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HIGHER ORDER NEVANLINNA FUNCTIONS AND THE INVERSE THREE SPECTRA PROBLEM

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Abstract. The three spectra problem of recovering the Sturm-Liouville equation by the spectrum of the Dirichlet-Dirichlet boundary value problem on $[0, a]$, the Dirichlet-Dirichlet problem on $[0, a/2]$ and the Neumann-Dirichlet problem on $[a/2, a]$ is considered. Sufficient conditions of solvability and of uniqueness of the solution to such a problem are found.

Keywords: spectrum, eigenvalue, Dirichlet boundary condition, Neumann boundary condition, Marchenko equation, Nevanlinna function.

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1. INTRODUCTION

In [15] the so-called three spectra inverse problem was introduced (see also [6], and generalizations in [1–5, 9, 16, 22]). There the spectrum of a boundary value problem generated by the Sturm-Liouville equation on a finite interval is given together with the spectra of the boundary value problems generated by the same equation on complementary subintervals and the aim is to find the equation.

Let us describe a functional equation which appears in both the Hochstadt-Liebermann problem (see [7, 8, 13, 17, 19–21] for different versions of it) and in the three spectra inverse problem.

It is more convenient to measure distance on the right half of the interval in the opposite direction. Then we can write our problem as follows:

$$-y_j'' + q_j(x)y_j = \lambda^2 y_j, \quad x \in [0, a/2], \quad j = 1, 2, \tag{1.1}$$

$$y_1(0) = 0, \tag{1.2}$$

$$y_2(0) = 0, \tag{1.3}$$

$$y_1(a/2) = y_2(a/2), \tag{1.4}$$

$$y_1'(a/2) + y_2'(a/2) = 0. \tag{1.5}$$

Here $q_j \in L_2(0, a/2)$ $(j = 1, 2)$ are real valued. We denote the eigenvalues of problem (1.1)–(1.5) by $\{\lambda_k\}_{-\infty,\ k\neq 0}^{\infty}$ $(\lambda_{-k} = -\lambda_k)$. We introduce $s_j(\lambda, x)$, the solution of the Sturm-Liouville equation (1.1) which satisfies the conditions $s_j(\lambda, 0) = 0$, $s_j'(\lambda, 0) = 1$.

Let us consider also the following problems on the subintervals:

$$-y_j'' + q_j(x)y_j = \lambda^2 y_j, \quad x \in [0, a/2]\,, \tag{1.6}$$

$$y_j(0) = 0, \tag{1.7}$$

$$y_j\,(a/2) = 0, \tag{1.8}$$

the spectra $\{\nu_k^{(j)}\}_{-\infty,k\neq 0}^{\infty}$ $(j = 1, 2)$ of which coincide with the sets of zeros of $s_j(\lambda, a/2)$ and problem

$$-y_j'' + q_j(x)y_j = \lambda^2 y_j, \quad x \in [0, a/2]\,, \tag{1.9}$$

$$y_j(0) = 0, \tag{1.10}$$

$$y_j'\,(a/2) = 0, \tag{1.11}$$

the spectra $\{\mu_k^{(j)}\}_{-\infty,k\neq 0}^{\infty}$ $(j = 1, 2)$ of which coincide with the sets of zeros of $s_j'(\lambda, a/2)$.

Let us look for a solution to problem (1.1)–(1.5) in the form $y_1 = C_1 s_1(\lambda, x)$, $y_2 = C_2 s_2(\lambda, x)$ where C_j are constants. Then (1.4) and (1.5) imply

$$C_1 s_1\,(\lambda, a/2) = C_2 s_2(\lambda, a/2),$$

$$C_1 s_1'\,(\lambda, a/2) + C_2 s_2'(\lambda, a/2) = 0.$$

This system of equations possesses a nontrivial solution at the zeros of the characteristic function

$$\Phi(\lambda) = s_1(\lambda, a/2)s_2'(\lambda, a/2) + s_2(\lambda, a/2)s_1'(\lambda, a/2). \tag{1.12}$$

The set of zeros $\{\lambda_k\}_{-\infty,k\neq 0}^{\infty}$ of this function is the spectrum of problem (1.1)–(1.5).

Let us notice that equation (1.12) is a particular case of the second of equations (2.18) in Theorem 2.1 of [11] which appears in the spectra theory of quantum graphs.

Equation (1.12) plays an important role in the theory of the three spectra problem and in the theory of the Hochstadt-Liebermann problem. Since knowing $q_1(x)$ we can solve equation (1.1) with $j = 1$ and find $s_1(\lambda, a/2)$ and $s_1'(\lambda, a/2)$, knowing $\{\lambda_k\}_{-\infty,k\neq 0}^{\infty}$ we can find $\Phi(\lambda)$, the Hochstadt-Liebermann problem can be formulated as follows: given $\Phi(\lambda)$, $s_1(\lambda, a/2)$ and $s_1'(\lambda, a/2)$ find $q_2(x)$.

The three spectra problem of [15] can be formulated as follows: given $\Phi(\lambda)$, $s_1(\lambda, a/2)$ and $s_2(\lambda, a/2)$ find $q_1(x)$ and $q_2(x)$. In the same way as in [15] one can solve the following problem: given $\Phi(\lambda)$, $s_1'(\lambda, a/2)$ and $s_2'(\lambda, a/2)$ find $q_1(x)$ and $q_2(x)$.

In the present paper we introduce the notion of the Nevanlinna function of higher order and essentially positive Nevanlinna functions of higher order (Section 2) which we use in Section 3 to describe boundary value problems with conditions at two or more interior points of the interval. There we compare the spectrum of the Dirichlet problem on the whole interval with the union of the spectra of the Dirichlet problems on the complementary subintervals.

