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A TWO CONES SUPPORT THEOREM

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Abstract. We show that if the Radon transform of a distribution f vanishes outside of an acute cone C_0, the support of the distribution is contained in the union of C_0 and another acute cone C_1, the cones are in a suitable position, and f vanishes distributionally in the direction of the axis of C_1, then actually $\operatorname{supp} f \subset C_0$. We show by examples that this result is sharp.

Keywords: Radon transforms, support theorems, distributions.

Mathematics Subject Classification: 44A12, 46F10.

1. INTRODUCTION

One of the most studied aspects of Radon transforms are the support theorems, from the seminal results of Helgason [8] and Ludwig [12], to recent studies [7], and many studies in between, as described in [17]. The support theorems are very important in integral geometry [9]. In most cases the question is whether if the Radon transform of a distribution vanishes outside of a compact convex set K, and some extra conditions are satisfied, then the distribution itself has support in K.

The aim of this article is to consider whether if the Radon transform of a distribution vanishes outside a cone then the support should be contained in the cone. In a way this study continues the work of Boman [2] and Boman and Lindskog [3], where cones appear in the extra conditions for compact convex support results. Our main result, given in Section 4, is *a two cone support theorem* that says that if the Radon transform of a distribution f vanishes outside of an acute cone C_0, the support of the distribution is contained in the union of the two suitable located acute cones, $C_0 \cup C_1$, and it vanishes distributionally in the direction of the axis of C_1, then actually $\operatorname{supp} f \subset C_0$.

It is interesting to observe the special role played by acute cones in the support theorems of [2], of [3], and the one in this article. There seems to be a geometric reason for this, since, in a way the acute cones are a natural geometric generalization of the convex compact sets. Moreover, if we add a point at infinity and consider the one point compatification of $\mathbb{R}^n$ then near the infinity point the acute cones, with the extra point added, resemble the convex compact sets near an ordinary point.

The article is completed in Section 5, where we give several examples to show that our result is sharp in many ways. Preliminary questions about cones and distributions are considered in Sections 2 and 3.

2. NOTATION

In this note a *cone* is a subset of $\mathbb{R}^n$ of the following form.

Definition 2.1. Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, with $|\mathbf{w}| = 1$, and let $\alpha \geq 0$. The cone with vertex at $\mathbf{v}$, direction $\mathbf{w}$, and angle α is the set

$$C_{\mathbf{v};\mathbf{w};\alpha} = \{\mathbf{x} \in \mathbb{R}^n : (\mathbf{x} - \mathbf{v}) \cdot \mathbf{w} \geq \cos\alpha \ |\mathbf{x} - \mathbf{v}|\} . \tag{2.1}$$

We shall mostly consider *acute* non-degenerate cones, that is, the ones where $0 < \alpha < \pi/2$; notice that if $\alpha = 0$ the cone reduces to the half line formed by those points of the form $\mathbf{v}+t\mathbf{w}$, $t \geq 0$, while if $\alpha = \pi/2$ then the cone becomes a half space.

The following definition will also be useful.

Definition 2.2. Let $C_0 = C_{\mathbf{v}_0;\mathbf{w}_0;\alpha_0}$ and $C_1 = C_{\mathbf{v}_1;\mathbf{w}_1;\alpha_1}$ be two cones. We shall say C_0 and C_1 are in position S if $C_0 \setminus \{\mathbf{v}_0\} \subset \operatorname{Int}\left(C_{\mathbf{v}_0;-\mathbf{w}_1;\pi/2}\right)$.

The ensuing simple result would be needed in our analysis.

