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ON THE BAIRE CLASSIFICATION OF CONTINUOUS MAPPINGS DEFINED ON PRODUCTS OF SORGENFREY LINES

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Abstract. We study the Baire measurability of functions defined on $\mathbb{R}^T$ which are continuous with respect to the product topology on a power $\mathbb{S}^T$ of Sorgenfrey lines.

Keywords: Baire-one function, Sorgenfrey line, equiconnected space.

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1. INTRODUCTION

The collection of all continuous mappings between topological spaces X and Y will be denoted by $\mathrm{C}(X,Y)$. A mapping $f : X \to Y$ belongs to the *first Baire class*, $f \in \mathrm{B}_1(X,Y)$, if there exists a sequence $(f_n)_{n=1}^{\infty}$ of mappings from $\mathrm{C}(X,Y)$ which is convergent to f pointwise on X.

The Sorgenfrey line $\mathbb{S}$ is the set of all reals equipped with the topology $\mathcal{S}$ generated by the basis of all half-open intervals $[a,b)$. Since the topology $\mathcal{S}$ is finer than the standard topology $\mathcal{E}$ on the real line, each continuous function $f : \mathbb{R} \to Y$ with values in arbitrary space Y is continuous in the topology $\mathbb{S}$ too. It is easy to see that the converse proposition is not true: the characteristic function $\chi_{[0,1)}$ of the half-open interval $[0,1)$ is continuous in $\mathcal{S}$ but is discontinuous at the points $x = 0$ and $x = 1$ in the topology $\mathcal{E}$; at the same time is is easy to see that $\chi_{[0,1)}$ belongs to the first Baire class on $\mathbb{R}$. Bade [1] proved that each real-valued continuous function on $\mathbb{S}^2$ belongs to the first Baire class in the topology $\mathcal{E}$. Moreover, Bade noticed that Mrówka [7] obtained the inclusion $\mathrm{C}(\mathbb{S}^{\mathfrak{n}},\mathbb{R}) \subseteq \mathrm{B}_1(\mathbb{R}^{\mathfrak{n}},\mathbb{R})$ for every cardinal $\mathfrak{n}$. Since all functions in the above-mentioned results take values in the real line, it is natural to consider other range spaces which lead to the following questions.

Question 1.1. Let T be a set and $f : \mathbb{S}^T \to Y$ be a continuous mapping. Does the inclusion $f \in \mathrm{B}_1(\mathbb{R}^T, Y)$ hold if

a) Y is a topological vector space,
b) Y is a locally convex space,
c) Y is a metrizable topological vector space?

In this paper we show that the answer to Question 1.1c) is positive for any T; the answer to b) is positive for $|T| \le \aleph_0$; and the answer to Question 1.1a) is positive in the case $|T| < \aleph_0$.

2. CLASSIFICATION OF MAPPINGS ON QUARTER-STRATIFIABLE SPACES

Definition 2.1. A topological space X is said to be *equiconnected* if there exists a continuous function $\lambda : X \times X \times [0,1] \to X$ such that

(i) $\lambda(x, y, 0) = x$;
(ii) $\lambda(x, y, 1) = y$;
(iii) $\lambda(x, x, t) = x$

for all $x, y \in X$ and $t \in [0,1]$.

The class of all equiconnected spaces contains the class of all topological vector spaces: the equality $\lambda(x, y, t) = (1-t)x + ty$ for $x, y \in X$ and $t \in [0,1]$ defines the required continuous function.

Now we recall the concept of the λ-sum in an equiconnected space (X, λ) (see [4]). For every $n \in \mathbb{N}$ we put

$$S_n = \{(\alpha_k)_{k=1}^n \in \mathbb{R}^n : \alpha_1 + \cdots + \alpha_n = 1, \quad \alpha_1, \ldots, \alpha_n \ge 0\}.$$

We define inductively a sequence of mappings $\lambda_n : X^n \times S_n \to X$. For $n = 1$ we set

$$\lambda_1(x, 1) = x$$

for all $x \in X$. If $n \in \mathbb{N}$, $x_1, \ldots, x_{n+1} \in X$ and $(\alpha_1, \ldots, \alpha_{n+1}) \in S_{n+1}$, then we set

$$\begin{aligned}&\lambda_{n+1}(x_1, \ldots x_{n+1}, \alpha_1, \ldots, \alpha_{n+1})\\ &= \lambda_n\Big(\lambda\Big(x_1, x_2, \frac{\alpha_2}{\alpha_1 + \alpha_2}\Big), x_3, \ldots x_{n+1}, \alpha_1 + \alpha_2, \alpha_3 \ldots, \alpha_{n+1}\Big),\end{aligned}$$

in the case $\alpha_1 + \alpha_2 > 0$, and

$$\lambda_{n+1}(x_1, \ldots x_{n+1}, \alpha_1, \ldots, \alpha_{n+1}) = \lambda_n(x_2, x_3, \ldots x_{n+1}, \alpha_2, \alpha_3 \ldots, \alpha_{n+1}),$$

in the case $\alpha_1 + \alpha_2 = 0$.

