Opuscula Math. 36, no. 2 (2016), 145–152
http://dx.doi.org/10.7494/OpMath.2016.36.2.145 Opuscula Mathematica

BOUNDS ON THE INVERSE SIGNED TOTAL DOMINATION NUMBERS IN GRAPHS

M. Atapour, S. Norouzian, S.M. Sheikholeslami, and L. Volkmann

Communicated by Mariusz Meszka

Abstract. Let $G = (V, E)$ be a simple graph. A function $f : V \to \{-1, 1\}$ is called an inverse signed total dominating function if the sum of its function values over any open neighborhood is at most zero. The inverse signed total domination number of G, denoted by $\gamma^0_{st}(G)$, equals to the maximum weight of an inverse signed total dominating function of G. In this paper, we establish upper bounds on the inverse signed total domination number of graphs in terms of their order, size and maximum and minimum degrees.

Keywords: inverse signed total dominating function, inverse signed total domination number.

Mathematics Subject Classification: 05C69.

1. INTRODUCTION

In the whole paper, G is a simple graph without isolated vertices and with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). For every vertex $v \in V$, the open neighborhood $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and the *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \bigcup_{v\in S} N(v)$. The *minimum* and *maximum* degree of G are respectively denoted by $\delta(G) = \delta$ and $\Delta(G) = \Delta$. If $X \subseteq V(G)$, then $G[X]$ is the subgraph of G induced by X. For disjoint subsets X and Y of vertices of a graph G, we let $E(X, Y)$ denote the set of edges between X and Y. For a tree T, a leaf of T is a vertex of degree 1 and a support vertex is a vertex adjacent to a leaf. The set of leaves and the set of support vertices in T are denoted by $L(T)$ and $S(T)$, respectively. Consult [3] for terminology and notation which are not defined here.

For a real-valued function $f : V \to \mathbb{R}$ the *weight* of f is $\omega(f) = \sum_{v\in V} f(v)$, and for $S \subseteq V$ we define $f(S) = \sum_{v\in S} f(v)$, so $\omega(f) = f(V)$. For a vertex v in V, we denote $f(N(v))$ by $f[v]$. Let $f : V \to \{-1, 1\}$ be a function which assigns to each vertex of G an element of the set $\{-1, 1\}$. Zelinka [4] defined the function f to be a *signed total dominating function* (STDF) of G if $f[v] \geq 1$ for every $v \in V$. The signed total domination number, denoted by $\gamma_{st}(G)$, of G is the minimum weight of a STDF on G. A signed total dominating function of weight $\gamma_{st}(G)$ is called a $\gamma_{st}(G)$-*function*.

A function $f : V \to \{-1, 1\}$ is said to be an *inverse signed total dominating function* (ISTDF) of G if $f[v] \leq 0$ for every $v \in V$. The *inverse signed total domination number* of G, denoted by $\gamma^0_{st}(G)$, is the maximum weight of an inverse signed total dominating function of G. An inverse signed total dominating function of weight $\gamma^0_{st}(G)$ is called a $\gamma^0_{st}(G)$-*function*. Huang *et al.* [2] introduced the concept of an inverse signed total domination number and obtained the exact values of this parameter for paths, cycles, complete graphs, stars and wheels. In this paper, we establish upper bounds on the inverse signed total domination number of graphs in terms of their order, size and maximum and minimum degree.

Throughout this paper, if f is a STDF (respectively, ISTDF) of G, then we let P and M denote the sets of those vertices in G which are assigned +1 and -1 under f, respectively, and we let $|P| = p$ and $|M| = m$. Thus, $w(f) = |P| - |M| = n - 2|M|$.

For any $\gamma_{st}(G)$-function f of G, we can define an ISTDF on G by assigning $+1$ to every vertex in M and -1 to every vertex in P that implies

$$\gamma^0_{st}(G) \geq -\gamma_{st}(G). \tag{1.1}$$

We make use of the following results in this paper.