Lemma 2.3. *Let $C_0 = C_{\mathbf{v}_0;\mathbf{w}_0;\alpha_0}$ be an acute cone and let K be a compact set. Let $\mathbf{x} \in K \setminus C_0$. Then there exists a cone $C_1 = C_{\mathbf{v}_1;\mathbf{w}_1;\alpha_1}$ such that C_0 and C_1 are in position S, $C_0 \cup K \subset C_0 \cup C_1$, and such that*

$$\mathbf{w}_1 \cdot (\mathbf{x} - \mathbf{v}_1) > \mathbf{w}_1 \cdot (\mathbf{y} - \mathbf{v}_1) \quad \textit{for all} \quad \mathbf{y} \in C_0. \tag{2.2}$$

Proof. We can separate the point $\mathbf{x}$ and the convex set C_0 by a hyperplane [19, Chp. 18], that is, there exists $\mathbf{w}_1 \in \mathbb{R}^n$, which we may take with $|\mathbf{w}_1| = 1$, such that $\mathbf{w}_1 \cdot (\mathbf{x} - \mathbf{v}_0) > 0$, while $\mathbf{w}_1 \cdot (\mathbf{y} - \mathbf{v}_0) < 0$ for all $\mathbf{y} \in C_0 \setminus \{\mathbf{v}_0\}$. Notice now that (2.2) will be satisfied for any $\mathbf{v}_1$; observe also that C_0 and C_1 are in position S. We just then take $\mathbf{v}_1 = \mathbf{x} - c\mathbf{w}_1$, where $c > 0$, in such a way that for some angle α_1 the acute cone $C_1 = C_{\mathbf{v}_1;\mathbf{w}_1;\alpha_1}$ satisfies $C_0 \cup K \subset C_0 \cup C_1$, which is possible since K is compact. □

3. SEVERAL FACTS ABOUT DISTRIBUTIONS

Let $f \in \mathcal{D}'(\mathbb{R}^n)$ be a distribution. In general one cannot restrict f to manifolds of smaller dimension, and, in particular, one cannot restrict it to a hyperplane. However, it is possible to employ the ensuing procedure to consider the restriction to certain *families* of hyperplanes.

Let us consider $\mathbb{R}^n$ as $\mathbb{R}^{n-1} \times \mathbb{R}$, so that the elements of $\mathbb{R}^n$ are written as $(\mathbf{y}, x) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Let $\phi \in \mathcal{D}\left(\mathbb{R}^{n-1}\right)$; then the evaluation $\langle f(\mathbf{y}, x), \phi(\mathbf{y})\rangle_{\mathbf{y}}$ makes sense as a distribution of the variable $x \in \mathbb{R}$, $g(x) = \langle f(\mathbf{y}, x), \phi(\mathbf{y})\rangle_{\mathbf{y}}$, given as

$$\langle g(x), \varphi(x)\rangle_x = \langle f(\mathbf{y}, x), \phi(\mathbf{y}) \varphi(x)\rangle_{(\mathbf{y},x)}, \quad \varphi \in \mathcal{D}(\mathbb{R}). \tag{3.1}$$

Sometimes the distribution g will have a value in the sense of Łojasiewicz [13, 14] at a point $x = x_0$ and thus the distribution f can be restricted to the hyperplane $H_{x_0} : (\mathbf{y}, x_0), \mathbf{y} \in \mathbb{R}^{n-1}$, and this is related to the wave front set [10] of f at H_{x_0}. However we want to emphasize that, even if the value $g(x_0)$ does not exist for any x_0, the evaluation $\langle f(\mathbf{y}, x), \phi(\mathbf{y})\rangle_{\mathbf{y}}$ is *always defined distributionally.*

Naturally, if f belongs to a smaller space of distributions, say if $f \in \mathcal{S}'(\mathbb{R}^n)$, then $\langle f(\mathbf{y}, x), \phi(\mathbf{y})\rangle_{\mathbf{y}}$ will be defined not only for $\phi \in \mathcal{D}\left(\mathbb{R}^{n-1}\right)$, but for $\phi \in \mathcal{S}\left(\mathbb{R}^{n-1}\right)$, and, furthermore, the evaluation belongs to $\mathcal{S}'(\mathbb{R})$. Partial distributional evaluations involving linear manifolds of other dimensions can be handled similarly, as, for example, evaluations of the type $\langle f(\mathbf{y}, x), \varphi(x)\rangle_x$.

A particularly interesting situation arises when $\operatorname{supp} f$ is contained in an acute cone.