Definition 2.2. For any $n \in \mathbb{N}$, $(\alpha_1, \ldots, \alpha_n) \in S_n$ and for any $x_1, \ldots x_n$ from an equiconnected space (X, λ) the element $\lambda_n(x_1, \ldots x_n, \alpha_1, \ldots, \alpha_n)$ is called *the convex combination of elements $x_1, \ldots x_n$ with coefficients $\alpha_1, \ldots, \alpha_n$.*

Definition 2.3. Let $(I, \leq)$ be a completely ordered set, $(x_i)_{i\in I}$ be a family of points of an equiconnected space (X, λ) and let $(\alpha_i)_{i\in I}$ be a collection of non-negative scalars with

(1) $\{i \in I : \alpha_i \neq 0\} = \{i_k : 1 \leq k \leq n\}$;
(2) $i_1 < i_2 < \ldots < i_n$;
(3) $\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_n} = 1$.

Then $\lambda_n(x_{i_1}, \ldots, x_{i_n}, \alpha_{i_1}, \ldots, \alpha_{i_n})$ is called *the λ-sum of $(x_i)_{i\in I}$ with the coefficients* $(\alpha_i)_{i\in I}$ and is denoted by $\sum_{i\in I}{}^{\lambda}\alpha_i x_i$.

We observe that $\sum_{i\in I}{}^{\lambda}\alpha_i x_i = \sum_{i\in I}\alpha_i x_i$ for any topological vector space X.
If A is a subset of an equiconnected space (X, λ), then

$$\lambda^0(A) = A, \quad \lambda^n(A) = \lambda(\lambda^{n-1}(A), A, [0,1]) \text{ for } n \in \mathbb{N},$$
$$\lambda^\infty(A) = \bigcup_{n=1}^{\infty} \lambda^n(A).$$

Definition 2.4. An equiconnected space (X, λ) is *locally convex* [2] if for any $x \in X$ and a neighborhood U of x there is a neighborhood V of x such that $\lambda^\infty(V) \subseteq U$.

Definition 2.5. A topological space $(X, \mathcal{S})$ is called *metrically quarter-stratifiable* (see [2, Definition 2.1] and [2, Theorem 2.2]) if it admits a weaker metrizable topology τ (called its *stratifying topology*) with a sequence of τ-open coverings $\mathcal{U}_n = (U_{i,n} : i \in I_n)$ of X and a sequence $((x_{i,n} : i \in I_n))_{n=1}^\infty$ of families of points from X such that for every $x \in X$ we have

$$\forall (i_n)_{n=1}^\infty \Big(x \in \bigcap_{n=1}^{\infty} U_{i_n,n} \implies x_{i_n,n} \to x\Big). \tag{2.1}$$

Let us notice that (2.1) is equivalent to the following:

$$\begin{aligned} &\forall\, U - \text{a neighborhood of } x \text{ in } (X, \mathcal{S})\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0\ \forall i \in I_n \\ &\qquad\qquad\qquad\qquad (x \in U_{i,n} \implies x_{i,n} \in U). \end{aligned} \tag{2.2}$$

Indeed, it is easy to see that (2.2)⇒(2.1). Now let $x \in X$ and (2.1) holds. Assume that (2.2) is not valid and take a neighborhood U of x, an increasing sequence $(k_n)_{n=1}^\infty$ of numbers and a sequence $(j_n)_{n=1}^\infty$ of indexes $j_n \in I_{k_n}$ such that $x \in U_{j_n,k_n}$ but $x_{j_n,k_n} \notin U$ for every $n \in \mathbb{N}$. Since $(U_{i,n} : i \in I_n)$ is a covering of X for every n, we choose $s_n \in I_n$ such that $x \in U_{s_n,n}$. For all $n \in \mathbb{N}$ we set $i_n = j_m$ if $n = k_m$ for some $m \in \mathbb{N}$ and $i_n = s_n$, otherwise. Then the sequence $(i_n)_{n=1}^\infty$ does not satisfy the implication from (2.1), which implies a contradiction.

