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A MODEL FOR THE INVERSE 1-MEDIAN PROBLEM ON TREES UNDER UNCERTAIN COSTS

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Abstract. We consider the problem of justifying vertex weights of a tree under uncertain costs so that a prespecified vertex become optimal and the total cost should be optimal in the uncertainty scenario. We propose a model which delivers the information about the optimal cost which respect to each confidence level $\alpha \in [0,1]$. To obtain this goal, we first define an uncertain variable with respect to the minimum cost in each confidence level. If all costs are independently linear distributed, we present the inverse distribution function of this uncertain variable in $O(n^2 \log n)$ time, where n is the number of vertices in the tree.

Keywords: location problem, uncertain variable, inverse optimization problem, tree.

Mathematics Subject Classification: 90B10, 90B80, 90C27.

1. INTRODUCTION

Inverse location theory has become an interesting topic in Operations Research and has been investigated mainly in the two approaches, the *inverse median* and the *inverse center* problems. For the inverse 1-median problem on trees, Burkard *et al.* [6] and Galavii [9] modeled the problem as a linear knapsack problem and thus solved this problem in linear time. Burkard *et al.* [7] investigated the inverse 1-median problem on a cycle and solved the problem in $O(n^2)$ time by exploring the concavity of linear constraints. The inverse p-median problem on general graphs is NP-hard, see Bonab *et al.* [5]. However, the inverse 2-median problem on a tree can be solved in polynomial time and this problem can be solved in linear time if the underlying tree is a star. Concerning inverse 1-center problems, Cai *et al.* [8] were the first who showed the NP-hardness of the inverse 1-center problem whereas the classical 1-center problem is solvable in polynomial time. Therefore, it is interesting to consider some cases where the inverse 1-center problem can be solved in polynomial time. The inverse 1-center problem on unweighted trees with variable edge lengths was investigated in depth and efficiently solved, see Alizadeh and Burkard [1–3]. Nguyen and Chassein [15] considered a generalization of the inverse 1-center problem on trees, the problem on cactus graphs, and showed the NP-completeness. Moreover, the inverse 1-center problem on weighted trees was studied and solved under various norms, see [17, 18]. Nguyen and Anh [16] further solved the inverse 1-center problem with variable vertex weights in $O(n^2)$ time.

While most problems considered fixed parameters, real-life scenarios lead us to investigate the problem with uncertainty. For example, the cost of modification may depend on the condition of the road, the weather, or the time of a day. Therefore, a nondeterministic model, like probability or fuzzy set theory, should be taken into account. In the situation where there exists no sample to estimate the probability distribution of a given nondeterministic variable, we have to invite experts to evaluate the belief degree such that the event occurs. Liu [13, 14] has developed an uncertainty theory to deal with the belief degree which has been recently applied to model many nondeterministic problems. Gao [10] gave the concept of uncertain α-shortest path on graphs with uncertain edge lengths and pointed out that the uncertain α-shortest path is equivalent to the shortest path in a corresponding deterministic environment. Then Zhou *et al.* [19] investigated the inverse shortest path problem on uncertain graphs. They proposed a crisp model corresponding to some given confidence levels and proved that the model is equivalent to a linear program.

Although uncertainty theory plays an important role in real life applications and the inverse location problem has been intensively studied in operations research, according to the best of our knowledge the connection between these two topics has not been studied so far. We consider in this paper the inverse 1-median problem on a tree under uncertain cost coefficients, i.e., the cost will be evaluated by the experts. The paper is organized as follows. Section 2 recalls preliminaries concerning the inverse 1-median problem on trees and uncertainty theory. We investigate in Section 3 a combination of these two theories. In other words, we first construct an uncertain variable representing the minimum cost of the inverse 1-median problem on a tree with respect to each confidence level. Then we develop an $O(n^2 \log n)$ algorithm to find the inverse distribution function of the uncertain variable.