Lemma 3.1. *If $f \in \mathcal{D}'(\mathbb{R}^n)$ satisfies $\operatorname{supp} f \subset C_{\mathbf{0};\mathbf{w};\alpha}$, where $\mathbf{w} = (\mathbf{0}, 1)$ and where $0 < \alpha < \pi/2$, then one can perform the evaluation $\langle f(\mathbf{y}, x), \phi(\mathbf{y})\rangle_{\mathbf{y}}$ distributionally for any $\phi \in \mathcal{E}\left(\mathbb{R}^{n-1}\right)$, that is, for* any *smooth function ϕ defined in $\mathbb{R}^{n-1}$.*

Proof. This happens because in this case for each $\varphi \in \mathcal{D}(\mathbb{R})$ the distribution $\langle f(\mathbf{y}, x), \varphi(x)\rangle_x$ has compact support. □

The next thing we would like to discuss has to do with the moments $\mu_{\mathbf{k}} = \left\langle f(\mathbf{x}), \mathbf{x}^{\mathbf{k}}\right\rangle$ of a distribution $f \in \mathcal{D}'(\mathbb{R}^n)$, where $\mathbf{k} \in \mathbb{N}^n$, and where $\mathbf{x}^{\mathbf{k}} = x_1^{k_1} \cdots x_n^{k_n}$. Naturally the moments of a general distribution of $\mathcal{D}'(\mathbb{R}^n)$ do not need to exist, but they will if f has an appropiate decay at infinity, say if f has compact support, or, more generally, if $f \in \mathcal{K}'(\mathbb{R}^n)$ [1)]. Suppose now that all the moments of a distribution f vanish; is it true that f must also vanish as well? In general the answer is *no,* since it is easy to find a function $\rho \in \mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \rho(\mathbf{x}) \mathbf{x}^{\mathbf{k}} d\mathbf{x} = 0$ for all $\mathbf{k} \in \mathbb{N}^n$: we just need to ask that $\mathbf{0}$ does not belong to the support of the Fourier transform $\widehat{\rho}$ [2)]. On the other hand, if $\operatorname{supp} f$ is a *compact* set, then $\mu_{\mathbf{k}} = 0$ for all $\mathbf{k} \in \mathbb{N}^n$ implies that $f = 0$, since the polynomials are dense in $\mathcal{E}(\mathbb{R}^n)$. The following result on vanishing moments would be useful for our analysis.

Lemma 3.2. *Let $f \in \mathcal{D}'\left(\mathbb{R}^{n-1} \times \mathbb{R}\right)$ with $\operatorname{supp} f \subset C_{\mathbf{0};\mathbf{w};\alpha}$, where $\mathbf{w} = (\mathbf{0}, 1)$ and where $0 < \alpha < \pi/2$. Suppose the distributional evaluation $\langle f(\mathbf{y}, x), \mathbf{y}^{\mathbf{m}}\rangle_{\mathbf{y}}$ vanishes for all $\mathbf{m} \in \mathbb{N}^{n-1}$ and for $a < x < b$. Then $f = 0$ in $\mathbb{R}^{n-1} \times (a, b)$, that is, $f(\mathbf{y}, x) = 0$ for $a < x < b$.*

[1)] The space $\mathcal{K}'(\mathbb{R}^n)$ plays an important role in the asymptotic analysis of distributions [6].

[2)] Actually [4] given any sequence $\{\mu_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}^n}$ there exists a function $\lambda \in \mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \lambda(\mathbf{x}) \mathbf{x}^{\mathbf{k}} d\mathbf{x} = \mu_{\mathbf{k}}$ for all $\mathbf{k} \in \mathbb{N}^n$, but in general no solution exists with $\operatorname{supp} \lambda$ compact.

Proof. Indeed, if $\varphi \in \mathcal{D}(\mathbb{R})$ is a test function such that $\operatorname{supp}\varphi \subset (a,b)$ then $\langle f(\mathbf{y},x), \varphi(x)\rangle_x$ has compact support and vanishing moments, and thus $\langle f(\mathbf{y},x), \varphi(x)\rangle_x = 0$. Notice that in this case $\langle f(\mathbf{y},x), \mathbf{y}^{\mathbf{m}}\rangle_{\mathbf{y}}$ is a well defined distribution of x for any $\mathbf{m}$ because of the Lemma 3.1. □