Definition 2.6. A topological space X is *strongly countably dimensional* if there exist a sequence $(X_n)_{n=1}^\infty$ of closed subspaces of X such that $X = \bigcup_{n=1}^\infty X_n$ and $\dim X_n < \infty$ for every $n \in \mathbb{N}$.

Theorem 2.7. *Let $(X,\mathcal{S})$ be a metrically quarter-stratifiable space with its stratifying metrizable topology τ, (Y,λ) be an equiconnected space. If one of the conditions holds*

(i) *(X,τ) is strongly countably dimensional, or*
(ii) *Y is locally convex,*

then

$$\mathrm{C}((X,\mathcal{S}),Y)\subseteq \mathrm{B}_1((X,\tau),Y).$$

Proof. Let $f:(X,\mathcal{S})\to Y$ be a continuous mapping.

(i) Let $(X_m)_{m=1}^\infty$ be a sequence of closed subspaces of (X,τ) such that $X=\bigcup_{m=1}^\infty X_m$ and $\dim X_m<\infty$ for every $m\in\mathbb{N}$. Since X is metrically quarter-stratifiable, we take a sequence $((U_{i,n}:i\in I_n))_{n=1}^\infty$ of τ-open coverings of X and a sequence $((x_{i,n}:i\in I_n))_{n=1}^\infty$ of families of points from X. By [3, Theorem 5.1.10], for every $n\in\mathbb{N}$ we may choose a locally finite refinement $\mathcal{V}_n=(V_{j,n}:j\in J_n)$ of the open covering $(U_{i,n}:i\in I_n)$ of the paracompact strongly countably dimensional space (X,τ) such that for all $m\in\mathbb{N}$ and $x\in X_m$ there exists a neighborhood U of x with

$$|\{j\in J_n: U\cap V_{j,n}\neq\emptyset\}|\le m.$$

Fix $n\in\mathbb{N}$ and take a locally finite partition of unity $(\varphi_{j,n}:j\in J_n)$ on (X,τ) subordinated to $\mathcal{V}_n$. For every $j\in J_n$ we denote by $u_{j,n}$ an element $x_{i,n}$ with $V_{j,n}\subseteq U_{i,n}$. Assume that the set J_n is completely ordered and define $f_n:X\to Y$ by the formula

$$f_n(x)=\sum_{j\in J_n}^{\lambda}\varphi_{j,n}(x)f(u_{j,n}). \tag{2.3}$$

It follows from [4, Theorem 3.2] that f_n is continuous on (X,τ).

We prove now that $f_n(x)\to f(x)$ for every $x\in X$. Fix $x_0\in X$ and consider a neighborhood W of $y_0=f(x_0)$ in Y. Let m be a number such that

$$|\{j\in J_n: x_0\in V_{j,n}\}|\le m$$

for every $n\in\mathbb{N}$. The continuity of λ_m, the equality

$$\lambda_m(y_0,\dots,y_0,\alpha_1,\dots,\alpha_m)=y_0$$

for all $(\alpha_1,\dots,\alpha_m)\in S_m$ and compactness of S_m imply the existence of a neighborhood W_1 of y_0 in Y with

$$\lambda_m(y_1,\dots,y_m,\alpha_1,\dots,\alpha_m)\in W$$

for all $y_1,\dots,y_m\in W_1$ and $(\alpha_1,\dots,\alpha_m)\in S_m$. Since the mapping f is continuous at the point x_0, there exists an $\mathcal{S}$-open neighborhood U of x_0 such that $f(x)\in W_1$ for every $x\in U$. Using (2.2) we choose a number n_0 such that $u_{j,n}\in U$ for all $n\ge n_0$ and $j\in J_n$ with $x_0\in V_{j,n}$.

We show that $f_n(x_0)\in W$ for every $n\ge n_0$. Let $n\ge n_0$ be fixed and

$$J=\{j\in J_n: x_0\in V_{j,n}\}=\{j_1,j_2,\dots j_k\},$$

where $j_1 < j_2 < \ldots < j_k$ and $k \le m$. Denote

$$\alpha_1 = \varphi_{j_1,n}(x_0), \ldots, \alpha_k = \varphi_{j_k,n}(x_0), y_1 = f(x_{j_1,n}), \ldots, y_k = f(x_{j_k,n}).$$

Then $y_1, \ldots y_k \in W_1$. Hence,

$$f_n(x_0) = \lambda_k(y_1, \ldots, y_k, \alpha_1, \ldots \alpha_k) = \lambda_m(y_1, \ldots, y_k, y_0, \ldots, y_0, \alpha_1, \ldots \alpha_k, 0, \ldots, 0) \in W$$

by [4, Proposition 2.3].