In Section 4 we deal with a direct three spectra problem. We show that

$$\frac{\Phi(\sqrt{z})}{s_1'(\sqrt{z},a/2)s_2'(\sqrt{z},a/2)} \quad \text{and} \quad \frac{s_1(\sqrt{z},a/2)s_2(\sqrt{z},a/2)}{\Phi(\sqrt{z})}$$

belong to the class of essentially positive Nevanlinna functions (see Definition 2.3 below), while

$$\frac{s_1(\sqrt{z},a/2)s_2'(\sqrt{z},a/2)}{\Phi(\sqrt{z})}$$

belongs to the class of essentially positive Nevanlinna functions of the second order (see Definition 2.6 below). A consequence of these facts appears in certain mutual order in the location of zeros of $\Phi(\lambda)$ and zeros of the product $s_1(\lambda,a/2)s_2'(\lambda,a/2)$.

In Section 5 we consider the following inverse problem. Given the spectra of problem (1.1)–(1.5), of problem (1.6)–(1.8) with $j=1$ and of problem (1.9)–(1.11) with $j=2$, in other words given $\Phi(\lambda)$, $s_1(\lambda,a/2)$ and $s_2'(\lambda,a/2)$, find $q_1(x)$ and $q_2(x)$. We give sufficient conditions of solvability and uniqueness of solutions to such a problem.

2. NEVANLINNA FUNCTIONS

In the sequel we will use the notion of the Nevanlinna function, also called the R-function in [10]. In this paper we deal only with meromorphic functions.

Definition 2.1 (see e.g. [14, Definition 5.1.20])**.** The meromorphic function θ is said to be a Nevanlinna function, or an R-function, or an $\mathcal{N}$-function if:

(i) θ is analytic in the half-planes $\mathrm{Im}\lambda > 0$ and $\mathrm{Im}\lambda < 0$;
(ii) $\theta(\overline{\lambda}) = \overline{\theta(\lambda)}$ if $\mathrm{Im}\lambda \neq 0$;
(iii) $\mathrm{Im}\lambda\, \mathrm{Im}\theta(\lambda) > 0$ for $\mathrm{Im}\lambda \neq 0$.

Lemma 2.2 (see e.g. [14, Lemma 5.1,22])**.** *If θ is a Nevanlinna function, then so are the functions $-\frac{1}{\theta}$ and $(\frac{1}{\theta}+c)^{-1}$ for any real constant c.*

Proof. This easily follows from

$$\mathrm{Im}\left(-\frac{1}{\theta(\lambda)}\right) = \frac{\mathrm{Im}\theta(\lambda)}{|\theta(\lambda)|^2} \quad \text{and} \quad \mathrm{Im}\left(\frac{1}{\theta(\lambda)}+\mathrm{c}\right)^{-1} = \frac{-\mathrm{Im}\frac{1}{\theta(\lambda)}}{\left|\frac{1}{\theta(\lambda)}+\mathrm{c}\right|^2}. \qquad \square$$

Definition 2.3 (see e.g. [18] or [14, Definition 5.1.26])**.**
1. The class $\mathcal{N}^{ep}$ of essentially positive Nevanlinna functions is the set of all functions $\theta \in \mathcal{N}$ which are analytic in $\mathbb{C} \setminus [0,\infty)$ with the possible exception of finitely many poles.
2. The class $\mathcal{N}^{ep}_{+}$ ($\mathcal{N}^{\mathrm{ep}}_{-}$) is the set of all functions $\theta \in \mathcal{N}^{ep}$ such that for some $\gamma \in \mathbb{R}$ we have $\theta(\lambda) > 0$ ($\theta(\lambda) < 0$) for all $\lambda \in (-\infty,\gamma)$.

Lemma 2.4 (see e.g. [18])**.** *Let $\theta \in \mathcal{N}^{ep}_{+}$. Then the zeros $\{a_k\}_{k=1}^{\infty}$ and poles $\{b_k\}_{k=1}^{\infty}$ of θ interlace, i.e.*

$$-\infty < b_1 < a_1 < b_2 < a_2 < \ldots .$$

Lemma 2.5 (see [18] or [14, Theorem 11.1.6])**.**

$$\theta \in \mathcal{N}^{\mathrm{ep}}_{\pm} \quad \textit{if and only if} \quad -\frac{1}{\theta} \in \mathcal{N}^{\mathrm{ep}}_{\mp}.$$

Proof. Since poles and zeros of $\theta \in \mathcal{N}^{\mathrm{ep}}_{+}$ interlace by Lemma 2.4, there is $\gamma \leq 0$ such that θ has no poles or zeros in $(-\infty, \gamma)$ and we have that $\theta(\lambda) > 0$ for all $\lambda \in (-\infty, \gamma)$. This means that $-\frac{1}{\theta}$ also has no poles and zeros in $(-\infty, \gamma)$ and $-\frac{1}{\theta(\lambda)} < 0$ for all $\lambda \in (-\infty, \gamma)$ □

Definition 2.6. A function $\frac{f(z)}{g(z)}$ is said to belong to $\mathcal{N}^{ep}_{+,p}$ (essentially positive Nevanlinna class of order p) if there exist functions $g_1(z)$, $g_2(z)$,..., $g_p(z)$ such that

$$\frac{f(z)}{g_1(z)} \in \mathcal{N}^{ep}_{+}, \quad \frac{g_1(z)}{g_2(z)} \in \mathcal{N}^{ep}_{+}, \ \dots \ , \ \frac{g_{p-1}(z)}{g_p(z)} \in \mathcal{N}^{ep}_{+}, \quad \frac{g_p(z)}{g(z)} \in \mathcal{N}^{ep}_{+}.$$

Here $\mathcal{N}^{ep}_{+,0} =: \mathcal{N}^{ep}_{+}$.