We shall denote by Rf the Radon transform of a function f, its integral over hyperplanes. The hyperplanes can be parametrized by the pairs $(\theta, t) \in \mathbb{S}^{n-1} \times \mathbb{R}$, where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$, and where $(\theta,t) \to H_{(\theta,t)}$, $H_{(\theta,t)} = \{\mathbf{x} : \mathbf{x}\cdot\theta = t\}$. If f is a locally integrable function, this means that

$$(Rf)(\theta,t) = \int_{H_{(\theta,t)}} f(\mathbf{x})\, \mathrm{d}\mathbf{x}, \tag{3.2}$$

where d$\mathbf{x}$ is the measure on the hyperplane $H_{(\theta,t)}$; naturally one needs growth restrictions at infinity for those integrals to exist. When f is a distribution its Radon transform is defined by duality as explained in [15] and [16, Chpt. 10][3)], but it is not possible to define the Radon transform as an operator that sends all distributions of $\mathcal{S}'(\mathbb{R}^n)$ to ordinary distributions, since $R(\mathcal{S}'(\mathbb{R}^n)) = (\mathcal{S}_t(\mathbb{R}^n))'$, and the test function space $\mathcal{S}_t(\mathbb{R}^n)$ does not contain the standard test function space $\mathcal{D}(\mathbb{R}^n)$ [15]. *Some* locally integrable functions will have Radon transforms that are also locally integrable functions and *some* distributions of $\mathcal{S}'(\mathbb{R}^n)$ will have Radon transforms that belong to $\mathcal{D}'(\mathbb{R}^n)$, but not all of them do. The Radon transform of a given f –whether an ordinary function or a distribution– will not exist, as a function or a distribution, for all pairs (θ,t), in general[4)].

We may also employ the idea of distributional evaluation along the family of hyperplanes obtained as t varies to understand the Radon transform of a distribution in the neighborhood of a given hyperplane; this is especially useful in certain *geometric* circumstances. In the particular case when $f \in \mathcal{D}'(\mathbb{R}^{n-1}\times\mathbb{R})$ with $\operatorname{supp} f \cap C_{\mathbf{0};\mathbf{w};\pi/2} \subset C_{\mathbf{0};\mathbf{w};\alpha}$, where $\mathbf{w} = (\mathbf{0},1)$ and where $0<\alpha<\pi/2$, then $(Rf)(\mathbf{w},t)$ will be defined for $t>0$ by distributional evaluation, as follows from Lemma 3.1 if $\phi(\mathbf{y}) = 1$ for all $\mathbf{y}$; actually in the case when f is a continuous function with $\operatorname{supp} f \cap C_{\mathbf{0};\mathbf{w};\pi/2} \subset C_{\mathbf{0};\mathbf{w};\alpha}$, then $(Rf)(\mathbf{w},t)$ will be given by a *convergent* integral in (3.2) if $t>0$. Similarly, if $\operatorname{supp} f \subset C_0 \cup C_1$, where C_1 is another acute cone such that C_0 and C_1 are in position S, then $(Rf)(\theta,t)$ will be defined for θ in a neighborhood of $\mathbf{w} = (\mathbf{0},1)$ and $t>0$.

4. THE TWO CONES SUPPORT THEOREM

Our main result is the following two cones support theorem.

3) Other *equivalent* definitions are also considered in [16].

4) The Radon transform of f will be defined for all (θ,t) for functions of rapid decay at infinity [17], or as a standard distribution in all of $\mathbb{S}^{n-1}\times\mathbb{R}$ for distributions of rapid decay [3], or if $f \in \mathcal{K}'(\mathbb{R}^n)$ [5].

Theorem 4.1. *Let* $f \in \mathcal{S}'(\mathbb{R}^n)$. *Suppose there exist two cones* $C_0 = C_{\mathbf{v}_0;\mathbf{w}_0;\alpha_0}$ *and* $C_1 = C_{\mathbf{v}_1;\mathbf{w}_1;\alpha_1}$ *in position S that satisfy the following conditions:*

a) *The Radon transform* Rf *vanishes outside* C_0;
b) $\operatorname{supp} f \subset C_0 \cup C_1$; *and*
c) *The distribution of* $\mathbf{y} \in H$ *given as* $g_b(\mathbf{y}) = f(\mathbf{y}+b\mathbf{w}_1)$, *where* H *is the hyperplane* $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \cdot \mathbf{w}_1 = 0\}$, *satisfies* $\lim_{b\to\infty} g_b = 0$ *in the space* $\mathcal{E}'(H)$ [5].

