n''(x) &= M_{i-1}\frac{s_{n,i} - x}{h_i} + M_i\frac{x - s_{n,i-1}}{h_i}, \\ S_n'(x) &= -M_{i-1}\frac{(s_{n,i} - x)^2}{2h_i} + M_i\frac{(x - s_{n,i-1})^2}{2h_i} + C_i, \\ S_n(x) &= M_{i-1}\frac{(s_{n,i} - x)^3}{6h_i} + M_i\frac{(x - s_{n,i-1})^3}{6h_i} + C_i(x - s_{n,i-1}) + D_i, \end{aligned}$$

where C_i and D_i are some constants. Using the interpolation conditions, we obtain

$$S_n'(x) = -M_{i-1}\frac{(s_{n,i} - x)^2}{2h_i} + M_i\frac{(x - s_{n,i-1})^2}{2h_i} + \frac{y_i - y_{i-1}}{h_i} - (M_i - M_{i-1})\frac{h_i}{6}.$$

The function S_n' is continuous on the interval $[-1,1]$. Hence $S_n'(s_{n,i}-) = S_n'(s_{n,i}+)$, $i = 1,\dots,n-1$, and we obtain

$$\mu_j M_{j-1} + 2M_j + \lambda_j M_{j+1} = d_j,\ j = 0,1,\dots,n, \text{ where } M_{-1} := M_{n+1} := 0. \tag{2.3}$$

The matrix of this system has dominated the main diagonal. Hence, by the Gerschgorin theorem, the spline S_n is unique ([1]) and $S_n''(x) = \sum_{k=0}^{n} a_k f_k(x)$ with not all $a_n = 0$.

Let $n = 2^{k+1} = 2m$. We have $\mu_j = \lambda_j = \frac{1}{2}$, $d_j = 0$ for $j = 0,1,\dots,m-2,m,$ $m+2,\dots,2m$ and $d_{m-1} = 3m^2 = -d_{m+1}$.

Further we need the following result.

Theorem 2.1 (S. Demko [10])**.** *Let $A = [a_{ij}]$ be an $m \times m$ band matrix with bandwidth k, i.e. $a_{ij} = 0$ for $\mid i - j \mid \geq k$ and $\|A\|_p$ denote matrix l_p-norm. Suppose there exist $1 \leq p \leq \infty$ and M such that $\|A\|_p \leq 1$ and $\|A^{-1}\|_p \leq M$. Then there exist constants $C = C_{k,M}$ and $q = q_{k,M}$, $0 < q < 1$, such that for $A^{-1} = [b_{ij}]$*

$$\mid b_{ij} \mid \leq C\, q^{|i-j|}, \quad i,j = 1,\dots,m.$$

Let $A = [a_{ij}]$ be the matrix of the first m equations of the system (2.3) and $B = [b_{ij}] = A^{-1}$. By means of Theorem 2.1 we prove that there exist constants C and $0 < q < 1$ such that

$$| M_{m-k} |\leq Cm^2 q^k, \quad k = 1, 2, \ldots \tag{2.4}$$

Then for $k \geq \sqrt{m}$

$$| M_{m-k} |\to 0 \quad \text{as} \quad m \to \infty.$$

Let $0 < \alpha < 1$. We choose m such that $\alpha > \frac{1}{\sqrt{m}}$. Hence for each $\varepsilon > 0$ there exists $\delta > 0$ such that $| M_{m-k} |< \varepsilon$ for all integers $m > \delta$ and $k \in [\sqrt{m}, m]$, and

$$M_{m-k} = S''_{2m}(s_{n,m-k}) = S''_{2m}\Big(- \frac{k}{m}\Big), \quad \frac{k}{m} \geq \frac{1}{\sqrt{m}}.$$

Since S''_{2m} is piecewise linear, then $| S''_{2m}(x) |< \varepsilon$ for $x \in [-1, \alpha] \cup [\alpha, 1]$, and we obtain

$$\lim_{m\to\infty} S''_{2m}(x) = \lim_{m\to\infty} s_{2m}(x) = \lim_{m\to\infty} \sum_{k=0}^{2m} a_k f_k(x) = 0.$$

The convergence is uniform on the intervals $[-1, -\alpha]$ and $[\alpha, 1]$ $(0 < \alpha < 1)$.

Now let $n = 2^{\mu} + \nu$ and $\nu = 2^{\mu-1} = m$. In this case the quantities M_j satisfy the following system of equations

$$\begin{cases} 2M_0 + M_1 = 0, \\ M_{j-1} + 4M_j + M_{j+1} = d_j, \ j = 1, \ldots, 3m - 1, \\ M_{3m-1} + 2M_{3m} = 0, \end{cases} \tag{2.5}$$

where $d_j = 0$ for $j = 1, \ldots, 2m - 2, 2m + 2, \ldots, 3m - 1$, and

$$d_{2m-1} = 24m^2, \quad d_{2m} = -8m^2, \quad d_{2m+1} = -6m^2.$$

To obtain M_{2m}, we use the Cramer formulas for the system (2.5). After elementary calculations, we write the nominator and the denominator in a form of block triangular matrices and we prove that $|M_{2m}| \to \infty$ as $m \to \infty$.