It is clear that $\mathcal{N}^{ep}_{+,r} \subset \mathcal{N}^{ep}_{+,p}$ for $p > r$.

Theorem 2.7. *Let* $\frac{f(z)}{g(z)} \in \mathcal{N}^{ep}_{+,p}$. *Then zeros* $\{a_k\}_{k=1}^{m}$ $(m \leq \infty)$ *of* $f(z)$ *and zeros* $\{b_k\}_{k=1}^{m_1}$ $(m_1 \leq \infty)$ *of* $g(z)$ *satisfy the conditions:*

1. *the interval* $(-\infty, b_1]$ *does not contain elements of* $\{a_k\}_{k=1}^{(\infty)}$,
2. *for* $k \geq 2$ *each interval* $(-\infty, b_k)$ *contains not more than* $k-1$ *and not less than* $k-1-p$ *elements of* $\{a_k\}_{k=1}^{(\infty)}$.

Proof. Denote by $\{\tau_k^{(j)}\}_{k=1}^{\infty}$ $(j = 1, 2, \dots, p)$ the sequence of zeros of $g_j(z)$. Statement 1 is obvious due to

$$-\infty < b_1 < \tau_1^{(1)} < \tau_1^{(2)} < \dot{<} \tau_1^{(p)} < a_1 < a_2 < \dots .$$

The interval $(-\infty, b_k)$ containes exactly $k-1$ elements of $\{\tau_k^{(1)}\}_{k=1}^{\infty}$. The interval $(\tau_{k-1}^{(1)}, \tau_k^{(1)})$ containes exactly one element of $\{\tau_k^{(2)}\}_{k=1}^{\infty}$, namely $\tau_{k-1}^{(2)}$ which can lie either inside $(\tau_{k-1}^{(1)}, b_k)$ or outside of it. Thus, the number of elements of $\{\tau_k^{(2)}\}_{k=1}^{\infty}$ belonging to $(-\infty, b_k)$ is either $k-1$ or $k-2$. Thus, our theorem is proved for $p = 1$. In the case of $p > 1$ we consider the following cases:

1. $\tau_{k-1}^{(2)} \in (\tau_{k-1}^{(1)}, b_k)$. If $\tau_{k-1}^{(3)} < b_k$, then since $\tau_k^{(3)} > \tau_k^{(2)} > \tau_k^{(1)} > b_k$ we conclude that the interval $(-\infty, b_k)$ contains $k-1$ element of $\{\tau_k^{3)}\}_{k=1}^{\infty}$. If $\tau_{k-1}^{(3)} > b_k$, then since $\tau_{k-2}^{(3)} < \tau_{k-1}^{(2)} < b_k$ we conclude that $(-\infty, b_k)$ contains $k-2$ element of $\{\tau_k^{3)}\}_{k=1}^{\infty}$.
2. $\tau_{k-1}^{(2)} \in [b_k, \tau_k^{(1)})$. If $\tau_{k-2}^{(3)} \in [b_k, \tau_{k-1}^{(2)})$, then since $\tau_{k-3}^{(3)} < \tau_{k-2}^{(2)} < \tau_{k-1}^{(1)} < b_k$ we conclude that $(-\infty, b_k)$ contains $k-3$ element of $\{\tau_k^{(3)}\}_{k=1}^{\infty}$. If $\tau_{k-2}^{(3)} < b_k$, then since $\tau_{k-1}^{(3)} > \tau_{k-1}^{(2)} > b_k$ and $(-\infty, b_k)$ contains $k-2$ element of $\{\tau_k^{(3)}\}_{k=1}^{\infty}$.

Thus, our theorem is true for $p = 2$. The case of any $p > 2$ can be treated in the same way. □

Theorem 2.8. *Let* $\frac{h(z)}{f(z)} \in \mathcal{N}_{+}^{ep}$ *and* $\frac{h(z)}{g(z)} \in \mathcal{N}_{+}^{ep}$. *Then zeros* $\{a_k\}_{k=1}^{m}$ $(m \leq \infty)$ *of* $f(z)$ *and zeros* $\{b_k\}_{k=1}^{m_1}$ $(m_1 \leq \infty)$ *of* $g(z)$ *satisfy the condition: each interval* $(-\infty, b_k)$ *contains* $k-1$ *or* k *elements of* $\{a_k\}_{k=1}^{(\infty)}$.

