Opuscula Math. 36, no. 1 (2016), 81–101
http://dx.doi.org/10.7494/OpMath.2016.36.1.81 Opuscula Mathematica

EIGENVALUE PROBLEMS FOR ANISOTROPIC EQUATIONS INVOLVING A POTENTIAL ON ORLICZ-SOBOLEV TYPE SPACES

Ionela-Loredana Stăncuţ and Iulia Dorotheea Stîrcu

Communicated by Vicentiu D. Radulescu

Abstract. In this paper we consider an eigenvalue problem that involves a nonhomogeneous elliptic operator, variable growth conditions and a potential V on a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with a smooth boundary. We establish three main results with various assumptions. The first one asserts that any $\lambda > 0$ is an eigenvalue of our problem. The second theorem states the existence of a constant $\lambda_* > 0$ such that any $\lambda \in (0, \lambda_*]$ is an eigenvalue, while the third theorem claims the existence of a constant $\lambda^* > 0$ such that every $\lambda \in [\lambda^*, \infty)$ is an eigenvalue of the problem.

Keywords: anisotropic Orlicz-Sobolev space, potential, critical point, weak solution, eigenvalue.

Mathematics Subject Classification: 35D30, 35J60, 58E05.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded domain with a smooth boundary $\partial\Omega$. Consider that $a_i : (0, \infty) \to \mathbb{R}$, $i \in \{1, \ldots, N\}$, are functions such that the mappings $\varphi_i : \mathbb{R} \to \mathbb{R}$, $i \in \{1, \ldots, N\}$, defined by

$$\varphi_i(t) = \begin{cases} a_i(|t|)t, & \text{for} \quad t \neq 0, \\ 0, & \text{for} \quad t = 0, \end{cases}$$

are odd, increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$, $\lambda > 0$ is a real number, $V(x)$ is a potential and q_1, q_2, $m : \overline{\Omega} \to (2, \infty)$ are continuous functions. This paper is devoted to the study of the anisotropic eigenvalue problem

$$\begin{cases} -\sum_{i=1}^N \partial_i(\varphi_i(\partial_i u)) + V(x)|u|^{m(x)-2}u = \lambda(|u|^{q_1(x)-2} + |u|^{q_2(x)-2})u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where the potential $V : \Omega \to \mathbb{R}$ satisfies $V \in L^{r(x)}(\Omega)$ with $r(x) \in C(\overline{\Omega})$.

Considering that the operator in the divergence form is nonhomogeneous we introduce an Orlicz-Sobolev space setting for problems of type (1.1). In fact, given that our problem contains an equation of anisotropic type, we seek weak solutions in a more general Orlicz-Sobolev type space, namely an anisotropic Orlicz-Sobolev space. At the same time we note the presence of the continuous exponent functions m, q_1 and q_2 which leads us to use a suitable variable exponent Lebesgue space setting.

We should note that, as the Orlicz spaces, denoted by $L_\Phi(\Omega)$, are a generalization of the Lebesgue spaces $L^p(\Omega)$, so the Orlicz-Sobolev spaces, denoted by $W^m L_\Phi(\Omega)$, are a generalization of the Sobolev spaces $W^{m,p}(\Omega)$. Consequently, several properties of Sobolev spaces have been extended to Orlicz-Sobolev spaces (see [2, 9, 10, 24]). Due to the interest regarding the Orlicz-Sobolev spaces, motivated by their applicability in many fields of mathematics, in the last decades there appeared many papers involving such spaces. These spaces consist of functions which have weak derivatives and fulfill some integrability conditions. The Orlicz-Sobolev spaces are used to model various phenomena among which are the image restoration (see [6]), and modeling of electrorheological fluids (see [1, 5, 12, 13, 18, 31]). Both applications are based on variable exponent type Laplace operators.

In what follows we make a brief introduction for each of the following spaces: Orlicz spaces, Orlicz-Sobolev spaces, anisotropic Orlicz-Sobolev spaces, and variable exponent Lebesgue spaces.

We firstly recall some basic facts about Orlicz spaces. We refer to [2, 3, 7, 8, 14, 15, 22, 26] for more details.

Define

$$\Phi_i(t) = \int_0^t \varphi_i(s)ds, \quad (\Phi_i)^*(t) = \int_0^t (\varphi_i)^{-1}(s)ds \quad \text{for all } t \in \mathbb{R},\ i \in \{1, \dots, N\}.$$

We notice that Φ_i, $i \in \{1, \dots, N\}$, are *Young functions*, that is, $\Phi_i(0) = 0$, Φ_i are convex, and $\lim_{x\to\infty} \Phi_i(x) = +\infty$. Also, whereas $\Phi_i(x) = 0$ if and only if $x = 0$, $\lim_{x\to 0} \frac{\Phi_i(x)}{x} = 0$, and $\lim_{x\to\infty} \frac{\Phi_i(x)}{x} = +\infty$, then Φ_i are called *N-functions*. The functions $(\Phi_i)^*$, $i \in \{1, \dots, N\}$, are called the *complementary functions* of Φ_i, $i \in \{1, \dots, N\}$, and are defined as

$$(\Phi_i)^*(t) = \sup\{st - \Phi_i(s); s \geq 0\} \quad \text{for all } t \geq 0.$$

We observe that $(\Phi_i)^*$, $i \in \{1, \dots, N\}$, are also N-functions. Furthermore, *Young's inequality* holds true:

$$st \leq \Phi_i(s) + (\Phi_i)^*(t) \quad \text{for any } s, t \geq 0.$$

The Orlicz spaces $L_{\Phi_i}(\Omega)$, $i \in \{1, \dots, N\}$ are the spaces of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L_{\Phi_i}} := \sup \left\{ \int_\Omega uv\, dx; \int_\Omega (\Phi_i)^*(|v|)dx \leq 1 \right\} < \infty.$$