Then $\operatorname{supp} f \subset C_0$.

Proof. We shall first show that $\operatorname{supp} f \subset \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \cdot \mathbf{w}_1 \le \mathbf{v}_0 \cdot \mathbf{w}_1\}$, by using the following argument, that can be traced[6] to [18], and also presented in [3]. Indeed, we may suppose that $\mathbf{w}_1 = (\mathbf{0}, 1)$, $\mathbf{v}_1 = 0$, so that H is just $\mathbb{R}^{n-1}$, when we denote the elements of $\mathbb{R}^n$ as $(\mathbf{y}, x)$, $\mathbf{y} \in \mathbb{R}^{n-1}$, $x \in \mathbb{R}$. Consider now the function

$$G(\mathbf{a}, b) = \langle f(\mathbf{y}, \mathbf{a} \cdot \mathbf{y} + b), 1 \rangle_{\mathbf{y}}, \tag{4.1}$$

for $\mathbf{a} \in \mathbb{R}^{n-1}$ and $b \in \mathbb{R}$. Our hypotheses imply that there exists $\varepsilon > 0$ such that this is an evaluation in $\mathcal{E}'(\mathbb{R}^{n-1}) \times \mathcal{E}(\mathbb{R}^{n-1})$ if $|\mathbf{a}| < \varepsilon$ and $b > \mathbf{v}_0 \cdot \mathbf{w}_1$, and in that case

$$G(\mathbf{a}, b) = 0, \tag{4.2}$$

since $G(\mathbf{a}, b)$ is actually equal to a constant, $c_{\mathbf{a}} = \left(1 + |\mathbf{a}|^2\right)^{-1/2}$, times the Radon transform $Rf(\theta, c_{\mathbf{a}} b)$, where $\theta = c_{\mathbf{a}}(-\mathbf{a}, 1)$. Thus if $b > \mathbf{v}_0 \cdot \mathbf{w}_1$, $\mathbf{m} \in \mathbb{N}^{n-1}$, $|\mathbf{m}| = M$,

$$\frac{\partial^M}{\partial b^M}\left(\langle f(\mathbf{y}, b), \mathbf{y}^{\mathbf{m}} \rangle_{\mathbf{y}}\right) = \left\langle \frac{\partial^M f}{\partial x^M}(\mathbf{y}, b), \mathbf{y}^{\mathbf{m}} \right\rangle_{\mathbf{y}} = \left. \frac{\partial^M G(\mathbf{a}, b)}{\partial a_1^{m_1} \cdots \partial a_{n-1}^{m_{n-1}}} \right|_{\mathbf{a}=\mathbf{0}} = 0, \tag{4.3}$$

so that $\langle f(\mathbf{y}, b), \mathbf{y}^{\mathbf{m}} \rangle_{\mathbf{y}}$ is a distribution of the variable b that is a polynomial of degree at most $M - 1$ in the interval $(\mathbf{v}_0 \cdot \mathbf{w}_1, \infty)$; but $\lim_{b\to\infty} \langle f(\mathbf{y}, b), \mathbf{y}^{\mathbf{m}} \rangle_{\mathbf{y}} = 0$, and hence it follows that this distribution vanishes in that interval, that is, $\langle f(\mathbf{y}, b), \mathbf{y}^{\mathbf{m}} \rangle_{\mathbf{y}} = 0$ for $b > \mathbf{v}_0 \cdot \mathbf{w}_1$. We can now apply the Lemma 3.2 to conclude that $f(\mathbf{y}, b) = 0$ for $b > \mathbf{v}_0 \cdot \mathbf{w}_1$. It follows that $\operatorname{supp} f \subset C_0 \cup K$, where K is compact.

Finally we employ the Lemma 2.3 and the first part of the proof to conclude that $\operatorname{supp} f \subset C_0$. □

5. EXAMPLES

In this section we shall give several examples to show the limitations of our results.