Proof. Denote by $\{d_k\}_{k=1}^{\infty}$ the zeros of $h(z)$. Then

$$a_{k-1} < d_{k-1} < b_k < d_k < a_{k+1}$$

and

$$d_{k-1} < a_k < d_k.$$

If $a_k \in (d_{k-1}, b_k)$, then $(-\infty, b_k)$ contains k elements of $\{a_k\}_{k=1}^{(\infty)}$. If $a_k \in [b_k, d_k)$, then $(-\infty, b_k)$ contains $k-1$ elements of $\{a_k\}_{k=1}^{(\infty)}$. □

3. SPECTRAL PROBLEM OF VIBRATIONS OF A SMOOTH STRING CLAMPED AT MORE THAN ONE INTERIOR POINT

We obtain an example where functions of $\mathcal{N}_{+,p}^{ep}$ can be used considering the problem of vibrations of a smooth string clamped at a few interior points. Let $0 = a_0 < a_1 < a_2 < \ldots < a_n = a$ and consider the Dirichlet problems

$$\begin{cases} -y'' + q(x)y = \lambda^2 y, \\ y(a_0) = y(a_n) = 0, \end{cases} \tag{3.1}$$

$$\begin{cases} -y'' + q(x)y = \lambda^2 y, \\ y(a_{j-1}) = y(a_j) = 0, \end{cases} \quad \text{where} \quad j = 1, 2, \ldots, n, \tag{3.2}$$

generated by the same real $q \in L_2(0, a)$. Denote by $s_j(\lambda, x)$ the solution of the differential equation in (3.2) which satisfies $s_j(\lambda, a_{j-1}) = s_j'(\lambda, a_{j-1}) - 1 = 0$ $(j = 1, 2, \ldots, n+1)$. Then $s_j(\lambda, a_j - a_{j-1})$ $(j = 1, 2, \ldots, n)$ are the characteristic functions of problems (3.2). The case of $n = 2$ was considered in [15].

Theorem 3.1. *For* $n \geq 2$,

$$\frac{\prod_{j=1}^{n} s_j(\sqrt{z}, a_j - a_{j-1})}{s_1(\sqrt{z}, a)} \in \mathcal{N}_{+,n-2}^{ep}.$$

Proof. Let us consider the boundary value problems

$$\begin{cases} -y'' + q(x)y = \lambda^2 y, \\ y(a_j) = y(a_n) = 0. \end{cases} \tag{3.3}$$

The characteristic function of problem (3.1) is $s_1(\sqrt{z}, a)$, the characteristic function of problems (3.2) are $s_j(\sqrt{z}, a_j - a_{j-1})$ $(j = 1, 2, \ldots, n)$, the characteristic function of problem (3.3) is $s_{j+1}(\sqrt{z}, a)$. Evidently,

$$\frac{s_1(\sqrt{z}, a_1) s_2(\sqrt{z}, a)}{s_1(\sqrt{z}, a)} \in \mathcal{N}_{+}^{ep},$$

$$\frac{s_1(\sqrt{z},a_1)s_2(\sqrt{z},a_2-a_1)s_3(\sqrt{z},a)}{s_1(\sqrt{z},a_1)s_2(\sqrt{z},a)}=\frac{s_2(\sqrt{z},a_2-a_1)s_3(\sqrt{z},a)}{s_2(\sqrt{z},a)}\in\mathcal{N}_+^{ep},$$

$$\frac{s_1(\sqrt{z},a_1)s_2(\sqrt{z},a_2-a_1)s_3(\sqrt{z},a_3-a_2)s_4(\sqrt{z},a)}{s_1(\sqrt{z},a_1)s_2(\sqrt{z},a_2-a_1)s_3(\sqrt{z},a)}=\frac{s_3(\sqrt{z},a_3-a_2)s_4(\sqrt{z},a)}{s_3(\sqrt{z},a)}\in\mathcal{N}_+^{ep}.$$

□

Theorems 2.7 and 3.1 imply the following corollary.

Corollary 3.2. *Let $n \geq 2$. The spectrum $\{\lambda_k\}_{-\infty,k\neq 0}^{\infty}$ of problem* (3.1) *is related with the union $\{\eta_k\}_{-\infty,k\neq 0}^{\infty} = \bigcup_{j=1}^{n}\{\nu_k^{(j)}\}_{-\infty,k\neq 0}^{\infty}$ of the spectra of problems* (3.2) *in the sence that each interval $(-\infty,\lambda_k^2)$ contains not more than $k-1$ and not less than $k+1-n$ elements of $\{\eta_k^2\}_{k=1}^{\infty}$.*

Lemma 3.3. *If any two of the three equalities $s_1(\lambda,a)=0$, $s_1(\lambda,a_j)=0$, $s_j(\lambda,a)=0$ are true, then the third of them is true also.*

Proof. This follows from the identity

$$s_1(\lambda,a)=s_1(\lambda,a_j)s'_{n+1}(\lambda,a_j)+s'_1(\lambda,a_j)s_{n+1}(\lambda,a_j)$$

which is an analogue of (1.12) because the sets of zeros of $s_{n+1}(\lambda,a_j)$ and of $s_j(\lambda,a)$ coincide □

Corollary 3.4. *Let $n>2$ and $k>n-1$. Then if $\lambda_k=\eta_{k-n+1}=\eta_{k-n+2}=\ldots=\eta_{k-1}$ then $\lambda_k=\eta_k$.*

Proof. Condition $\lambda_k=\eta_{k-n+1}=\eta_{k-n+2}=\ldots=\eta_{k-1}$ means that among $s_j(\lambda_k,a_j-a_{j-1})$ $(j=1,2,\ldots,n)$ at least $n-1$ are equal to zero. Suppose $s_p(\lambda_k,a_p-a_{p-1})\neq 0$. Then, by Lemma 3.3, $s_1(\lambda_k,a)=0$ and $s_1(\lambda_k,a_{p-1})=0$ imply $s_p(\lambda_k,a)=0$. Now $s_p(\lambda_k,a)=0$ and $s_{p+1}(\lambda_k,a)=0$ imply $s_p(\lambda_k,a_p-a_{p-1})=0$, a contradiction. □

4. DIRECT THREE SPECTRA PROBLEM

Here we compare the spectrum of problem (1.1)–(1.5) with the union of the spectrum of problem (1.6)–(1.8) with $j=1$ and the spectrum of problem (1.9)–(1.11) with $j=2$.