Remark 2.2.

1) Theorem 1.1 remains true for the periodic Franklin system, and its proof is analogous to that above.

2) Let

$$g_0(x) = 1, \quad g_1(x) = x, \quad g_n(x) = \int_0^x f_{n-1}(t)dt,$$

where f_n is the n-th Franklin function, $n = 2, 3, \ldots$. Applying the Gram-Schmidt process to this system, we obtain the Ciesielski orthonormal system of splines of degree 2. Proceeding in the same way, we get an orthonormal system of splines of degree 3. Repeating this process, we may obtain an orthonormal system of splines of degree k, where $k = 1, 2, \ldots$. These systems were introduced by Z. Ciesielski [8]. We may prove Theorem 1.1 for them (cf. [8, 18]). In this case we interpolate the function F from Theorem 1.1 by splines of odd degree and the function $G(x) = 1$ for $x \in [-1, 1] \setminus \{0\}$ and $G(0) = 0$ by splines of even degree.

3. THE GEVORKYAN PROBLEM FOR THE SPACE V OF FUNCTIONS $f \in C[0, 1]$ WITH $f(0) = 0$

Let $\{\Delta_n\}_{n=1}^{\infty}$, $\Delta_n = \{0 = x_0 < x_1 < \ldots < x_n = 1\}$, be a sequence of partitions of the interval $[0, 1]$ with $\Delta_{n-1} \subset \Delta_n$ $(n = 1, 2 \ldots)$ and

$$\lim_{n\to\infty} \max_{1\le i\le n} (x_i - x_{i-1}) = 0.$$

We define the Franklin system for the space V with the sequence $\{\Delta_n\}_{n=1}^{\infty}$ as follows: $f_1 = \sqrt{3}\,x$, g_{n+1} is a cubic spline such that $g_{n+1}(x_j) = 0$ for $x_j \in \Delta_n$ and $g_{n+1}(x_k) = 1$ for $x_k \in \Delta_{n+1} \setminus \Delta_n$, $g_n''(0) = g_n'(1) = 0$. Then we put

$$f_{n+1}(x) = \frac{g_{n+1}''}{\|g_{n+1}''\|}, \quad n = 1, 2 \ldots.$$

The system $\{f_n\}_{n=1}^{\infty}$ is orthonormal in the space $L^2[0, 1]$ and it forms a basis in the space V (see [1, 6, 9, 10]). Now we define the sequence $\{S_n(x)\}_{n=1}^{\infty}$ of cubic splines such that $S_n(x_k) = 1$ for $x_k \in \Delta_n$, $k = 1, 2, \ldots, n$, $S_n(0) = S_n''(0) = S_n'(1) = 0$. As in Theorem 1.1, we may prove that S_n is unique. $S_n''(x) = \sum_{n=1}^{n} a_k f_k(x)$, $a_1 = -\sqrt{3}$.

Let $M_j = S_n''(x_j)$, $j = 1, \ldots, n$. Using the Demko theorem, as in the proof of Theorem 1.1, we prove that for any $\alpha \in (0, 1)$ the sequence $\{S_n''(x)\}_{n=1}^{\infty}$ is uniformly convergent to zero in the interval $[\alpha, 1]$. Hence

$$\lim_{m\to\infty} S_m''(x) = \sum_{n=1}^{\infty} a_n f_n(x) = 0 \ \text{ on } \ [0, 1] \quad \text{and} \quad a_1 \neq 0,$$

and thus we have proved the following result.

Theorem 3.1. *There is a nonzero Franklin series* $\sum_{n=1}^{\infty} a_n f_n$ *in the space* V *with the sequence of partial sums converging to zero in the interval* $[0, 1]$.

4. PROBLEMS

Let $\{\Delta_n\}_{n=1}^{\infty}$ be a given sequence of partitions of the interval $I = [-1, 1]$, $\Delta_n = \{-1 = t_{n,0} < t_{n,1} < \ldots < t_{n,n} = 1\}$ with $\Delta_n \subset \Delta_{n+1}$, i.e. each point of Δ_n is a point of Δ_{n+1}. We assume that

$$\lim_{n\to\infty} \max_{1\le i\le n} (t_{n,i} - t_{n,i-1}) = 0.$$

We define an orthonormal system of piecewise functions with respect to the sequence of partitions $\{\Delta_n\}_{n=1}^{\infty}$ analogous to the Franklin system. We call it the general Franklin system (see [12]). We have the following problems: to prove a counterpart of Theorem 1.3 for

a) any general Franklin system (see [17]),
b) any complete orthonormal system in the space $C[0,1]$.

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