2. PRELIMINARIES

In this section we first focus on the concept of an inverse 1-median problem on trees with variable vertex weights, then we recall the definition of uncertain variables in order to connect these two theories.

2.1. INVERSE 1-MEDIAN PROBLEM ON TREE (INVT)

Let $T = (V, E)$ be a tree with n vertices. Each edge $e \in E$ has a positive length and each vertex $v \in V$ is associated with a nonnegative weight w_v. The distance between two vertices u and v is the length of the shortest path connecting these two vertices and is denoted by $d(u, v)$. The 1-median function at a vertex v is defined as $\sum_{v' \in V} w_{v'} d(v, v')$. A vertex v^0 is a 1-median of T if $\sum_{v' \in V} w_{v'} d(v^0, v') \leq \sum_{v' \in V} w_{v'} d(v, v')$ for all vertices $v \in V$.

In what follows, we restate the inverse 1-median model proposed by Burkard *et al.* [6] and Galavii [9]. Given a tree $T = (V, E)$ and a prespecified vertex v^*. The weight of each vertex v can be increased or decreased by an amount p_v or q_v, i.e., the new weight of v is defined as $\tilde{w}_v := w_v + p_v - q_v$. Assume that the modified weights are non-negative and increasing (decreasing) one unit weight of v costs c_e^+ (c_e^-), the inverse 1-median problem on tree is formulated as follows.

— The prespecified vertex v^* becomes a 1-median of the modified tree, i.e.,

$$\sum_{v' \in V} \tilde{w}_{v'} d(v^*, v') \leq \sum_{v' \in V} \tilde{w}_{v'} d(v, v')$$

for all $v \in V$.
— The cost $\sum_{v \in V}(c_v^+ p_v + c_v^- q_v)$ is minimized.
— The modifications are limited within their bounds, i.e., $0 \leq p_v \leq \bar{p}_v$ and $0 \leq q_v \leq \bar{q}_v$.

Solution methods of (InvT) are based on an optimality criterion. Let T_1, T_2,..., T_k be the subtrees induced by deleting all edges incident to v^* from T, where k is the degree of v^*. Denote by $W := \sum_{v \in T} w_v$ and $W_i := \sum_{v \in T_i} w_v$ for $i = 1, \dots, k$, then we get the following theorem.

Theorem 2.1 (Median Criterion, see Goldman [11] and Hua [12]). *The vertex v^* is a 1-median of T if and only if $W_i \leq \frac{W}{2}$ for all $i = 1, \dots, k$.*

It is easy to see that the conditions for a vertex to be a 1-median of T does not depend on the edge lengths. If v^* is not a 1-median of T, there exists exactly one subtree, say T_{i_0}, that violates the optimality criterion in Theorem 2.1. Then there exists an optimal solution such that the vertex weights in $T_{i_0}(T \backslash T_{i_0})$ is reduced(increased). Therefore, we set $x_v := p_v$ and $\bar{x}_v := \bar{p}_v$ ($x_v := q_v$ and $\bar{x}_v := \bar{q}_v$) if $v \in T_{i_0}$ ($v \in T \backslash T_{i_0}$). We say that the weight of a vertex v in T_{i_0} ($T \backslash T_{i_0}$) is modified by x_v if its weight is reduced (increased) by an amount x_v. After renumbering the vertex and making some elementary computations, we can formulate the inverse 1-median problem on T as follows:

$$\begin{aligned} & \min \sum_{i=1}^{n} c_i x_i \\ \text{s.t.} & \sum_{i=1}^{n} x_i = 2D \\ & 0 \leq x_i \leq \bar{x}_i \quad \text{for all} \quad i = 1, \dots, n \end{aligned} \tag{2.1}$$

Here, $D := W_{i_0} - \frac{W}{2}$ is the optimality gap.

The inverse 1-median problem on trees has been solved efficiently in linear time by applying the algorithm of Balas and Zemel [4], see Burkard *et al.* [6] and Galavii [9].