Thus, $(L_{\Phi_i}(\Omega), \|\cdot\|_{L_{\Phi_i}})$, $i \in \{1,\dots,N\}$, are Banach spaces whose norms are equivalent to the Luxemburg norms

$$\|u\|_{\Phi_i} := \inf \left\{ k > 0; \int\limits_{\Omega} \Phi_i \left(\frac{u(x)}{k} \right) dx \leq 1 \right\}.$$

Holder's inequality in Orlicz spaces is as follows:

$$\int\limits_{\Omega} uv\, dx \leq 2\|u\|_{L_{\Phi_i}} \|v\|_{L_{(\Phi_i)^*}} \quad \text{for any } u \in L_{\Phi_i}(\Omega) \text{ and } v \in L_{(\Phi_i)^*}(\Omega),\ i \in \{1,\dots,N\}.$$

Now, we are going to briefly describe *the Orlicz-Sobolev spaces* $W^1L_{\Phi_i}(\Omega)$, $i \in \{1,\dots,N\}$, defined by

$$W^1L_{\Phi_i}(\Omega) := \{u \in L_{\Phi_i}(\Omega); \partial_{x_i} u \in L_{\Phi_i}(\Omega),\ i = 1,\dots,N\},$$

which are Banach spaces endowed with the norms

$$\|u\|_{1,\Phi_i} := \|u\|_{\Phi_i} + \||\nabla u|\|_{\Phi_i}, \quad i \in \{1,\dots,N\}.$$

In addition, the Orlicz-Sobolev spaces $W_0^1L_{\Phi_i}(\Omega)$, $i \in \{1,\dots,N\}$, are the closure of $C_0^1(\Omega)$ in $W^1L_{\Phi_i}(\Omega)$. By Lemma 5.7 in [15], we obtain that on $W_0^1L_{\Phi_i}(\Omega)$ can be considered some equivalent norms

$$\|u\|_i := \||\nabla u|\|_{\Phi_i}, \quad i \in \{1,\dots,N\}.$$

Furthermore, the above norms are equivalent to the norms

$$\|u\|_{i,1} = \sum_{j=1}^{N} \|\partial_j u\|_{\Phi_i}, \quad i \in \{1,\dots,N\}$$

(see Proposition 1 in [17]).

An important role in handling the Orlicz-Sobolev spaces is played by

$$(p_i)_0 := \inf_{t>0} \frac{t\varphi_i(t)}{\Phi_i(t)} \quad \text{and} \quad (p_i)^0 := \sup_{t>0} \frac{t\varphi_i(t)}{\Phi_i(t)}, \quad i \in \{1,\dots,N\}.$$

In this paper we assume that for each $i \in \{1,\dots,N\}$ we have

$$1 < (p_i)_0 \leq \frac{t\varphi_i(t)}{\Phi_i(t)} \leq (p_i)^0 < \infty \quad \text{for any } t \geq 0. \tag{1.2}$$

The above inequalities imply that each Φ_i, $i \in \{1,\dots,N\}$, satisfies the Δ_2-*condition*, namely

$$\Phi_i(2t) \leq K\Phi_i(t) \quad \text{for any } t \geq 0, \tag{1.3}$$

where K is a positive constant (see Proposition 2.3 in [21]).

We assume also that for each $i \in \{1, \ldots, N\}$ the function Φ_i satisfies the following condition:

$$\text{the function } [0, \infty) \ni t \to \Phi_i(\sqrt{t}) \text{ is convex.} \tag{1.4}$$

The latter two conditions guarantee that for each $i \in \{1, \ldots, N\}$ the Orlicz spaces $L_{\Phi_i}(\Omega)$ are uniformly convex spaces, and consequently reflexive Banach spaces (see Proposition 2.2 in [21]). Therefore, the Orlicz-Sobolev spaces $W_0^1 L_{\Phi_i}(\Omega)$, $i \in \{1, \ldots, N\}$, are reflexive Banach spaces, as well.

Next, we introduce *the anisotropic Orlicz-Sobolev space* $W_0^1 L_{\overrightarrow{\Phi}}(\Omega)$, as the closure of $C_0^1(\Omega)$ under the norm

$$\|u\|_{\overrightarrow{\Phi}} = \sum_{i=1}^{N} |\partial_i u|_{\Phi_i},$$

where $\overrightarrow{\Phi} : \overline{\Omega} \to \mathbb{R}^N$ denotes the vectorial function $\overrightarrow{\Phi} = (\Phi_1, \ldots, \Phi_N)$. In [17] it was argued that $W_0^1 L_{\overrightarrow{\Phi}}(\Omega)$ is a reflexive Banach space.

Now, we introduce $\overrightarrow{P^0}, \overrightarrow{P_0} \in \mathbb{R}^N$ as

$$\overrightarrow{P^0} = ((p_1)^0, \ldots, (p_N)^0), \quad \overrightarrow{P_0} = ((p_1)_0, \ldots, (p_N)_0),$$

and $(P^0)_+, (P_0)_+, (P_0)_- \in \mathbb{R}^+$ as

$$\begin{aligned}
(P^0)_+ &= \max\{(p_1)^0, \ldots, (p_N)^0\},\\
(P_0)_+ &= \max\{(p_1)_0, \ldots, (p_N)_0\},\\
(P_0)_- &= \min\{(p_1)_0, \ldots, (p_N)_0\}.
\end{aligned}$$

We also always assume that

$$\sum_{i=1}^{N} \frac{1}{(p_i)_0} > 1,$$

and define $(P_0)^*, P_{0,\infty} \in \mathbb{R}^+$ by

$$(P_0)^* = \frac{N}{\sum_{i=1}^{N} \frac{1}{(p_i)_0} - 1}, \quad P_{0,\infty} = \max\{(P_0)_+, (P_0)^*\}.$$

Finally, we recall some definitions and basic properties of Lebesgue spaces with variable exponent. To the best of our knowledge, these spaces were introduced in the literature for the first time in 1931 by Orlicz [25]. Then, Nakano continued (in the 1950s) this survey in [23] with a systematic study of spaces with variable exponent (called modular spaces), and later the investigation was carried on by Polish mathematicians (see for instance Musielak [22]). Taking [16] as a starting point, where Kováčik and Rákosník have analyzed the spaces $L^{p(x)}$ and $W^{k,p(x)}$, respectively, many results were obtained regarding this type of variable exponent spaces. Of the recent works, which treat problems involving various classes of nonlinear equations in such spaces, or more general, we find the papers [27, 29, 30], and the book [28].