Theorem 1.1 ([1])**.** *If G is a graph of order n with minimum degree $\delta \geq 2$ and maximum degree Δ, then*

$$\gamma_{st}(G) \geq \left(\frac{\lceil \frac{\delta-1}{2} \rceil - \lfloor \frac{\Delta-1}{2} \rfloor + 1}{\lceil \frac{\Delta-1}{2} \rceil + \lfloor \frac{\delta-1}{2} \rfloor + 1} \right) n,$$

and this bound is sharp.

Observation 1.2. *Let f be an ISTDF of G and let $v \in V(G)$. If $\deg(v)$ is even, then $f[v] \leq 0$, while if $\deg(v)$ is odd, then $f[v] \leq -1$.*

Proposition 1.3. *For any graph G of order n, $\gamma^0_{st}(G) \equiv n \pmod 2$.*

Proof. Let $f : V(G) \to \{-1, +1\}$ be a $\gamma^0_{st}(G)$- function. Since $\gamma^0_{st}(G) = \omega(f) = |P| - |M|$ and $n = |P| + |M|$, we have $\gamma^0_{st}(G) = n - 2|M|$ and so $\gamma^0_{st}(G) \equiv n \pmod 2$. □

Proposition 1.4. *Let G be a graph of order $n \geq 2$ without isolated vertices. Then $\gamma^0_{st}(G) = -n$ if and only if G is a 1-regular graph.*

Proof. If G is a 1-regular graph, then obviously, the function f that assigns -1 to every vertex of G is a $\gamma^0_{st}(G)$- function.

Conversely, let $\gamma^0_{st}(G) = -n$. We show that $\Delta(G) \leq 1$. Assume, to the contrary, that $\deg(v) \geq 2$ for some $v \in V(G)$. If v has a neighbour of degree 1, say u, then define $f : V(G) \to \{-1, +1\}$ by $f(u) = +1$ and $f(x) = -1$ otherwise. If all neighbours of v have degree at least two, then define $f : V(G) \to \{-1, +1\}$ by $f(v) = +1$ and $f(x) = -1$ otherwise. It is easy to see that in each case, f is an ISTDF of G implying that $\gamma^0_{st}(G) \geq 2 - n$ which is a contradiction. Hence, $\delta(G) = \Delta(G) = 1$ and so G is 1-regular. □

2. BOUNDS ON THE INVERSE SIGNED TOTAL DOMINATION NUMBERS

In this section, we present bounds on inverse signed total domination numbers of graphs in terms of their order, size, maximum and minimum degree.

Lemma 2.1. *If G is s graph with minimum degree δ and maximum degree Δ and f is a $\gamma_{st}^0(G)$-function, then*

$$|P|\left\lceil\frac{\delta}{2}\right\rceil \le |E(P,M)| \le |M|\left\lfloor\frac{\Delta}{2}\right\rfloor.$$

Proof. Let $v \in P$. The condition $f[v] \le 0$ leads to $2|N(v)\cap M| \ge \deg(v)$ and therefore $|N(v)\cap M| \ge \lceil\frac{\deg(v)}{2}\rceil \ge \lceil\frac{\delta}{2}\rceil$. We deduce that $|E(P,M)]| \ge |P|\lceil\frac{\delta}{2}\rceil$.

Now let $v \in M$. Since $f[v] \le 0$, we have $2|N(v)\cap P| \le \deg(v)$ and therefore $|N(v)\cap P| \le \lfloor\frac{\deg(v)}{2}\rfloor \le \lfloor\frac{\Delta}{2}\rfloor$. This leads to $|E(P,M)| \le |M|\lfloor\frac{\Delta}{2}\rfloor$, and the proof is complete. □

Theorem 2.2. *If G is a graph of order n with minimum degree $\delta \ge 1$ and maximum degree Δ, then*

$$\gamma_{st}^0(G) \le \left(\frac{\lfloor\frac{\Delta}{2}\rfloor - \lceil\frac{\delta}{2}\rceil}{\lfloor\frac{\Delta}{2}\rfloor + \lceil\frac{\delta}{2}\rceil}\right)n.$$

Proof. Lemma 2.1 implies that $|P|\lceil\frac{\delta}{2}\rceil \le |M|\lfloor\frac{\Delta}{2}\rfloor$. Using this inequality and $|P| = \frac{n+\gamma_{st}^0(G)}{2}$ and $|M| = \frac{n-\gamma_{st}^0(G)}{2}$, the desired bound is easy to verify. □

We show next that the bound given in Theorem 2.2 is sharp. For this purpose, we shall need the following two observations proved by Henning [1].