2.2. UNCERTAIN VARIABLE

This subsection recall the concepts of uncertain variables which have been studied by Liu [13]. While the probability theory is applicable when samples are available, the uncertainty theory concerns the belief degree, i.e., the sample size is not large enough or no sample is available.

Let Γ be a nonempty set, and let $\mathcal{L}$ be a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is called an event. In order to present an axiomatic definition of uncertain measure, it is necessary to assign to each event Λ a number $\mathcal{M}\{\Lambda\}$ which indicates the level that Λ will occur. In order to ensure that the number $\mathcal{M}\{\Lambda\}$ has certain mathematical properties, we have the following four axioms:
Axiom 1. (Normality) $\mathcal{M}\{\Gamma\} = 1$.
Axiom 2. (Monotonicity) $\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}$ whenever $\Lambda_1 \subset \Lambda_2$.
Axiom 3. (Self-Duality) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.
Axiom 4. (Countable Subadditivity) For every countable sequence of events $\{\Lambda_i\}$, we have

$$\mathcal{M}\Big\{\bigcup_{i=1}^{n} \Lambda_i\Big\} \leq \sum_{i=1}^{n} \mathcal{M}\{\Lambda_i\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. Moreover, an uncertain variable ξ is a measurable function from an uncertainty space to the set of real numbers. The distribution function of an uncertain variable ξ is defined as follows.

Definition 2.2. The uncertainty distribution of an uncertain variable ξ is, by definition, a function

$$\Phi(x) := \mathcal{M}\{\xi \leq x\}.$$

for $x \in \mathbb{R}$.

To illustrate the distribution function of an uncertain variable, let us consider the following example.

Example 2.3. An uncertain variable ξ is called linear, or $\xi \sim \mathcal{L}(a, b)$, if it has a linear uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leq a, \\ (x-a)/(b-a), & \text{if } a \leq x \leq b, \\ 1, & \text{if } x \geq b. \end{cases}$$

The inverse uncertainty distribution of an uncertain variable ξ is defined as $\Phi^{-1}(\alpha)$. An uncertainty distribution Φ is regular if $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0, 1)$. We can observe that if $\xi \sim \mathcal{L}(a, b)$, then its distribution function is regular and $\Phi^{-1}(\alpha) = (1 - \alpha)a + \alpha b$ for $\alpha \in (0, 1)$.

Definition 2.4. The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$\mathcal{M}\Big\{\bigcup_{i=1}^{m} (\xi_i \in B_i)\Big\} = \min_{i=1}^{m} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets B_1, B_2, $\ldots$, B_m of real numbers.

We get the following rule that states the inverse distribution function of the combination of independent variables.

Proposition 2.5. *Let $\xi_1, \xi_2, \dots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \dots, \Phi_m$. If the function $f(x_1, x_2, \dots, x_m)$ is strictly increasing with respect to $x_1, \dots, x_k$ and strictly decreasing with respect to $x_{k+1}, x_{k+2} \dots, x_m$, then*

$$\xi = f(\xi_1, \xi_2, \dots, \xi_m)$$

is an uncertain variable with the inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \dots, \Phi_k^{-1}(\alpha), \Phi_{k+1}^{-1}(1-\alpha), \dots, \Phi_m^{-1}(1-\alpha)).$$

3. THE INVERSE 1-MEDIAN PROBLEM ON TREES UNDER UNCERTAIN COSTS (IMUC)

In this section we assume that the cost cooefficients of (InvT), viz. c_i for $i = 1, \dots, n$, are independent uncertain variables. For simplicity, we denote by ξ_i the uncertain variable with respect to cost c_i for $i = 1, \dots, n$. Then one can write the cost function in (2.1) as $\sum_{i=1}^n \xi_i x_i$. One model related to the inverse 1-median problem under uncertainty is the crisp model

$$\begin{aligned} &\min \mathbb{E}\Big(\sum_{i=1}^n \xi_i x_i\Big) \\ &\text{s.t.} \sum_{i=1}^n x_i = 2D, \\ &\qquad 0 \le x_i \le \bar{x}_i \quad \text{for all} \quad i = 1, \dots, n. \end{aligned} \tag{3.1}$$

where D is defined in (2.1).