Set

$$C_+(\overline{\Omega}) = \left\{p \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} p(x) > 1\right\}$$

and denote, for every $p \in C_+(\overline{\Omega})$,

$$p^+ = \sup_{x\in\Omega} p(x) \quad \text{and} \quad p^- = \inf_{x\in\Omega} p(x).$$

For any $p \in C_+(\overline{\Omega})$ we define *the variable exponent Lebesgue space*

$$L^{p(x)}(\Omega) = \left\{u;\ u \text{ is a measurable real-valued function with } \int_\Omega |u|^{p(x)} dx < \infty\right\}.$$

On this space we define the so-called *Luxemburg norm*

$$|u|_{p(x)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}$$

and emphasize that $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ is a separable and reflexive Banach space. If $0 < |\Omega| < \infty$ and $p_1, p_2 \in C_+(\overline{\Omega})$ are variable exponents satisfying $p_1(x) \le p_2(x)$ almost everywhere in Ω, then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the following Hölder-type inequality

$$\left|\int_\Omega uv\ dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) |u|_{p(x)} |v|_{p'(x)} \le 2|u|_{p(x)}|v|_{p'(x)} \tag{1.5}$$

holds true.

Also, we define $p(\cdot)$-*modular* of the $L^{p(\cdot)}(\Omega)$ spaces, which is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)} dx.$$

If $(u_n), u \in L^{p(\cdot)}(\Omega)$, then the following relations hold true:

$$|u|_{p(\cdot)} > 1 \Rightarrow |u|_{p(\cdot)}^{p^-} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^+}, \tag{1.6}$$

$$|u|_{p(\cdot)} < 1 \Rightarrow |u|_{p(\cdot)}^{p^+} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^-}, \tag{1.7}$$

$$|u_n - u_0|_{p(\cdot)} \to 0 \Leftrightarrow \rho_{(\cdot)}(u_n - u) \to 0. \tag{1.8}$$

Look into [16] for more details of these facts and further properties of the variable exponent Lebesgue spaces.

2. THE MAIN RESULTS

In this paper we look for weak solutions of problem (1.1) in a subspace of the anisotropic Orlicz-Sobolev space $W_0^1 L_{\vec{\Phi}}(\Omega)$, namely

$$E := \left\{ u \in W_0^1 L_{\vec{\Phi}}(\Omega);\ \int\limits_\Omega |V(x)||u|^{m(x)} dx < \kappa,\ \text{with } \kappa > 0 \text{ real constant} \right\}.$$

Define the functionals $J_V, I : E \to \mathbb{R}$ by

$$J_V(u) = \int\limits_\Omega \sum_{i=1}^N \Phi_i(|\partial_i u|) dx + \int\limits_\Omega \frac{V(x)}{m(x)} |u|^{m(x)} dx,$$

$$I(u) = \int\limits_\Omega \frac{1}{q_1(x)} |u|^{q_1(x)} dx + \int\limits_\Omega \frac{1}{q_2(x)} |u|^{q_2(x)} dx.$$

By standard arguments, $J_V, I \in C^1(E, \mathbb{R})$ and the Fréchet derivatives are given by

$$\langle J_V'(u), v\rangle = \int\limits_\Omega \sum_{i=1}^N a_i(|\partial_i u|)\partial_i u \partial_i v\ dx + \int\limits_\Omega V(x)|u|^{m(x)-2} uv\ dx,$$

$$\langle I'(u), v\rangle = \int\limits_\Omega |u|^{q_1(x)-2} uv\ dx + \int\limits_\Omega |u|^{q_2(x)-2} uv\ dx,$$

for all $u, v \in E$.

The energy functional corresponding to problem (1.1) is defined as $T_\lambda : E \to \mathbb{R}$,

$$T_\lambda(u) = J_V(u) - \lambda I(u).$$

It is obvious that $T_\lambda \in C^1(E, \mathbb{R})$ with

$$\langle T_\lambda'(u), v\rangle = \langle J_V'(u), v\rangle - \lambda \langle I'(u), v\rangle$$

for all $u, v \in E$.

We say that $\lambda \in \mathbb{R}$ is an *eigenvalue* of problem (1.1) if and only if there exists $u \in E \setminus \{0\}$ a *critical point* of T_λ, or, in other words, *weak solution* of problem (1.1) corresponding to the eigenvalue λ.

The main results of the present paper are given by the following three theorems.

Theorem 2.1. *Assume that the functions $q_1, q_2, m \in C(\overline{\Omega})$ satisfy the hypothesis*

$$2 < (P^0)_+ < q_2^- \le q_2^+ \le m^- \le m^+ \le q_1^- \le q_1^+ < q_1^+ \cdot {r^-}' < (P_0)^*, \tag{2.1}$$

where ${r^-}' = \frac{r^-}{r^- - 1}$. Then any $\lambda > 0$ is an eigenvalue of problem (1.1).

Theorem 2.2. *Assume that the functions* $q_1, q_2, m \in C(\overline{\Omega})$ *verify the condition*

$$2 < q_2^- \leq q_2^+ \leq q_1^- \leq q_1^+ \leq m^- \leq m^+ < m^+ \cdot r^{-\prime} < (P_0)_- \leq (P_0)^*, \tag{2.2}$$

where $r^{-\prime} = \frac{r^-}{r^- - 1}$. *Then there is* $\lambda_* > 0$ *so that every* $\lambda \in (0, \lambda_*]$ *is an eigenvalue of problem* (1.1).