Observation 2.3. *If k and n are integers with $k < n$ and n is even, then we can construct a k-regular graph on n vertices.*

Observation 2.4. *Let k, m and p be integers satisfying $1 \le k \le mp$, $m|k$ and $p|k$. Then there exists a bipartite graph of size k with partite sets P and M such that $|P| = p$ and $|M| = m$, and each vertex in P has degree $\frac{k}{p}$ while each vertex in M has degree $\frac{k}{m}$.*

Theorem 2.5. *Let δ and Δ be integers with $2 \le \delta \le \Delta$. Then there exists a graph G such that*

$$\gamma_{st}^0(G) = \left(\frac{\lfloor\frac{\Delta}{2}\rfloor - \lceil\frac{\delta}{2}\rceil}{\lfloor\frac{\Delta}{2}\rfloor + \lceil\frac{\delta}{2}\rceil}\right)n,$$

where $\delta = \delta(G)$ and $\Delta = \Delta(G)$.

Proof. Let $x = \lfloor\frac{\Delta}{2}\rfloor$, $y = \lceil\frac{\delta}{2}\rceil$, $\lambda = 2\lceil\frac{\Delta}{\delta}\rceil$, $m = \lambda x$, $p = \lambda y$ and $k = \lambda xy$. Then, $\frac{k}{m} = y$ and $\frac{k}{p} = x$ and so $1 \le k \le pm$. By Observation 2.4, there exists a bipartite graph H of size k with partite sets P and M such that $|P| = p$ and $|M| = m$, and each vertex in P has degree $\lceil\frac{\delta}{2}\rceil$ while each vertex in M has degree $\lfloor\frac{\Delta}{2}\rfloor$. Furthermore, p is even and $p = \lambda y > \lfloor\frac{\delta}{2}\rfloor$. Hence, by Observation 2.3, we can construct a $\lfloor\frac{\delta}{2}\rfloor$-regular graph with vertex set P. Similarly, m is even and $m = \lambda x > \lceil \frac{\Delta}{2} \rceil$. Hence, by Observation 2.3, we can construct a $\lceil \frac{\Delta}{2} \rceil$-regular graph with vertex set M. Adding the edges of both these graphs to H produces a graph G in which every vertex of P has $\lceil \frac{\delta}{2} \rceil$ neighbours in M and $\lfloor \frac{\delta}{2} \rfloor$ neighbors in P, while every vertex of M has $\lceil \frac{\Delta}{2} \rceil$ neighbours in M and $\lfloor \frac{\Delta}{2} \rfloor$ neighbours in P. In particular, every vertex in P has degree δ and every vertex in M has degree Δ. Let $f : V(G) \longrightarrow \{-1, +1\}$ be a function that assigns 1 to all vertices in P and -1 to all vertices in M. By construction, f is an ISTDF. Hence, $\gamma^0_{st}(G) \geq w(f) = |P| - |M| = \lambda(x-y) = \lambda \left(\lfloor \frac{\Delta}{2} \rfloor - \lceil \frac{\delta}{2} \rceil\right)$. Since G has order $n = \lambda(x+y) = \lambda \left(\lfloor \frac{\Delta}{2} \rfloor + \lceil \frac{\delta}{2} \rceil\right)$, it follows from Theorem 2.2 that $\gamma^0_{st}(G) = \left(\frac{\lfloor \frac{\Delta}{2} \rfloor - \lceil \frac{\delta}{2} \rceil}{\lfloor \frac{\Delta}{2} \rfloor + \lceil \frac{\delta}{2} \rceil}\right) n$. □

Next we give a sharp upper bound on the inverse signed total domination number of a graph in terms of its order.