(ii) We consider sequences $((U_{i,n} : i \in I_n))_{=1}^{\infty}$ of τ-open coverings of X and $((x_{i,n} : i \in I_n))_{n=1}^{\infty}$ of families of points from X such that (2.2) holds. Since (X, τ) is a paracompact space, for every $n \in \mathbb{N}$ we take a locally finite refinement $(V_{j,n} : j \in J_n)$ of $(U_{i,n} : i \in I_n)$. For every $j \in J_n$ by $u_{j,n}$ we denote a point $x_{i,n}$ such that $V_{j,n} \subseteq U_{i,n}$. Let $(\varphi_{j,n} : j \in J_n)$ be a locally finite partition of unity on (X, τ) subordinated to the covering $(V_{j,n} : j \in J_n)$. For every $n \in \mathbb{N}$ and $x \in X$ we define a mapping $f_n : X \to Y$ by the equality (2.3) and observe that f_n is continuous on (X, τ) by [4, Theorem 3.2].

It remains to show that $f_n(x) \to f(x)$ on X. Fix $x_0 \in X$ and a neighborhood W of $y_0 = f(x_0)$. Since Y is locally convex, there exists a neighborhood W_1 of y_0 in Y such that

$$\lambda^{\infty}(W_1) \subseteq W.$$

Since f is continuous on the space $(X, \mathcal{S})$ at x_0, we may choose a neighborhood U of x_0 in $(X, \mathcal{S})$ such that $f(x) \in W_1$ for every $x \in U$. Using (2.2) we take a number n_0 with $u_{j,n} \in U$ for all $n \ge n_0$ and $j \in J_n$ with $x_0 \in V_{j,n}$. In order to show that $f_n(x_0) \in W$ for all $n \ge n_0$ we fix $n \ge n_0$ and set

$$J = \{j \in J_n : x_0 \in V_{j,n}\} = \{j_1, j_2, \ldots, j_k\},$$

where $j_1 < j_2 < \ldots < j_k$. Denote

$$\alpha_1 = \varphi_{j_1,n}(x_0), \ldots, \alpha_k = \varphi_{j_k,n}(x_0), y_1 = f(x_{j_1,n}), \ldots, y_k = f(x_{j_k,n}).$$

Then $y_1, \ldots y_k \in W_1$ and

$$f_n(x_0) = \lambda_k(y_1, ..., y_k, \alpha_1, ..., \alpha_k) \in \lambda^k(W_1) \subseteq W.$$

Therefore, $f \in \mathrm{B}_1((X, \tau), Y)$. □

3. MAPPINGS ON A PRODUCT OF THE SORGENFREY LINES

Lemma 3.1. *For any at most countable set T the product $\mathbb{S}^T$ is metrically quarter-stratifiable and a topology τ from Definition 2.5 is the Tykhonoff topology on $\mathbb{R}^T$.*

Proof. We consider the case $|T| = \aleph_0$. For all $n, k \in \mathbb{N}$ we set $I_{n,k} = \mathbb{Z}$, $I_n = \prod_{k=1}^{\infty} I_{n,k} = \mathbb{Z}^{\aleph_0}$ and $X_n = \mathbb{R}$. Now for all $n \in \mathbb{N}$ and $i = (i_k)_{k=1}^{\infty} \in I_n$ we put

$$U_{i,n} = \prod_{k=1}^{n} \Big(\frac{i_k - 1}{n}, \frac{i_k + 1}{n}\Big) \times \prod_{k=n+1}^{\infty} X_k,$$
$$x_{i,n} = \Big(\frac{i_1 + 1}{n}, \ldots, \frac{i_n + 1}{n}, 0, 0, \ldots\Big).$$

We verify that the defined sequences $((U_{i,n} : i \in I_n))_{n=1}^{\infty}$ of open coverings of $\mathbb{R}^{\aleph_0}$ and points $((x_{i,n} : i \in I_n))_{n=1}^{\infty}$ satisfy the properties indicated in Definition 2.5. Let $x = (\xi_1, \xi_2, \dots) \in \mathbb{R}^{\aleph_0}$ and let $U = U_1 \times \dots \times U_m \times X_{m+1} \times \dots$ be a basic neighborhood of x in $\mathbb{S}^{\aleph_0}$. For every $k = 1, \dots, m$ we take $n_k \in \mathbb{N}$ such that for all $n \geq n_k$ and $i_k \in I_{n,k}$ the inclusion $\xi_k \in (\frac{i_k-1}{n_k}, \frac{i_k+1}{n_k})$ implies that $\frac{i_k+1}{n_k} \in U_k$. We set $n_0 = \max\{m, n_1, \dots, n_m\}$. Assume that $n \geq n_0$ and $i \in I_n$ be such that $x \in U_{i,n}$. Then, obviously, $x_{i,n} \in U$. □

Theorem 2.7 and Lemma 3.1 immediately imply the following result.