We will use the following simple result.

Lemma 4.1. *Let Φ be given by* (1.12). *Then the function*

$$\frac{s_2(\sqrt{z},a/2)s_1(\sqrt{z},a/2)}{\Phi(\sqrt{z})}$$

is an essentially positive Nevanlinna function.

Proof. It is known that $\frac{s_1(\sqrt{z},a/2)}{s_1'(\sqrt{z},a/2)}$ and $\frac{s_2(\sqrt{z},a/2)}{s_2'(\sqrt{z},a/2)}$ are essentially positive Nevanlinna functions. Therefore, such is also

$$\frac{s_2(\sqrt{z},a/2)s_1(\sqrt{z},a/2)}{\Phi(\sqrt{z})} = \left(\left(\frac{s_1(\sqrt{z},a/2)}{s_1'(\sqrt{z},a/2)}\right)^{-1} + \left(\frac{s_2(\sqrt{z},a/2)}{s_2'(\sqrt{z},a/2)}\right)^{-1}\right)^{-1}$$

by arguments similar to the proof of Lemma 2.2 □

Theorem 4.2. *The sequences* $\{\lambda_k\}_{-\infty,k\neq 0}^{\infty}$ *and* $\{\zeta_k\}_{-\infty,k\neq 0}^{\infty} \overset{def}{=} \{\nu_k^{(1)}\}_{-\infty,k\neq 0}^{\infty} \cup \{\mu_k^{(2)}\}_{-\infty,k\neq 0}^{\infty}$ *are interlaced as follows: each interval* $(-\infty,(\lambda_k)^2)$ *contains* k *or* $k-1$ *elements of the sequence* $\{(\zeta_k)^2\}_{k=1}^{\infty}$.

Proof. This follows from

$$\frac{s_1(\sqrt{z},a/2)s_2(\sqrt{z},a/2)}{\Phi(\sqrt{z})} \in \mathcal{N}_+^{ep}, \quad \frac{s_1(\sqrt{z},a/2)s_2(\sqrt{z},a/2)}{s_1(\sqrt{z},a/2)s_2'(\sqrt{z},a/2)} \in \mathcal{N}_+^{ep}$$

and Theorem 2.8. □

In the next section we will use the following known result.

Lemma 4.3 (see, e.g. [12, Theorem 3.4.1] with $a=\pi$ or [14, Corollaries 12.2.10 and 12.5.2]). *If* $q_j \in L_2(0,a/2)$, *then the sequences* $\{\lambda_k\}_{-\infty,k\neq 0}^{\infty}$, $\{\mu_k^{(j)}\}_{-\infty,k\neq 0}^{\infty}$, $\{\nu_k^{(j)}\}_{-\infty,k\neq 0}^{\infty}$, *which are the sets of zeros of the functions*

$$s(\lambda,a) = \frac{\sin\lambda a}{\lambda} - \frac{A_0\cos\lambda a}{\lambda^2} + \frac{\psi(\lambda)}{\lambda^2},$$

where $A_0 = A_1 + A_2$, $A_j = \frac{1}{2}\int_0^{a/2} q_j(x)dx$, $\psi \in \mathcal{L}^a$,

$$s_j'(\lambda,a/2) = \cos\frac{\lambda a}{2} + A_j\frac{\sin\frac{\lambda a}{2}}{\lambda} + \frac{\tilde{\psi}_j(\lambda)}{\lambda},$$

$$s_j(\lambda,a/2) = \frac{\sin\frac{\lambda a}{2}}{\lambda} - A_j\frac{\cos\frac{\lambda a}{2}}{\lambda^2} + \frac{\psi_j(\lambda)}{\lambda^2}$$

with $\psi_j \in \mathcal{L}^{a/2}$, $\tilde{\psi}_j \in \mathcal{L}^{a/2}$, *behave asymptotically as follows:*

$$\lambda_k = \frac{\pi k}{a} + \frac{A_0}{\pi k} + \frac{\beta_k}{k}, \tag{4.1}$$

$$\mu_k^{(j)} = \frac{\pi(2k-1)}{a} + \frac{A_j}{\pi k} + \frac{\tilde{\beta}_k^{(j)}}{k}, \tag{4.2}$$

$$\nu_k^{(j)} = \frac{2\pi k}{a} + \frac{A_j}{\pi k} + \frac{\beta_k^{(j)}}{k}, \tag{4.3}$$

where $\{\beta_k\}_{-\infty,k\neq 0}^{\infty} \in l_2$, $\{\beta_k^{(j)}\}_{-\infty,k\neq 0}^{\infty} \in l_2$, $\{\tilde{\beta}_k^{(j)}\}_{-\infty,k\neq 0}^{\infty} \in l_2$ $(j=1,2)$.

5. THREE SPECTRA INVERSE PROBLEM

Now we consider a three spectra inverse problem in which the spectrum of problem (1.1)–(1.5) is given together with f the spectrum of problem (1.6)–(1.8) with $j = 1$ and the spectrum of problem (1.9)–(1.11) with $j = 2$. The aim is to recover q_j $(j = 1, 2)$. In other words, in this case the sets of zeros of $\Phi(\lambda)$, $s_1(\lambda, a/2)$ and $s_2'(\lambda, a/2)$ are given.

Definition 5.1. An entire function of exponential type $\leq a$ is said to belong to the Paley-Wiener class $\mathcal{L}^a$ if its restriction onto the real axis belongs to $L_2(-\infty, \infty)$.