Theorem 2.6. *If G is a graph of order n, then $\gamma^0_{st}(G) \leq n - 2\sqrt{n}$ with equality if and only if G is obtained from a complete graph K_t with vertex set $\{v_1, v_2, \ldots, v_t\}$ by adding the set of vertices $\bigcup_{i=1}^t \{x_{i_1}, \ldots, x_{i_{t-1}}\}$ and edges $v_i x_{i_j}$ for each $1 \leq i \leq t$ and $1 \leq j \leq t-1$.*

Proof. Let f be a $\gamma^0_{st}(G)-$ function. Then,

$$\gamma^0_{st}(G) = \omega(f) = |P| - |M| = n - 2|M|.$$

Each vertex in P is adjacent to at least one vertex in M. Thus, by the pigeonhole principle, at least one vertex v of M is adjacent to at least $\frac{|P|}{|M|}$ vertices of P. It follows, therefore, that

$$\frac{n - |M|}{|M|} - (|M| - 1) \leq f[v] \leq 0,$$

and so $n - |M|^2 \leq 0$ which implies that $\gamma^0_{st}(G) = n - 2|M| \leq n - 2\sqrt{n}$.

Assume that G is obtained from a complete graph K_t with vertex set $\{v_1, v_2, \ldots, v_t\}$ by adding the set of vertices $\bigcup_{i=1}^t \{x_{i_1}, \ldots, x_{i_{t-1}}\}$ and edges $v_i x_{i_j}$ for each $1 \leq i \leq t$ and $1 \leq j \leq t-1$. Then G is a graph of order t^2. Define $f : V(G) \to \{-1, 1\}$ by $f(v) = -1$ if $v \in V(K_t)$ and $f(v) = 1$ otherwise. It is easy to see that f is an ISTDF of G which implies that

$$\gamma^0_{st}(G) \geq \omega(f) = t(t-1) - t = t^2 - 2t.$$

Therefore, $\gamma^0_{st}(G) = t^2 - 2t$.

Let now G be a graph of order n with $\gamma^0_{st}(G) = n - 2\sqrt{n}$ and let f be a $\gamma^0_{st}(G)-$ function. Then $|M| = \sqrt{n}$ and $|P| = n - \sqrt{n}$ and therefore $|P| = |M|^2 - |M|$. Each vertex in P is adjacent to at least one vertex in M. Thus,

$$|E(P, M)| \geq |P| = |M|^2 - |M|. \tag{2.1}$$

On the other hand, since $f[v] \leq 0$, for each $v \in V$, we have $|N(v) \cap M| \geq |N(v) \cap P|$ and so each vertex in M is adjacent to at most $|M| - 1$ vertices in P, which implies that

$$|E(P, M)| \leq |M|(|M| - 1). \tag{2.2}$$

By (2.1) and (2.2), we have $|E(P,M)| = |P| = |M|^2 - |M|$. Thus, each vertex in M is adjacent to exactly $|M| - 1$ vertices of P and each vertex of P is adjacent to exactly one vertex of M. Also, $G[M]$ is a complete graph and P is an independent set as desired. □

Now we give an upper bound on the inverse signed total domination number of a graph in terms of its order, size and minimum degree.

Theorem 2.7. *Let G be a graph of order n, size m and minimum degree $\delta \geq 1$. Then $\gamma^0_{st}(G) \leq \frac{4}{3}m - n$, and if $\delta \geq 2$, then $\gamma^0_{st}(G) \leq \frac{2m}{\delta} - n$.*