Theorem 2.3. *Assume that the functions* $q_1, q_2, m \in C(\overline{\Omega})$ *fulfill the hypothesis*

$$2 < q_2^- \leq q_2^+ \leq m^- \leq m^+ \leq q_1^- \leq q_1^+ < q_1^+ \cdot r^{-\prime} < (P_0)_- \leq (P_0)^*, \tag{2.3}$$

where $r^{-\prime} = \frac{r^-}{r^- - 1}$. *Then there is* $\lambda^* > 0$ *so that every* $\lambda \in [\lambda^*, \infty)$ *is an eigenvalue of problem* (1.1).

Remark 2.4. If in our problem we take $q_1(x) = q_2(x) = q(x)$ for every $x \in \overline{\Omega}$ and $V \equiv 0$ for every $x \in \Omega$, we obtain the problem dealt in [17]. Therefore, we are motivated to state, for our more general problem, to a certain extent, some similar results to those in paper [17], although in the present paper we encounter more technical difficulties.

3. PROOF OF THEOREM 2.1

We begin by proving two auxiliary lemmas.

Lemma 3.1. *Assume that the hypothesis of Theorem 2.1 is satisfied. Then there exist* $\eta > 0$ *and* $\alpha > 0$ *such that* $T_\lambda(u) \geq \alpha > 0$ *for any* $u \in E$ *with* $\|u\|_{\vec{\Phi}} = \eta$.

Proof. We have

$$\begin{aligned}
T_\lambda(u) &= J_V(u) - \lambda I(u) \\
&= \int_\Omega \sum_{i=1}^N \Phi_i(|\partial_i u|)dx + \int_\Omega \frac{V(x)}{m(x)}|u|^{m(x)}dx \\
&\quad - \lambda \int_\Omega \left(\frac{1}{q_1(x)}|u|^{q_1(x)} + \frac{1}{q_2(x)}|u|^{q_2(x)} \right) dx \\
&\geq \int_\Omega \sum_{i=1}^N \Phi_i(|\partial_i u|)dx - \frac{1}{m^-} \int_\Omega |V(x)||u|^{m(x)}dx \\
&\quad - \frac{\lambda}{q_2^-} \int_\Omega \left(|u|^{q_1(x)} + |u|^{q_2(x)} \right) dx.
\end{aligned}$$

Now, taking into account that $q_2(x) \leq m^- \leq m(x) \leq m^+ \leq q_1(x)$ we obtain

$$|u(x)|^{m(x)} \leq |u(x)|^{m^-} + |u(x)|^{m^+} \leq 2\left(|u(x)|^{q_1(x)} + |u(x)|^{q_2(x)} \right) \text{ for all } x \in \overline{\Omega},\ u \in E.$$

On the other hand, since $q_i^- \le q_i(x) \le q_i^+$, $i = 1,2$, we derive that

$|u(x)|^{q_1(x)} + |u(x)|^{q_2(x)} \le |u(x)|^{q_1^-} + |u(x)|^{q_1^+} + |u(x)|^{q_2^-} + |u(x)|^{q_2^+}$ for all $x \in \overline{\Omega},\ u \in E$.

Therefore, we get

$$\begin{aligned} T_\lambda(u) \ge & \int\limits_\Omega \sum_{i=1}^N \Phi_i(|\partial_i u|)dx - \frac{2}{m^-} \int\limits_\Omega |V(x)| \left(|u|^{q_1^-} + |u|^{q_1^+} + |u|^{q_2^-} + |u|^{q_2^+} \right) dx \\ & - \frac{\lambda}{q_2^-} \int\limits_\Omega \left(|u|^{q_1^-} + |u|^{q_1^+} + |u|^{q_2^-} + |u|^{q_2^+} \right) dx. \end{aligned}$$

Further, the inequalities $q_i^\pm \le q_i^\pm \cdot r^{-\prime}$, $i = 1,2$, lead us to the fact that the embeddings $L^{q_i^\pm \cdot r^{-\prime}}(\Omega) \subset L^{q_i^\pm}(\Omega)$ are continuous, thus there exists $C_{ij} > 1$, $i,j = 1,2$, constants such that

$$|u|_{q_i^\pm} \le C_{ij} |u|_{q_i^\pm \cdot r^{-\prime}},\ i,j = 1,2, \quad \text{for all } u \in L^{q_i^\pm \cdot r^{-\prime}}(\Omega).$$

Since $V \in L^{r(x)}(\Omega)$, it is obvious that $V \in L^{r^-}(\Omega)$, as well. Hence, by Hölder-type inequality (1.5) we obtain

$$\int\limits_\Omega |V(x)||u|^{q_i^\pm} dx \le 2|V|_{r^-} |u|_{q_i^\pm \cdot r^{-\prime}}, \quad i = 1,2.$$

Then, we arrive at

$$\begin{aligned} T_\lambda(u) \ge & \int\limits_\Omega \sum_{i=1}^N \Phi_i(|\partial_i u|)dx - \frac{4|V|_{r^-}}{m^-} \left(|u|^{q_1^-}_{q_1^- \cdot r^{-\prime}} + |u|^{q_1^+}_{q_1^+ \cdot r^{-\prime}} + |u|^{q_2^-}_{q_2^- \cdot r^{-\prime}} + |u|^{q_2^+}_{q_2^+ \cdot r^{-\prime}} \right) \\ & - C_0 \left(|u|^{q_1^-}_{q_1^- \cdot r^{-\prime}} + |u|^{q_1^+}_{q_1^+ \cdot r^{-\prime}} + |u|^{q_2^-}_{q_2^- \cdot r^{-\prime}} + |u|^{q_2^+}_{q_2^+ \cdot r^{-\prime}} \right) \\ \ge & \int\limits_\Omega \sum_{i=1}^N \Phi_i(|\partial_i u|)dx - C \left(|u|^{q_1^-}_{q_1^- \cdot r^{-\prime}} + |u|^{q_1^+}_{q_1^+ \cdot r^{-\prime}} + |u|^{q_2^-}_{q_2^- \cdot r^{-\prime}} + |u|^{q_2^+}_{q_2^+ \cdot r^{-\prime}} \right), \end{aligned}$$

where $C_0 = \frac{\lambda}{q_2^-} \max \left\{ C_{11}^{q_1^-}, C_{12}^{q_1^+}, C_{21}^{q_2^-}, C_{22}^{q_2^+} \right\}$. In the same time, by (2.1) we have

$$2 < (P^0)_+ < q_i^\pm \cdot r^{-\prime} < (P_0)^* = \max\{(P_0)_+, (P_0)^*\} = P_{0,\infty}.$$