Lemma 5.2. *The set of zeros of the function*

$$\Phi(\lambda) = \cos\frac{\lambda a}{2}\frac{\sin\frac{\lambda a}{2}}{\lambda} - A_2\frac{\cos^2\frac{\lambda a}{2}}{\lambda^2} + A_1\frac{\sin^2\frac{\lambda a}{2}}{\lambda^2} + \frac{\psi(\lambda)}{\lambda^2}, \tag{5.1}$$

where A_j are real constants, $\psi \in \mathcal{L}^a$ can be given as the union of two sets (denote them by $\{\zeta_k^{(2)}\}_{-\infty,k\neq 0}^{\infty}$ and $\{\zeta_k^{(1)}\}_{-\infty,k\neq 0}^{\infty}$) such that $\zeta_{-k}^{(j)} = -\zeta_k^{(j)}$ $(j = 1, 2)$ and

$$\begin{aligned} \zeta_k^{(2)} &= \frac{2\pi k}{a} + \frac{A_2}{\pi k} + \frac{\gamma_k}{k}, \\ \zeta_k^{(1)} &= \frac{\pi(2k-1)}{a} + \frac{A_1}{\pi k} + \frac{\tilde{\gamma}_k}{k}, \end{aligned} \tag{5.2}$$

where $\{\gamma_k\}_{-\infty,k\neq 0}^{\infty} \in l_2$, $\{\tilde{\gamma}_k\}_{-\infty,k\neq 0}^{\infty} \in l_2$.

Proof. The function $\Phi(\lambda)$ can be presented as follows:

$$\Phi(\lambda) = \Phi_0(\lambda) + \frac{\psi_1(\lambda)}{\lambda^2},$$

where

$$\begin{aligned} \Phi_0(\lambda) &= \cos\frac{\lambda a}{2}\frac{\sin\frac{\lambda a}{2}}{\lambda} - A_2\frac{\cos^2\frac{\lambda a}{2}}{\lambda^2} + A_1\frac{\sin^2\frac{\lambda a}{2}}{\lambda^2} - A_1A_2\cos\frac{\lambda a}{2}\frac{\sin\frac{\lambda a}{2}}{\lambda^3} \\ &= \left(\frac{\sin\frac{\lambda a}{2}}{\lambda} - A_2\frac{\cos\frac{\lambda a}{2}}{\lambda^2}\right)\left(\cos\frac{\lambda a}{2} + A_1\frac{\sin\frac{\lambda a}{2}}{\lambda}\right) \end{aligned}$$

and $\Psi_1 \in \mathcal{L}^a$.

Let us consider circles $C_k(\rho)$ of radii $\frac{\rho}{k}$ centered at $\frac{2\pi k}{a} + \frac{A_2}{\pi k}$. If $\chi \in [0, 2\pi)$, then

$$\begin{aligned} \sin\left(\pi k + \frac{aA_2}{2\pi k} + \frac{a\rho}{2k}e^{i\chi}\right) &= (-1)^{k-1}\left(\frac{aA_2}{2\pi k} + \frac{\rho a}{2k}e^{i\chi}\right) + O\left(\frac{1}{k^3}\right), \\ \cos\left(\pi k + \frac{aA_2}{2\pi k} + \frac{a\rho}{2k}e^{i\chi}\right) &= (-1)^{k-1} + O\left(\frac{1}{k^2}\right), \\ \Phi_0\left(\frac{2\pi k}{a} + \frac{A_2}{\pi k} + \frac{\rho}{k}e^{i\chi}\right) &= \frac{\rho a^2}{2\pi k^2}e^{i\chi} + O\left(\frac{1}{k^3}\right). \end{aligned}$$

Therefore, for any $\epsilon \in (0, \rho a^2)$ and any $\chi \in [0, 2\pi)$ there exists $k_\epsilon \in \mathbb{N}$ such that for $k > k_\epsilon$

$$\left|\Phi_0\left(\frac{2\pi k}{a} + \frac{A_2}{\pi k} + \frac{\rho}{k}e^{i\chi}\right)\right| > \frac{\rho a^2 - \epsilon}{2\pi k^2}.$$

By Lemma 1.4.3 of [12] or Lemma 12.2.1 of [14],

$$\psi_1\left(\frac{2\pi k}{a} + \frac{A_2}{\pi k} + \frac{\rho}{k}e^{i\chi}\right) \underset{k\to\infty}{\to} 0$$

uniformly with respect to $\chi \in [0, 2\pi)$. Thus, we conclude that for k large enough

$$\left|\left(\frac{2\pi k}{a} + \frac{A_2}{\pi k} + \frac{\rho}{k}e^{i\chi}\right)^{-2} \psi_1\left(\frac{2\pi k}{a} + \frac{A_2}{\pi k} + \frac{\rho}{k}e^{i\chi}\right)\right|$$

$$< \frac{\rho a^2 - \epsilon}{2\pi k^2} < \left|\Phi_0\left(\frac{2\pi k}{a} + \frac{A_2}{\pi k} + \frac{\rho}{k}e^{i\chi}\right)\right|.$$

It is clear that the set of zeros of Φ_0 consists of two subsequences and one of these subsequences due to its asymptotics has exactly one element in each circle $C_k(\rho)$ for $k > k_\epsilon$. Hence by Rouche's theorem we conclude that for k large enough each such circle contains exactly one zero of $\Phi(\lambda)$. Since ρ can be chosen arbitrary small we conclude that there is a subsequence of zeros of $\Phi(\lambda)$ of the form

$$\zeta_k^{(2)} = \frac{2\pi k}{a} + \frac{A_2}{\pi k} + \frac{\Delta_k}{k}, \tag{5.3}$$

where $\Delta_k = o(1)$. Substituting (5.3) into $\Phi(\zeta_k^{(2)}) = 0$ and using (5.1) we obtain (5.2). In the same way we obtain the asymptotics for $\{\zeta_k^{(1)}\}_{-\infty,k\neq0}^{\infty}$. □