Proof. Let f be a $\gamma^0_{st}(G)-$ function. If $P = \emptyset$, then the result is true. Let $P \neq \emptyset$. Since $f[v] \leq 0$, we have $|N(v) \cap P| \leq |N(v) \cap M|$ for each $v \in V$. Therefore

$$\begin{aligned}2|E(G[P])| &= \sum_{v\in P} |N(v) \cap P| \leq \sum_{v \in P} |N(v) \cap M| = |E(P,M)| \\ &= \sum_{v\in M} |N(v) \cap P| \leq \sum_{v\in M} |N(v) \cap M| = 2|E(G[M])|\end{aligned}$$

and thus

$$2|E(G[P])| \leq |E(P,M)| \leq 2|E(G[M])|. \tag{2.3}$$

Each vertex in P is adjacent to at least one vertex in M. Thus, it follows from (2.3) that

$$2|E(G[M])| \geq |E(P,M)| \geq |P|.$$

Hence, we have

$$m \geq |E(G[M])| + |E(P,M)| \geq \frac{|P|}{2} + |P| = \frac{3|P|}{2} = \frac{3}{2}\left(\frac{n + \gamma^0_{st}(G)}{2}\right),$$

and this leads to the first bound immediately. In addition, (2.3) implies

$$m = |E(G[P])| + |E(P,M)| + |E(G[M])| \geq 2|E(G[P])| + |E(P,M)|.$$

Using this inequality and the identity

$$2|E(G[P])| = \sum_{v\in P} |N(v) \cap P| = \sum_{v\in P} \deg(v) - |E(P,M)|,$$

we obtain

$$m \geq \sum_{v \in P} \deg(v) \geq \delta|P| = \frac{\delta(n + \gamma^0_{st}(G))}{2}.$$

This leads to the second bound, and the proof is complete. □

Theorem 2.8. *For any tree T of order $n \geq 2$, $\gamma^0_{st}(T) \leq \frac{n-4}{3}$ with equality if and only if $n \equiv 1 \pmod 3$ and each vertex $v \in V(T) \setminus L(T)$ has even degree and is adjacent to $\frac{\deg(v)}{2}$ leaves.*

Proof. Let T be a tree of order $n \geq 2$. By Theorem 2.7 and the fact that each tree of order n has $n-1$ edges, we have $\gamma^0_{st}(T) \leq \frac{n-4}{3}$.

Let T be a tree of order $n \geq 2$, $n \equiv 1 \pmod 3$ such that each vertex $v \in V(T) \setminus L(T)$ has even degree and is adjacent to $\frac{\deg(v)}{2}$ leaves. Define $f : V(T) \to \{-1, 1\}$ by $f(v) = 1$ if $v \in L(T)$ and $f(v) = -1$ otherwise. It is easy to see that f is an ISTDF of T. Also, we have $|P| = |L(T)| = \frac{2n-2}{3}$ which implies that $\gamma^0_{st}(T) \geq \omega(f) = 2|P| - n = \frac{n-4}{3}$. Therefore, $\gamma^0_{st}(T) = \frac{n-4}{3}$.

Let now T be a tree of order $n \geq 2$ and $\gamma^0_{st}(T) = \frac{n-4}{3}$. Since $\gamma^0_{st}(T) = \frac{n-4}{3}$, we observe that $n \equiv 1 \pmod 3$, $|P| = \frac{2n-2}{3}$ and $|M| = \frac{n+2}{3}$. Each vertex in P is adjacent to at least one vertex in M. Thus, $|E(P, M)| \geq |P| = \frac{2n-2}{3}$. On the other hand, (2.3) implies $|E(P, M)| \leq 2|E(T[M])|$. Since $T[M]$ is simple and acyclic, we obtain $|E(T[M])| \leq |M| - 1 = \frac{n-1}{3}$. Therefore, we have $|E(P, M)| = \frac{2(n-1)}{3}$ which implies that $|N_T(v) \cap M| = 1$ for each $v \in P$ and $|N_T(v) \cap P| = |N_T(v) \cap M|$ for each $v \in M$ and so $|E(T[M])| = \frac{n-1}{3}$. Now we have

$$n - 1 = m = |E(P, M)| + |E(T[M])| + |E(T[P])| \geq \frac{2(n-1)}{3} + \frac{n-1}{3} = n - 1$$

which implies that $|E(T[P])| = 0$. Thus, P is an independent set and $T[M]$ is a tree. So we have $P \subseteq L(T)$. Since $|N_T(v) \cap P| = |N_T(v) \cap M|$ for each $v \in M$, it follows that $\deg_T(v)$ is even and so $P = L(T)$ and each vertex $v \in V(T) \setminus L(T)$ has even degree and is adjacent to $\frac{\deg(v)}{2}$ leaves. This completes the proof. □

The next result gives an upper bound on the inverse signed total domination number of a graph in terms of its degree sequence.