Consequently, using Lemma 1 in [17], we get that there exists $B_{ij} > 1$, $i,j = 1,2$, constants such that

$$|u|_{q_i^\pm \cdot r^{-\prime}} \le B_{ij} \|u\|_{\vec{\Phi}},\ i,j = 1,2, \quad \text{for all } u \in E.$$

So, we can see that

$$T_\lambda(u) \ge \int\limits_\Omega \sum_{i=1}^N \Phi_i(|\partial_i u|)dx - \left(B'_{11} \|u\|^{q_1^-}_{\vec{\Phi}} + B'_{12} \|u\|^{q_1^+}_{\vec{\Phi}} + B'_{21} \|u\|^{q_2^-}_{\vec{\Phi}} + B'_{22} \|u\|^{q_2^+}_{\vec{\Phi}} \right),$$

where B'_{ij}, $i,j = 1,2$, are positive constants. Next, we focus our attention on the case when $u \in E$ and $\|u\|_{\vec{\Phi}} < 1$. For such an element u, we have $\|\partial_i u\|_{\Phi_i} < 1$, for all $i \in \{1,\dots,N\}$. By a similar relation to the third relation of Lemma 1 in [19] we obtain

$$\sum_{i=1}^{N} \|\partial_i u\|_{\Phi_i}^{(p_i)^0} \leq \sum_{i=1}^{N} \int_\Omega \Phi_i(|\partial_i u|)dx \quad \text{for all } u \in E, \text{ with } \|u\|_{\vec{\Phi}} < 1. \tag{3.1}$$

Using the Jensen's inequality, applied to the convex function $a : \mathbb{R}^+ \to \mathbb{R}^+$, $a(t) = t^{(P^0)_+}$, $(P^0)_+ > 2$, we obtain

$$\begin{aligned} \frac{\|u\|_{\vec{\Phi}}^{(P^0)_+}}{N^{(P^0)_+ - 1}} &= N\left(\sum_{i=1}^{N} \frac{1}{N}\|\partial_i u\|_{\Phi_i}\right)^{(P^0)_+} \\ &\leq \sum_{i=1}^{N} \|\partial_i u\|_{\Phi_i}^{(P^0)_+} \leq \sum_{i=1}^{N} \|\partial_i u\|_{\Phi_i}^{(p_i)^0} \leq \sum_{i=1}^{N} \int_\Omega \Phi_i(|\partial_i u|)dx. \end{aligned} \tag{3.2}$$

By (3.1) and (3.2), we arrive at

$$\begin{aligned} T_\lambda(u) &\geq \frac{\|u\|_{\vec{\Phi}}^{(P^0)_+}}{N^{(P^0)_+ - 1}} - \left(B'_{11}\|u\|_{\vec{\Phi}}^{q_1^-} + B'_{12}\|u\|_{\vec{\Phi}}^{q_1^+} + B'_{21}\|u\|_{\vec{\Phi}}^{q_2^-} + B'_{22}\|u\|_{\vec{\Phi}}^{q_2^+}\right) \\ &= \left(\frac{1}{N^{(P^0)_+ - 1}} - B'_{11}\|u\|_{\vec{\Phi}}^{q_1^- - (P^0)_+} - B'_{12}\|u\|_{\vec{\Phi}}^{q_1^+ - (P^0)_+} - B'_{21}\|u\|_{\vec{\Phi}}^{q_2^- - (P^0)_+} \right. \\ &\quad \left. - B'_{22}\|u\|_{\vec{\Phi}}^{q_2^+ - (P^0)_+}\right) \cdot \|u\|_{\vec{\Phi}}^{(P^0)_+} \end{aligned}$$

for any $u \in E$, with $\|u\|_{\vec{\Phi}} < 1$.

Let $g : [0,1] \to \mathbb{R}$ be the function defined by

$$g(t) = \frac{1}{N^{(P^0)_+ - 1}} - B'_{11}t^{q_1^- - (P^0)_+} - B'_{12}t^{q_1^+ - (P^0)_+} - B'_{21}t^{q_2^- - (P^0)_+} - B'_{22}t^{q_2^+ - (P^0)_+}.$$

It is clear that g is positive in a neighbourhood of the origin, such that the choice of $\rho \in (0,1)$ is so small that $\alpha = \rho^{(P^0)_+} g(\rho) > 0$ and this completes the proof of lemma. □

Lemma 3.2. *Assume that the hypothesis of Theorem 2.1 is verified, and let η given in Lemma 3.1. Then there exists $e \in E$ with $\|e\|_{\vec{\Phi}} > \eta$ such that $T_\lambda(e) < 0$.*

Proof. Let $\Psi \in C_0^\infty(\Omega)$, $\Psi \geq 0$ and $\Psi \not\equiv 0$, be fixed and let $t > 1$. By a similar inequality to (11) in [19] we see that

$$\Phi_i(t|\partial_i \Psi|) \leq t^{(p_i)^0} \Phi_i(|\partial_i \Psi|) \leq t^{(P^0)_+} \Phi_i(|\partial_i \Psi|) \quad \text{for all } i \in \{1, \ldots, N\}.$$

Accordingly, we can write

$$T_\lambda(t\Psi) = \int_\Omega \sum_{i=1}^N \Phi_i(|\partial_i(t\Psi)|)dx + \int_\Omega \frac{V(x)}{m(x)}|t\Psi|^{m(x)}dx$$

$$- \lambda\left(\int_\Omega \frac{1}{q_1(x)}|t\Psi|^{q_1(x)}dx + \int_\Omega \frac{1}{q_2(x)}|t\Psi|^{q_2(x)}dx \right)$$

$$\leq t^{(P^0)_+} \int_\Omega \Phi_i(|\partial_i \Psi|)dx + \frac{t^{m^+}}{m^-} \int_\Omega |V(x)||\Psi|^{m(x)}dx$$

$$- \frac{\lambda t^{q_1^-}}{q_1^+} \int_\Omega |\Psi|^{q_1(x)}dx - \frac{\lambda t^{q_2^-}}{q_2^+} \int_\Omega |\Psi|^{q_2(x)}dx.$$