However, the crisp model (3.1) delivers only the expectation of the cost function but not the information about the objective value corresponding to each confidence level. Hence, we aim to construct in this paper an uncertain variable corresponding to the minimum cost in each confidence level. Then we consider the solution with respect to the expectation of this uncertain variable as the optimal solution of the model.

Denote by Δ the set of feasible solutions of (3.1), i.e.,

$$\Delta := \Big\{x = (x_1, x_2, \dots, x_n) : \sum_{i=1}^n x_i = 2D, 0 \le x_i \le \bar{x}_i \text{ for all } i = 1, \dots, n\Big\}.$$

Concerning each level of confidence α, we can define a corresponding optimal value C_α as follows.

Definition 3.1. C_α is the minimum cost of (IMUC) under the confidence level α, i.e.,

$$C_\alpha := \min\Big\{C \,|\, \mathcal{M}\Big\{\sum_{i=1}^{n} \xi_i x_i \leq C\Big\} \geq \alpha \quad \text{and} \quad (x_1, x_2, \ldots, x_n) \in \Delta\Big\}.$$

Assume that Φ_i is the distribution function of ξ_i for $i = 1, \ldots, n$. For a feasible solution $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \Delta$, the distribution function of $\sum_{i=1}^{n} \xi_i x_i$ is $\Psi_{\mathbf{x}} := \sum_{i=1}^{n} x_i \Phi_i$. As it is noted that $C_\alpha := \min\limits_{\mathbf{x}\in\Delta} \Psi_{\mathbf{x}}^{-1}(\alpha)$ and by Proposition 2.5, we can rewrite

$$C_\alpha := \min_{\mathbf{x}\in\Delta} \sum_{i=1}^{n} x_i \Phi_i^{-1}(\alpha).$$

Let us further assume from here on that, the uncertain variable ξ_i has a linear distribution, i.e., $\xi_i \sim \mathcal{L}(a_i, b_i)$, for $i = 1, \ldots, n$. To find C_α, we solve the inverse 1-median problem on T with a condifence level α. Precisely, we first replace uncertain variables by deterministic costs, say $c_i := (1-\alpha)a_i + \alpha b_i$ for $i = 1, \ldots, n$. Then we solve the corresponding inverse 1-median problem on T. It can be done in linear time by the algorithm of Balas and Zemel [4].

Now we construct an uncertain variable φ corresponding to the minimum cost of (IMUC) with each confidence level $\alpha \in [0, 1]$, i.e., $\{C_\alpha\}_{\alpha\in[0,1]}$, as follows.

$$\mathcal{M}\{\varphi \leq x\} := \begin{cases} 0, & \text{if } x \leq C_0, \\ \alpha, & \text{if } x = C_\alpha \in [C_0, C_1], \\ 1, & \text{if } x \geq C_1. \end{cases}$$

Denote the inverse distribution function of φ by Θ. We get the following property.

Lemma 3.2. *If ξ_i are independent and linearly distributed uncertain variables for $i = 1, \ldots, n$, then Θ is a piecewise linear function with at most $O(n^2)$ breakpoints.*

Proof. If $\xi_i \sim \mathcal{L}(a_i, b_i)$ for all $i = 1, \ldots, n$, then

$$\Theta(\alpha) = C_\alpha := \min\Big\{\sum_{i=1}^{n}(a_i + \alpha(b_i - a_i))x_i : (x_1, x_2, \ldots, x_n) \in \Delta\Big\}$$

for $\alpha \in [0, 1]$.

In order to find $\Theta(\alpha)$ for $\alpha \in [0, 1]$ we have to solve the corresponding continuous knapsack problem. Moreover, in a continuous knapsack problem we use the items with smaller costs first until the budget constraint is fulfilled. Therefore, the function Θ is piecewise-linear with breakpoints being the values of α in which the ordering of cost values changes. As there are $O(n^2)$ possible changes of the ordering of cost coefficients, the number of breakpoints is at most $O(n^2)$. □

We further get the following result.