Theorem 5.3. *Let three sequences* $\{\nu_k^{(1)}\}_{-\infty,k\neq0}^{\infty}$ $(\nu_{-k}^{(1)} = -\nu_k^{(1)})$, $\{\mu_k^{(2)}\}_{-\infty,k\neq0}^{\infty}$ $(\mu_{-k}^{(2)} = -\mu_k^{(2)})$ *and* $\{\lambda_k\}_{-\infty,k\neq0}^{\infty}$ $(\lambda_{-k} = -\lambda_k)$ *satisfy the following conditions:*

1. $\{\nu_k^{(1)}\}_{-\infty,k\neq0}^{\infty}$ *satisfy* (4.3) *with* $j = 1$, $\{\mu_k^{(2)}\}_{-\infty,k\neq0}^{\infty}$ *satisfy* (4.2) *with* $j = 2$ *and* $\{\lambda_k\}_{-\infty,k\neq0}^{\infty}$ *satisfy* (4.1), *where* $A_0 = A_1 + A_2$,
2. $$-\infty < (\lambda_1)^2 < (\zeta_1)^2 = (\mu_1^{(2)})^2 < (\lambda_2)^2 < (\zeta_2)^2 < \dots,$$

   where

   $$\{\zeta_k\}_{-\infty,k\neq0}^{\infty} := \{\nu_k^{(1)}\}_{-\infty,k\neq0}^{\infty} \cup \{\mu_k^{(2)}\}_{-\infty,k\neq0}^{\infty}.$$

Then there exists a unique pair of real valued functions $q_j(x) \in L_2(0, a/2)$ $(j = 1, 2)$ *which generate problem* (1.6)–(1.8) *with* $j = 1$ *and the spectrum* $\{\nu_k^{(1)}\}_{-\infty,k\neq0}^{\infty}$, *problem* (1.9)–(1.11) *with* $j = 2$ *and the spectrum* $\{\mu_k^{(2)}\}_{-\infty,k\neq0}^{\infty}$ *and problem* (1.1)–(1.5) *with the spectrum* $\{\lambda_k\}_{-\infty,k\neq0}^{\infty}$.

Proof. Let us construct

$$P(\lambda) := a \prod_{k=1}^{\infty} \left(\frac{a}{\pi k}\right)^2 (\lambda_k^2 - \lambda^2),$$
$$\tilde{\phi}_1(\lambda) := \frac{a}{2} \prod_{k=1}^{\infty} \left(\frac{a}{2\pi k}\right)^2 ((\nu_k^{(1)})^2 - \lambda^2),$$
$$\phi_2(\lambda) := \prod_{k=1}^{\infty} \left(\frac{a}{\pi(2k-1)}\right)^2 ((\mu_k^{(2)})^2 - \lambda^2),$$

We consider the function

$$\Phi(\lambda) := P(\lambda) - \tilde{\phi}_1(\lambda)\phi_2(\lambda). \tag{5.4}$$

Since (see, e.g. [12, Lemma 3.4.2], [14, Lemma 12.3.3])

$$P(\lambda) = \frac{\sin \lambda a}{\lambda} - \frac{A_0 \cos \lambda a}{\lambda^2} + \frac{\tau(\lambda)}{\lambda^2},$$

where τ belongs to $\mathcal{L}^a$, and

$$\phi_2(\lambda) = \cos\frac{\lambda a}{2} + A_2 \frac{\sin\frac{\lambda a}{2}}{\lambda} + \frac{\tau_2(\lambda)}{\lambda},$$

$$\tilde{\phi}_1(\lambda) = \frac{\sin\frac{\lambda a}{2}}{\lambda} - A_1 \frac{\cos\frac{\lambda a}{2}}{\lambda^2} + \frac{\tilde{\tau}_1(\lambda)}{\lambda^2}$$

with $\tau_2 \in \mathcal{L}^{a/2}$, $\tilde{\tau}_1 \in \mathcal{L}^{a/2}$, we conclude that $\Phi(\lambda)$ given by (5.4) can be represented as in (5.1). Therefore, Φ satisfies the conditions of Lemma 5.2 and the set of zeros of $\Phi(\lambda)$ consists of two subsequences which we denote by $\{\nu_k^{(2)}\}_{-\infty,k\neq 0}^{\infty}$ and $\{\mu_k^{(1)}\}_{-\infty,k\neq 0}^{\infty}$ which behave asymptotically as follows:

$$\nu_k^{(2)} = \frac{2\pi k}{a} + \frac{A_2}{\pi k} + \frac{\gamma_k}{k},$$
$$\mu_k^{(1)} = \frac{\pi(2k-1)}{a} + \frac{A_1}{\pi k} + \frac{\tilde{\gamma}_k}{k},$$

where $\{\gamma_k\}_{-\infty,k\neq 0}^{\infty} \in l_2$ and $\{\tilde{\gamma}_k\}_{-\infty,k\neq 0}^{\infty} \in l_2$.