By (2.1), it is clear that

$$\lim_{t\to\infty} T_\lambda(t\Psi) = -\infty.$$

Thus, for $t > 1$ sufficiently large, we can take $e = t\Psi$ so that $\|e\|_{\vec{\Phi}} > \eta$ and $T_\lambda(e) < 0$, that is what we wanted to show. □

Proof of Theorem 2.1. Taking account of Lemma 3.1 and Lemma 3.2 and the mountain pass theorem (see [4] with the variant given by Theorem 1.15 in [33]) we obtain the existence of a sequence $(u_n) \subset E$ such that

$$T_\lambda(u_n) \to \bar{c} > 0 \quad \text{and} \quad T'_\lambda(u_n) \to 0 \quad (\text{in } E^*) \text{ as } n \to \infty. \tag{3.3}$$

We are looking to prove that (u_n) is bounded in E. For this, we assume by contradiction that passing eventually to a subsequence, labeled again by (u_n), we have $\|u_n\|_{\vec{\Phi}} \to \infty$

and $\|u_n\|_{\vec{\Phi}} > 1$ for all n. Keeping in mind the above and relation (3.3) we deduce that for any n sufficiently large we have

$$\begin{aligned}
&1 + \bar{c} + \|u_n\|_{\vec{\Phi}} \\
&\geq T_\lambda(u_n) - \frac{1}{q_2^-}\langle T'_\lambda(u_n), u_n\rangle \\
&= \int\limits_\Omega \sum_{i=1}^N \Phi_i(|\partial_i u_n|)dx + \int\limits_\Omega \frac{V(x)}{m(x)}|u_n|^{m(x)}dx \\
&\quad - \lambda \int\limits_\Omega \left(\frac{1}{q_1(x)}|u_n|^{q_1(x)} + \frac{1}{q_2(x)}|u_n|^{q_2(x)}\right) dx \\
&\quad - \frac{1}{q_2^-}\int\limits_\Omega \sum_{i=1}^N a_i(|\partial_i u_n|)|\partial_i u_n|^2 dx - \frac{1}{q_2^-}\int\limits_\Omega V(x)|u_n|^{m(x)}dx \\
&\quad + \frac{\lambda}{q_2^-}\int\limits_\Omega \left(|u_n|^{q_1(x)} + |u_n|^{q_2(x)}\right) dx \\
&\geq \int\limits_\Omega \sum_{i=1}^N \Phi_i(|\partial_i u_n|)dx - \frac{1}{m^-}\int\limits_\Omega |V(x)||u_n|^{m(x)}dx - \frac{\lambda}{q_2^-}\int\limits_\Omega \left(|u_n|^{q_1(x)} + |u_n|^{q_2(x)}\right) dx \\
&\quad - \frac{1}{q_2^-}\int\limits_\Omega \sum_{i=1}^N \varphi_i(|\partial_i u_n|)\partial_i u_n dx - \frac{1}{q_2^-}\int\limits_\Omega |V(x)||u_n|^{m(x)}dx \\
&\quad + \frac{\lambda}{q_2^-}\int\limits_\Omega \left(|u_n|^{q_1(x)} + |u_n|^{q_2(x)}\right) dx \\
&= \int\limits_\Omega \sum_{i=1}^N \Phi_i(|\partial_i u_n|)dx - \frac{1}{q_2^-}\int\limits_\Omega \sum_{i=1}^N \varphi_i(|\partial_i u_n|)\partial_i u_n dx \\
&\quad - \left(\frac{1}{m^-} + \frac{1}{q_2^-}\right)\int\limits_\Omega |V(x)||u_n|^{m(x)}dx \\
&\geq \int\limits_\Omega \sum_{i=1}^N \Phi_i(|\partial_i u_n|)dx - \frac{1}{q_2^-}\int\limits_\Omega \sum_{i=1}^N \varphi_i(|\partial_i u_n|)\partial_i u_n dx - \kappa.
\end{aligned}$$

Next, taking into consideration the definitions of $(p_i)^0$, $i \in \{1, \ldots, N\}$, and $(P^0)_+$ we can write

$$(P^0)_+ \geq (p_i)^0 \geq \frac{|\partial_i u_n|\varphi_i(|\partial_i u_n|)}{\Phi_i(|\partial_i u_n|)} \quad \text{for all } i \in \{1, \ldots, N\}.$$

Hence, we have

$$\begin{aligned}1+\bar{c}+\|u_n\|_{\vec{\Phi}} &\geq \int_\Omega \sum_{i=1}^N \Phi_i(|\partial_i u_n|)dx - \frac{(P^0)_+}{q_2^-}\int_\Omega \sum_{i=1}^N \Phi_i(|\partial_i u_n|)dx - \kappa \\ &= \left(1-\frac{(P^0)_+}{q_2^-}\right)\int_\Omega \sum_{i=1}^N \Phi_i(|\partial_i u_n|)dx - \kappa.\end{aligned}$$

By using the Jensen's inequality for the convex function $b:\mathbb{R}^+\to\mathbb{R}^+$, $b(t)=t^{(P_0)_-}$, $(P_0)_->2$, we get

$$\frac{\|u_n\|_{\vec{\Phi}}^{(P_0)_-}}{N^{(P_0)_--1}} = N\left(\frac{\sum_{i=1}^N \|\partial_i u_n\|_{\Phi_i}}{N}\right)^{(P_0)_-} \leq \sum_{i=1}^N \|\partial_i u_n\|_{\Phi_i}^{(P_0)_-}. \tag{3.4}$$