Proposition 3.3. *There exists a confidence level $\alpha^* \in [0, 1]$ such that $\mathbb{E}[\varphi] = C_{\alpha^*}$.*

Proof. As the inverse distribution function Θ is a continuous piecewise linear function. The mean value theorem ensures that there exists α^* so that

$$\mathbb{E}[\varphi] = \int_0^1 \Theta(\alpha) d\alpha = \Theta(\alpha^*) =: C_{\alpha^*}. \qquad \square$$

Definition 3.4. An optimal solution of the inverse 1-median problem on T under the confidence level α^* is called the optimal solution of (IMUC).

It is noted that the optimal solution of (IMUC) can be easily found in linear time if we know exactly the presentation of $\Theta(\alpha)$ for $\alpha \in [0, 1]$.

Next we aim to find the inverse distribution function Θ in order to deliver the information of the minimum cost with respect to each confidence level $\alpha \in [0, 1]$. We first find the breakpoints of Θ in $O(n^2)$ time by considering the intersection in $(0, 1)$ of these affine linear functions $a_i + \alpha(b_i - a_i)$ for $i = 1, \ldots, n$. Then we sort the breakpoints increasingly in $O(n^2 \log n)$ time and get a sequence

$$0 < \alpha_1 < \ldots < \alpha_k < 1.$$

We start with finding $\Theta(0)$ in linear time and denote by $(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})$ the corresponding optimal solution. For $\alpha \in [0, \alpha_1]$, the optimal cost is $\Theta(\alpha) := \sum_{i=1}^n (a_i + \alpha(b_i - a_i))x_i^{(0)}$ with the corresponding minimizer $(x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})$. In order to find the minimizer $(x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)})$ of $\Theta(\alpha)$ for $\alpha \in [\alpha_1, \alpha_2]$, we focus on the affine linear functions which change the ordering at α_1. Denote by S_1 the set of index i such that $a_i + \alpha(b_i - a_i)$'s change the ordering at α_1. Take the uncertain variables corresponding to these affine linear functions and solve

$$\begin{aligned} \min & \sum_{i \in S_1} (a_i + (\alpha_1 + \varepsilon)(b_i - a_i))x_i, \\ \text{s.t.} & \sum_{i \in S_1} x_i = \sum_{i \in S_1} x_i^{(0)}, \\ & 0 \le x_i \le \bar{x}_i \quad \text{for all} \quad i \in S_1, \end{aligned} \tag{3.2}$$

for a sufficently small positive number ε, $\varepsilon < \alpha_2 - \alpha_1$.

The minimizer $x^{(1)}$ of $\Theta(\alpha)$ for $\alpha \in [\alpha_1, \alpha_2]$ is identified as follows:

– if $j \notin S_1$, $x_j^{(1)} = x_j^{(0)}$,
– $(x_j^{(1)})_{j \in S_1}$ is the minimizer of (3.2).

To find $\Theta(\alpha)$, $\alpha \in [\alpha_1, \alpha_2]$, and so on, we apply the similar procedure. In iteration i, it takes $O(S_i)$ time to update the corresponding optimal solution. Therefore, it costs totally $O(\sum_{i=1}^k |S_i|) = O(n)$ time to update the optimal solution in all intervals $[\alpha_i, \alpha_{i+1}]$ for $i = 0, \ldots, k - 1$. We get the main result.