Let us start with the fact that extra conditions are needed in any support theorem. Indeed, it is known that there are harmonic functions defined in all of $\mathbb{R}^n$ whose Radon transforms vanish *everywhere:* [20] in dimension 2, [1] in arbitrary dimensions. These

[5] The relation $\lim_{b\to\infty} g_b = 0$ can be understood in either the strong topology or in the weak topology of $\mathcal{E}'(H)$, since the convergent sequences for both topologies are the same [10, 19]. In the weak sense it means that $\lim_{b\to\infty} \langle f(\mathbf{y}+b\mathbf{w}_1), \varphi(\mathbf{y}) \rangle = 0$ for all $\varphi \in \mathcal{E}(H)$.

[6] The method in those references applies *only* to functions with compact support.

examples do not belong to $\mathcal{S}'(\mathbb{R}^n)$, but examples of distributions of $\mathcal{S}'(\mathbb{R}^n)$ whose Radon transforms vanish outside of a given point are easy to construct; notice, for example, that if $n = 2$ the Radon transform of $f(z) = z^{-k}$, $k = 2, 3, \ldots$ vanishes on all lines that do not contain the origin. Our first example uses analytic functions, too, to construct a function that vanishes inside a cone C but whose Radon transform vanishes outside of this cone.

Example 5.1. Let C be an acute closed cone in $\mathbb{R}^2 \simeq \mathbb{C}$. Let f_0 be a bounded analytic function defined in $\mathbb{C} \setminus C$, continuous in $\overline{\mathbb{C}} \setminus C$, and with $\lim_{z\to\infty} z^2 f_0(z) = 0$. Extend f_0 to f defined in all $\mathbb{C}$ by putting $f(z) = 0$ if $z \in C$. Then the Radon transform of f vanishes outside of the cone C, but $\operatorname{supp} f \subset \mathbb{C} \setminus C$.

The next example shows why the cone needs to be *acute.*

Example 5.2. Let $\phi \in \mathcal{D}\left(\mathbb{R}^{n-1}\right)$ with $\int_{\mathbb{R}^{n-1}} \phi(\mathbf{y})\, \mathrm{d}\mathbf{y} = 0$, and let $\varphi \in \mathcal{D}(\mathbb{R})$ be such that $\operatorname{supp}\varphi \nsubseteq (-\infty, 0]$. Then $f(\mathbf{y}, x) = \phi(\mathbf{y})\varphi(x)$ satisfies the hypotheses of the Theorem 4.1 with $C_0 = C_{\mathbf{0};-\mathbf{w}_1;\pi/2}$, $\mathbf{w}_1 = (\mathbf{0}, 1)$, and appropiate $C_1 = C_{\mathbf{v}_1;\mathbf{w}_1;\alpha_1}$. However, of course, $\operatorname{supp} f \nsubseteq C_0$.

The next example shows that f has to vanish in the direction of the axis of the cone C_1.

Example 5.3. Let
$$S(x) = H\left(1 - x^2\right) \operatorname{sgn} x, \tag{5.1}$$
where H is the Heaviside function, so that $H\left(1 - x^2\right)$ is the characteristic function of $(-1, 1)$. Consider $\mathbb{R}^n$ as $\mathbb{R}^{n-1} \times \mathbb{R}$, and the function f defined in $\mathbb{R}^{n-1} \times \mathbb{R}$ as
$$f(y_1, \ldots, y_{n-1}, x) = S(y_1) \cdots S(y_{n-1}). \tag{5.2}$$
Then for any α, $0 < \alpha < \pi/2$, if $\mathbf{w}_1 = (\mathbf{0}, 1)$, $\mathbf{v}_1 = (\mathbf{0}, -v)$, with v large enough, and $C_0 = C_{\mathbf{0};-\mathbf{w}_1;\alpha}$, $C_1 = C_{\mathbf{v}_1;\mathbf{w}_1;\alpha}$, then the cones are in position S, the Radon transform Rf vanishes outside C_0, $\operatorname{supp} f \subset C_0 \cup C_1$, but, by construction, $\operatorname{supp} f \nsubseteq C_0$.

Observe that in the example 5.3 one can use other odd functions S, such as, for example, the limit case $S(x) = \delta'(x)$.

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