On the other hand, let

$$\alpha_{i,n} = \begin{cases}(P^0)_+, & \text{if } \|\partial_i u_n\|<1,\\ (P_0)_-, & \text{if } \|\partial_i u_n\|>1.\end{cases}$$

That being defined we apply inequalities (C.9) and (C.10) in [8], then take into account relation (3.4) to see that

$$\begin{aligned}&1+\bar{c}+\|u_n\|_{\vec{\Phi}}\\ &\geq \left(1-\frac{(P^0)_+}{q_2^-}\right)\sum_{i=1}^N \|\partial_i u_n\|_{\Phi_i}^{\alpha_{i,n}} - \kappa\\ &= \left(1-\frac{(P^0)_+}{q_2^-}\right)\left[\sum_{i=1}^N \|\partial_i u_n\|_{\Phi_i}^{(P_0)_-} - \sum_{\{i;\ \alpha_{i,n}=(P^0)_+\}}\left(\|\partial_i u_n\|_{\Phi_i}^{(P_0)_-} - \|\partial_i u_n\|_{\Phi_i}^{(P^0)_+}\right)\right] - \kappa\\ &\geq \left(1-\frac{(P^0)_+}{q_2^-}\right)\left(\frac{1}{N^{(P^0)_--1}}\|u_n\|_{\vec{\Phi}}^{(P_0)_-} - N\right) - \kappa.\end{aligned}$$

Now, we divide by $\|u_n\|_{\vec{\Phi}}^{(P_0)_-}$ and obtain

$$\begin{aligned}\frac{1+\bar{c}}{\|u_n\|_{\vec{\Phi}}^{(P_0)_-}} + \frac{1}{\|u_n\|_{\vec{\Phi}}^{(P_0)_--1}} &\geq \frac{1}{N^{(P_0)_--1}}\left(1-\frac{(P^0)_+}{q_2^-}\right)\\ &\quad - \frac{N}{\|u_n\|_{\vec{\Phi}}^{(P_0)_-}}\left(1-\frac{(P^0)_+}{q_2^-}\right) - \frac{\kappa}{\|u_n\|_{\vec{\Phi}}^{(P_0)_-}}.\end{aligned}$$

Passing to the limit as $n\to\infty$ we obtain a contradiction. It results that (u_n) is bounded in E. This together with the fact that E is reflexive mean that there is a subsequence of (u_n), labeled again by (u_n), and an element $u_0 \in E$ such that $u_n \rightharpoonup u_0$ in E. On the other hand, the embeddings $E \subset L^{q_i(\cdot)}(\Omega)$, $i = 1, 2$, are compact and thus $u_n \to u_0$ in $L^{q_i(\cdot)}(\Omega)$, $i = 1, 2$. Thus, using the Hölder-type inequality (1.5), we deduce that

$$\int_\Omega |u_n|^{q_i(x)-2} u_n (u_n - u_0) dx \underset{n\to\infty}{\longrightarrow} 0, \quad i = 1, 2. \tag{3.5}$$

Given the definition of the subspace $E \subset W_0^1 L_{\vec{\Phi}}(\Omega)$, we can easily see that $V(x)|u_n|^{m(x)-1} \in L^{m'(x)}(\Omega)$. But the embedding $L^{m'(x)}(\Omega) \subset L^{q_1'(x)}(\Omega)$ is continuous, so $V(x)|u_n|^{m(x)-1} \in L^{q_1'(x)}(\Omega)$. Therefore, using again the Hölder-type inequality (1.5), we find that

$$\int_\Omega V(x)|u_n|^{m(x)-2} u_n (u_n - u_0) dx \leq 2 \left| V(x)|u_n|^{m(x)-1} \right|_{q_1'(x)} |u_n - u_0|_{q_1(x)} \underset{n\to\infty}{\longrightarrow} 0. \tag{3.6}$$

By relations (3.3), (3.5) and (3.6), we obtain

$$\sum_{i=1}^N \int_\Omega a_i(|\partial_i u_n|) \partial_i u_n (\partial_i u_n - \partial_i u_0) dx \underset{n\to\infty}{\longrightarrow} 0.$$

Taking into consideration that $u_n \rightharpoonup u_0$ in E, by the above relation we infer that

$$\int_\Omega (a_i(|\partial_i u_n|)\partial_i u_n - a_i(|\partial_i u_0|)\partial_i u_0)(\partial_i u_n - \partial_i u_0) dx \underset{n\to\infty}{\longrightarrow} 0. \tag{3.7}$$

Then, considering relation (3.7), the same arguments used at the end of Theorem 1 in [17] lead us to

$$\sum_{i=1}^N \int_\Omega \Phi_i \left(\left| \frac{\partial_i u_n - \partial_i u_0}{2} \right| \right) dx \underset{n\to\infty}{\longrightarrow} 0, \tag{3.8}$$

which means that $u_n \to u_0$ in E. This, together with (3.3) show that

$$T_\lambda(u_0) = \bar{c} > 0 \quad \text{and} \quad T_\lambda'(u_0) = 0.$$

In other words $u_0 \in E$ is a nontrivial weak solution of problem (1.1). □

4. PROOF OF THEOREM 2.2

We start by showing two auxiliary results.