Theorem 3.5. *The inverse distribution function of (IMUC) can be represented in $O(n^2 \log n)$ time.*

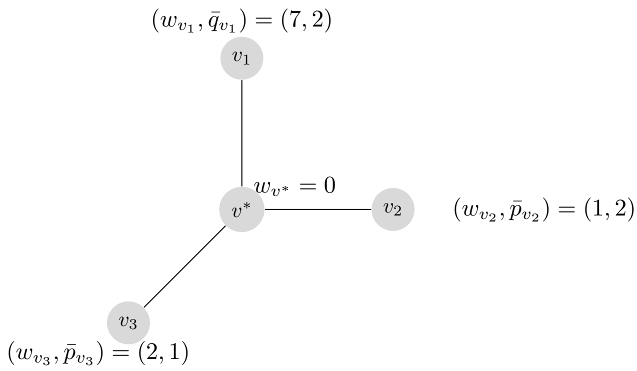

Fig. 1. An instance of (IMUC)

To illustrate the previous concept and algorithm, we consider the following example.

Example 3.6. Given a tree as in Figure 1 and v^* is the prespecified vertex and its weight is fixed. First, the subtree induced by $\{v_1\}$ violates the optimality criterion. Therefore, we have to reduce the weight of v_1 and increase the weight of v_2 and v_3. Let $\xi_1 \sim \mathcal{L}(1, 6)$, $\xi_2 \sim \mathcal{L}(2, 5)$, $\xi_3 \sim \mathcal{L}(4, 5)$ be the uncertain costs of modifying one unit weight of vertex v_1, v_2, v_3, respectively. We can write the inverse distribution function of the uncertain minimum cost as

$$\Theta(\alpha) := \min\{ (1 + 5\alpha)x_1 + (2 + 3\alpha)x_2 + (4 + \alpha)x_3 : x_1 + x_2 + x_3 = 4, \\ 0 \le x_1 \le 2, 0 \le x_2 \le 2, 0 \le x_3 \le 1\}.$$

The set of values such the cost coefficients change their ordering is $\{\alpha_1, \alpha_2\} = \{\frac{1}{2}, \frac{3}{4}\}$.

We first solve

$$\min\{x_1 + 2x_2 + 4x_3 : x_1 + x_2 + x_3 = 4, 0 \le x_1 \le 2, 0 \le x_2 \le 2, 0 \le x_3 \le 1\}$$

and get an optimal solution

$$(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) = (2, 2, 0).$$

Therefore, $\Theta(\alpha) = 6 + 16\alpha$ for $\alpha \in [0, \frac{1}{2}]$.

In order to find $(x_1^{(1)}, x_2^{(1)}, x_3^{(1)})$. We first detect the functions that intersect at $\alpha_1 = \frac{1}{2}$, they are in the ordering $2 + 3\alpha, 1 + 5\alpha$. Then solve

$$\min\left\{\left(2 + 3\left(\frac{1}{2} + \varepsilon\right)\right)x_2 + \left(1 + 5\left(\frac{1}{2} + \varepsilon\right)\right)x_1 : x_1 + x_2 = 4, 0 \le x_1 \le 2, 0 \le x_2 \le 2\right\}.$$

One also obtains

$$(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}) = (2, 2, 0).$$

Then $\Theta(\alpha) = 6 + 16\alpha$ for $\alpha \in [\frac{1}{2}, \frac{3}{4}]$.

In order to find $(x_1^{(2)}, x_2^{(2)}, x_3^{(2)})$ we first detect the functions that intersect at $\alpha_2 = \frac{3}{4}$, they are in the ordering $4 + \alpha, 1 + 5\alpha$. Then solve

$$\min \Big\{ (4 + \Big(\frac{3}{4} + \varepsilon\Big)\Big) x_3 + \Big(1 + 5\Big(\frac{3}{4} + \varepsilon\Big)\Big) x_1 : x_1 + x_3 = 2, 0 \le x_1 \le 2, 0 \le x_3 \le 1 \Big\}.$$

One also obtains

$$(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}) = (2, 1, 1).$$

Then $\Theta(\alpha) = 9 + 12\alpha$ for $\alpha \in [\frac{3}{4}, 1]$.