Theorem 2.9. *Let G be a graph of order n, with degree sequence $d_1 \leq d_2 \leq \ldots \leq d_n$. If G has n_{even} vertices of even degree, and if k is the greatest integer for which*

$$\sum_{i=1}^{k} d_i - \sum_{i=k+1}^{n} d_i \leq n_{even} - n,$$

then $\gamma^0_{st}(G) \leq 2k - n$ and this bound is sharp.

Proof. Let f be a $\gamma^0_{st}(G)-$ function and $p = |P|$. By Observation 1.2, we have

$$n_{\text{even}} - n \geq \sum_{v \in V} \sum_{u \in N(v)} f(u) = \sum_{v \in V} \deg(v) f(v)$$

$$= \sum_{v \in P} \deg(v) - \sum_{v \in M} \deg(v) \geq \sum_{i=1}^{p} d_i - \sum_{i=p+1}^{n} d_i.$$

By our choice of k, it follows that $p \leq k$ and so $\gamma^0_{st}(G) = 2p - n \leq 2k - n$.

In order to show that the bound is sharp, let G be obtained from the path $P_k = v_1 v_2 \ldots v_k$, $k \geq 3$, by adding the set $\{x_i, y_i | 2 \leq i \leq k-1\}$ of vertices and the set

$\{v_ix_i, v_iy_i | 2 \leq i \leq k-1\}$ of edges. Since the degree sequence of G is $\underbrace{1,\dots,1}_{2k-2}, \underbrace{4\dots,4}_{k-2}$ and

$$2(k-2) - (4k-8+2) = -2k+2 = n_{even} - n,$$

it follows that $2(k-2)$ is the greatest positive integer such that

$$\sum_{i=1}^{2(k-2)} d_i - \sum_{i=2k-3}^{n} d_i \leq n_{even} - n$$

and therefore

$$\gamma_{st}^0(G) \leq 2(2(k-2)) - n = k-4.$$

Define $f : V(G) \to \{-1, 1\}$ by $f(v) = -1$ if $v \in V(P_k)$ and $f(v) = 1$ otherwise. It is easy to see that f is an ISTDF of G which implies that

$$\gamma_{st}^0(G) \geq \omega(f) = 2(k-2) - k = k-4.$$

This completes the proof. □

REFERENCES

[1] M.A. Henning, *Signed total domination in graphs*, Discrete Math. **278** (2004), 109–125.

[2] Z. Huang, Z. Feng, H. Xing, *Inverse signed total domination numbers of some kinds of graphs*, Information Computing and Applications Communications in Computer and Information Science ICICA 2012, Part II, CCIS 308, 315–321, 2012.

[3] D.B. West, *Introduction to Graph Theory*, Prentice-Hall, Inc., 2000.

[4] B. Zelinka, *Signed total domination number of a graph*, Czechoslovak Math. J. **51** (2001), 225–229.

M. Atapour
m.atapour@bonabu.ac.ir

Department of Mathematics
Faculty of Basic Sciences
University of Bonab
Bonab, Iran, Po. Box: 5551761167

S. Norouzian

Azarbaijan Shahid Madani University
Department of Mathematics
Tabriz, I.R. Iran

S.M. Sheikholeslami
s.m.sheikholeslami@azaruniv.edu

Azarbaijan Shahid Madani University
Department of Mathematics
Tabriz, I.R. Iran

L. Volkmann
volkm@math2.rwth-aachen.de

RWTH-Aachen University
Lehrstuhl II für Mathematik
52056 Aachen, Germany

Received: October 22, 2014.
Accepted: July 9, 2015.