Theorem 3.2. *Let Y be an equiconnected space. Then*

$$\mathrm{C}(\mathbb{S}^n, Y) \subseteq \mathrm{B}_1(\mathbb{R}^n, Y)$$

for any $n \in \mathbb{N}$. If Y is a locally convex equiconnected space, then

$$\mathrm{C}(\mathbb{S}^{\aleph_0}, Y) \subseteq \mathrm{B}_1(\mathbb{R}^{\aleph_0}, Y).$$

Theorem 3.3. *Let Y be a metrizable connected and locally arcwise connected space. Then*

$$\mathrm{C}(\mathbb{S}^T, Y) \subseteq \mathrm{B}_1(\mathbb{R}^T, Y)$$

for any at most countable set T.

Proof. Consider a continuous mapping $f : \mathbb{S}^T \to Y$, where $|T| \leq \aleph_0$. Take an arbitrary open set $V \subseteq Y$ and a continuous function $g : Y \to \mathbb{R}$ with $V = g^{-1}((0, +\infty))$. Let $h = g \circ f$. Then $h \in \mathrm{B}_1(\mathbb{R}^T, \mathbb{R})$ by Theorem 3.2. Consequently, the preimage $f^{-1}(V) = h^{-1}((0, +\infty))$ is an F_σ-set in $\mathbb{R}^T$. Since the space $Z = f(\mathbb{S}^T)$ is separable as a continuous image of the separable space $\mathbb{S}^T$, Theorem 1 from [5] implies that $f \in \mathrm{B}_1(\mathbb{R}^T, Y)$. □

Theorem 3.4. *Let Y be a metrizable connected and locally arcwise connected space. Then*

$$\mathrm{C}(\mathbb{S}^T, Y) \subseteq \mathrm{B}_1(\mathbb{R}^T, Y)$$

for any set T.

Proof. The case $|T| \leq \aleph_0$ is considered in Theorem 3.3.

Assume that $|T| > \aleph_0$ and let $f : \mathbb{S}^T \to Y$ be a continuous mapping. By [6] there exists a countable set $T_0 \subseteq T$ such that for all $x, u \in \mathbb{S}^T$ the equality $x|_{T_0} = u|_{T_0}$ implies the equality $f(x) = f(u)$. Define the continuous mapping $\varphi : \mathbb{S}^T \to \mathbb{S}^{T_0}$ by the rule $\varphi(x) = x|_{T_0}$ for all $x \in \mathbb{S}^T$ and put $g(u) = f(x)$ if $u = \varphi(x)$ for some $x \in \mathbb{S}^T$. Observe that the mapping $g : \mathbb{S}^{T_0} \to Y$ is well-defined and continuous. According to Theorem 3.3 we have $g \in \mathrm{B}_1(\mathbb{R}^{T_0}, Y)$. Take a sequence of continuous mappings $g_n : \mathbb{R}^{T_0} \to Y$ which is convergent to g on $\mathbb{R}^{T_0}$ pointwise. For all $n \in \mathbb{N}$ and $x \in \mathbb{R}^T$ we put $h_n(x) = g_n(\varphi(x))$. Then

$$\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} g_n(\varphi(x)) = g(\varphi(x)) = f(x).$$

Hence, $f \in \mathrm{B}_1(\mathbb{R}^T, Y)$. □

Theorems 3.2–3.4 imply the ensuing corollary.

Corollary 3.5. *Let Y be a topological vector space. Then the inclusion* $\mathrm{C}(\mathbb{S}^T, Y) \subseteq \mathrm{B}_1(\mathbb{R}^T, Y)$ *is valid if one of the following conditions hold:*

a) $|T| < \aleph_0$,
b) *Y is a locally convex space and* $|T| \leq \aleph_0$,
c) *Y is metrizable.*

The following question is open.

Question 3.6. Does the inclusion $\mathrm{C}(\mathbb{S}^T, Y) \subseteq \mathrm{B}_1(\mathbb{R}^T, Y)$ hold for $|T| = \aleph_0$ and any topological vector space Y?

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