Therfore, the inverse distribution function can be writen as

$$\Theta(\alpha) := \begin{cases} 6 + 16\alpha, & \text{if } 0 \le \alpha \le \frac{3}{4}, \\ 9 + 12\alpha, & \text{if } \frac{3}{4} \le \alpha \le 1. \end{cases}$$

4. CONSLUSIONS

We have addressed in this paper the inverse 1-median problem on trees under uncertain costs. Here, we aim to present the optimal cost C_α with respect to each confidence level $\alpha \in [0, 1]$. To attain this goal, we develop an algorithm to find the inverse distribution function of an uncertain variable corresponding to minimum cost C_α, $\alpha \in [0, 1]$, in $O(n^2 \log n)$ time.

For future research, one can consider the inverse 1-center problem under uncertain costs. Moreover, other parameters, like edge lengths or vertex weights, should be uncertain variables and it is worthwhile to investigate the inverse location problem under these uncertainties.

REFERENCES

[1] B. Alizadeh, R.E. Burkard, *Combinatorial algorithms for inverse absolute and vertex 1-center location problems on trees*, Networks **58** (2011), 190–200.

[2] B. Alizadeh, R.E. Burkard, *Uniform-cost inverse absolute and vertex center location problems with edge length variations on trees*, Discrete Appl. Math. **159** (2011), 706–716.

[3] B. Alizadeh, R.E. Burkard, U. Pferschy, *Inverse 1-center location problems with edge length augmentation on trees*, Computing **86** (2009), 331–343.

[4] E. Balas, E. Zemel, *An algorithm for large zero-one knapsack problems*, Operations Research **28** (1980), 1130–1154.

[5] F.B. Bonab, R.E. Burkard, E. Gassner, *Inverse p-median problems with variable edge lengths*, Math. Meth. Oper. Res. **73** (2011), 263–280.

[6] R.E. Burkard, C. Pleschiutschnig, J.Z. Zhang, *Inverse median problems*, Discrete Optimization **1** (2004), 23–39.

[7] R.E. Burkard, C. Pleschiutschnig, J.Z. Zhang, *The inverse 1-median problem on a cycle*, Discrete Optimization **5** (2008), 242–253.

[8] M.C. Cai, X.G. Yang, J.Z. Zhang, *The complexity analysis of the inverse center location problem*, J. Global Optimization **15** (1999), 213–218.

[9] M. Galavii, *The inverse 1-median problem on a tree and on a path*, Electronic Notes in Disrete Mathematics **36** (2010), 1241–1248.

[10] Y. Gao, *Shortest path problem with uncertain arc lengths*, Computers and Mathematics with Applications **62** (2011), 2591–2600.

[11] A.J. Goldman, *Optimal center location in simple networks*, Transportation Science **5** (1971), 539–560.

[12] L.K. Hua, *Application of mathematical models to wheat harvesting*, Chinese Mathematics **2** (1962), 539–560.

[13] B. Liu, *Uncertainty Theory*, 2nd ed., Springer-Verlag, Berlin, 2007.

[14] B. Liu, *Some research problems in uncertainty theory*, Journal of Uncertain Systems **3** (2009), 3–10.

[15] K.T. Nguyen, A. Chassein, *Inverse eccentric vertex problem on networks*, Cent. Eur. Jour. Oper. Res. **23** (2015), 687–698.

[16] K.T. Nguyen, L.Q. Anh, *Inverse k-centrum problem on trees with variable vertex weights*, Math. Meth. Oper. Res. **82** (2015), 19–30.

[17] K.T. Nguyen, A.R. Sepasian, *The inverse 1-center problem on trees under Chebyshev norm and Hamming distance*, Journal of Combinatorial Optimization (2015), DOI: 10.1007/s10878-015-9907-5.

[18] K.T. Nguyen, *Reverse 1-center problem on weighted trees*, Optimization **65** (2016), 253–264.

[19] J. Zhou, F. Yang, K. Wang, *An inverse shortest path problem on an uncertain graph*, Journal of Networks **9** (2014), 2353–2359.

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