Let us notice that $\Phi(\lambda) > 0$ for $\lambda^2 \to -\infty$. Since $\Phi(\lambda_k) = -\tilde{\phi}_1(\lambda_k)\phi_2(\lambda_k)$, condition 2 implies

$$\Phi(\lambda_k)(-1)^k > 0.$$

Since $\Phi(\zeta_k) = P(\zeta_k)$, condition 2 implies

$$\Phi(\zeta_k)(-1)^k = P(\zeta_k)(-1)^k > 0.$$

Taking into account the asymptotics (above) we conclude that each interval $(-\infty, \lambda_1^2)$, (ζ_1^2, λ_2^2), (ζ_2^2, λ_3^2),..., contains exactly one zero of $\Phi(\sqrt{z})$. Now we denote the zero of

$\Phi(\sqrt{z})$ lying in $(-\infty,\lambda_1^2)$ by $(\mu_1^{(2)})^2$. We denote the zero of $\Phi(\sqrt{z})$ lying in $(\zeta_k^2,\lambda_{k+1}^2)$ by $(\nu_r^{(2)})^2$ if $\zeta_k=\mu_r^{(2)}$ and by $(\mu_{r+1}^{(1)})^2$ if $\zeta_k=\nu_r^{(1)}$. Therefore, $\{(\nu_k^{(2)})^2\}_{k=1}^{\infty}$ interlace with $\{(\mu_k^{(2)})^2\}_{k=1}^{\infty}$:

$$-\infty<(\mu_1^{(2)})^2<(\nu_1^{(2)})^2<(\mu_2^{(2)})^2<(\nu_2^{(2)})^2<\dots$$

Thus the sets $\{\nu_k^{(2)}\}_{-\infty,k\neq 0}^{\infty}$ and $\{\mu_k^{(2)}\}_{-\infty,k\neq 0}^{\infty}$ satisfy the conditions of Theorem 3.4.1 in [12] with $a=\pi$ (see also Theorem 12.6.2 in [14]) and there exists a unique real-valued function $q_2(x)\in L_2(0,\frac{a}{2})$ which generates the Dirichlet-Dirichlet and the Dirichlet-Neumann problems on $[0,\frac{a}{2}]$ with the spectra $\{\nu_k^{(2)}\}_{-\infty,k\neq 0}^{\infty}$ and $\{\mu_k^{(2)}\}_{-\infty,k\neq 0}^{\infty}$, respectively.

We can find $q_2(x)$ via procedure ([12, Theorem 3.4.1, p. 248] or [13, Theorem 12.6.2]) described below. Without loss of generality let us assume that $\mu_1^2>0$, otherwise we apply a shift. Using the functions $\phi_2(\lambda)$ and

$$\tilde{\phi}_2(\lambda):=\frac{a}{2}\prod_{k=1}^{\infty}\left(\frac{a}{2\pi k}\right)^2((\nu_k^{(2)})^2-\lambda^2),$$

we construct

$$e(\lambda)=(\phi_2(\lambda)+i\lambda\tilde{\phi}_2(\lambda))e^{-i\lambda\frac{a}{2}}$$

which is the so-called Jost-function of the corresponding prolonged Sturm-Liouville problem on the semi-axis:

$$-y''+q(x)y=\lambda^2 y,\quad x\in[0,\infty),$$

$$y(0)=0$$

with

$$q(x)=\begin{cases} lq_2(x) & \text{for } x\in[0,a/2],\\ 0 & \text{for } x\in(a/2,\infty).\end{cases}$$

Then we construct the S-function of the problem on the semi-axis:

$$S(\lambda)=\frac{e(\lambda)}{e(-\lambda)}$$

and the function

$$F(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}(1-S(\lambda))e^{i\lambda x}dx,$$

Solving the Marchenko equation

$$K_2(x,t)+F(x+t)+\int_{x}^{\infty}K_2(x,s)F(s+t)dp=0$$

we find $K_2(x,t)$ and the potential

$$q_2(x) = 2\frac{dK_2(x,x)}{dx},$$

which is a real-valued function and belongs to $L_2(0,a/2)$. This potential generates the Dirichlet-Neumann problem with the characteristic function $s'_2(\lambda,a/2) \equiv \phi_2(\lambda)$ and the spectrum $\{\mu_k^{(2)}\}_{-\infty,k\neq 0}^{\infty}$ and Dirichlet-Dirichlet problem with the characteristic function $s_2(\lambda,a/2) \equiv \tilde{\phi}_2(\lambda)$.

In the same way we construct $q_1(x)$ using the sequences $\{\mu_k^{(1)}\}_{-\infty,k\neq 0}^{\infty}$ and $\{\nu_k^{(1)}\}_{-\infty,k\neq 0}^{\infty}$. It is clear that the obtained $q_1(x)$ generates the Dirichlet-Neumann problem with the characteristic function

$$s'_1(\lambda,a/2) \equiv \phi_1(\lambda) =: \prod_{k=1}^{\infty}\left(\frac{a}{\pi(2k-1)}\right)^2((\mu_k^{(1)})^2-\lambda^2),$$

and the Dirichlet-Dirichlet problem with the characteristic function $s_1(\lambda,a/2) \equiv \tilde{\phi}_1(\lambda)$.

Now if we solve problem (1.1)–(1.5) with obtained q_1 and q_2, we find the characteristic function

$$\phi(\lambda) =: s'_1(\lambda,a/2)s_2(\lambda,a/2)+s'_2(\lambda,a/2)s_1(\lambda,a/2) = \phi_1(\lambda)\tilde{\phi}_2(\lambda)+\phi_2(\lambda)\tilde{\phi}_1(\lambda) = P(\lambda)$$

with the set of zeros $\{\lambda_k\}_{-\infty,k\neq 0}^{\infty}$. □

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