Lemma 4.1. *Assume that the hypothesis of Theorem 2.2 is satisfied. Then there exists $\lambda_* > 0$ so that for every $\lambda \in (0, \lambda_*]$ there exist $\rho, a > 0$ such that $T_\lambda(u) \geq a > 0$ for any $u \in E$ with $\|u\|_{\vec{\Phi}} = \rho$.*

Proof. Using the hypothesis (2.2), Lemma 1 in [17] shows that the embeddings

$$E \subset L^{m^{\pm}\cdot r^{-\prime}}(\Omega) \subset L^{q_i(\cdot)}(\Omega), \quad i = 1, 2, \tag{4.1}$$

are continuous. Hence, it follows that there is a constant $C_1 > 0$ such that

$$|u|_{q_1(\cdot)} \leq C_1 \|u\|_{\vec{\Phi}} \quad \text{for all } u \in E. \tag{4.2}$$

We fix $\rho \in (0, 1)$ such that $\rho < \frac{1}{C_1}$. Then, relation (4.2) implies that

$$|u|_{q_1(\cdot)} < 1 \quad \text{for all } u \in E, \text{ with } \|u\|_{\vec{\Phi}} = \rho. \tag{4.3}$$

Taking into account relations (4.3), (1.7) and (4.2) it results that

$$\int_\Omega |u|^{q_1(x)} dx \leq C_1^{q_1^-} \|u\|_{\vec{\Phi}}^{q_1^-}, \quad \text{for all } u \in E, \text{ with } \|u\|_{\vec{\Phi}} = \rho. \tag{4.4}$$

Similar arguments to those used in the proof of Lemma 3.1 show that there are some constants $C_{i,j} > 1$, $i, j = 1, 2$, so that

$$\begin{aligned} &\int_\Omega \frac{1}{q_1(x)} |u|^{q_1(x)} dx + \int_\Omega \frac{1}{q_2(x)} |u|^{q_2(x)} dx \\ &\leq \frac{1}{q_2^-} \left(C_{11}^{q_1^-} \|u\|_{\vec{\Phi}}^{q_1^-} + C_{12}^{q_1^+} \|u\|_{\vec{\Phi}}^{q_1^+} + C_{21}^{q_2^-} \|u\|_{\vec{\Phi}}^{q_2^-} + C_{22}^{q_2^+} \|u\|_{\vec{\Phi}}^{q_2^+} \right) \\ &\leq \frac{4c_0^{q_1^+}}{q_2^-} \|u\|_{\vec{\Phi}}^{q_2^-}, \end{aligned} \tag{4.5}$$

where $c_0 = \max\{C_{11}, C_{12}, C_{21}, C_{22}\}$. Also, relation (4.1) provides the existence of some constants $c_1, c_2 > 1$ such that

$$\begin{aligned} \int_\Omega |V(x)||u|^{m(x)} dx &\leq \int_\Omega |V(x)| \left(|u|^{m^-} + |u|^{m^+} \right) dx \\ &\leq |V|_{r^-} \left(|u|_{m^-\cdot r^{-\prime}}^{m^-} + |u|_{m^+\cdot r^{-\prime}}^{m^+} \right) \\ &\leq c^{m^+} |V|_{r^-} \left(\|u\|_{\vec{\Phi}}^{m^-} + \|u\|_{\vec{\Phi}}^{m^+} \right) \\ &\leq 2c^{m^+} |V|_{r^-} \|u\|_{\vec{\Phi}}^{m^-}, \end{aligned} \tag{4.6}$$

where $c = \max\{c_1, c_2\}$.

Therefore, by (3.2), (4.5) and (4.6) we deduce that

$$\begin{aligned} T_\lambda(u) &\geq \frac{\rho^{(P^0)_+}}{N^{(P^0)_+-1}} - \frac{2c^{m^+}|V|_{r^-}}{m^-}\rho^{m^-} - \frac{4\lambda c_0^{q_1^+}}{q_2^-}\rho^{q_2^-} \\ &\geq \rho^{q_2^-}\left(\frac{1}{N^{(P^0)_+-1}}\rho^{(P^0)_+-q_2^-} - \frac{2c^{m^+}|V|_{r^-}}{m^-}\rho^{m^--q_2^-} - \frac{4\lambda c_0^{q_1^+}}{q_2^-}\right). \end{aligned} \tag{4.7}$$

By setting the number

$$\lambda_* = \frac{q_2^-}{8c_0^{q_1^+}N^{(P^0)_+-1}}\rho^{(P^0)_+-q_2^-} - \frac{c^{m^+}q_2^-|V|_{r^-}}{2c_0^{q_1^+}m^-}\rho^{m^--q_2^-} \tag{4.8}$$

we get that for any $\lambda \in (0, \lambda_*]$ and $u \in E$ with $\|u\|_{\vec{\Phi}} = \rho$ the number $a = \frac{\rho^{(P^0)_+}}{2N^{(P^0)_+-1}}$ is such that

$$T_\lambda(u) \geq a > 0,$$

and this completes the proof of the lemma. □

Lemma 4.2. *Assume that the condition of Theorem 2.2 is verified. Then there is $\theta \in E$ such that $\theta \geq 0$, $\theta \not\equiv 0$ and $T_\lambda(t\theta) < 0$ for $t > 0$ sufficiently small.*

Proof. Firstly, we show that

$$\Phi_i(\sigma\ell) \leq \sigma^{(p_i)_0}\Phi_i(\ell) \quad \text{for all } \ell > 0,\ \sigma \in (0,1), \text{ and } i \in \{1,\dots,N\}. \tag{4.9}$$

Let $\sigma \in (0,1)$ be fixed. Using the definition of $(p_i)_0$, we have

$$\begin{aligned} \log(\Phi_i(\ell)) - \log(\Phi_i(\sigma\ell)) &= \int_{\sigma\ell}^{\ell} \frac{\varphi_i(s)}{\Phi_i(s)}ds \geq \int_{\sigma\ell}^{\ell} \frac{(p_i)_0}{s}ds \\ &= -\log\left(\sigma^{(p_i)_0}\right) \quad \text{for all } i \in \{1,\dots,N\}, \end{aligned}$$

that is, the relation (4.9) is true.

Next, from hypothesis (2.2) we obviously have $q_1^- \leq m^-$. Let $\epsilon > 0$ be so that $q_1^- + \epsilon \leq m^-$. In the same time, the fact that $q_1 \in C(\overline{\Omega})$ yields the existence of an open nonempty set $\omega \subset \Omega$ so that $|q_1(x) - q_1^-| < \epsilon$ for all $x \in \omega$. Or, in another train of thoughts, we have $q_1(x) < q_1^- + \epsilon \leq m^-$ for all $x \in \omega$.

On the other hand, let $\theta \in C_0^\infty(\Omega)$ be such that $\mathrm{supp}(\theta) \supset \overline{\omega}$, $\theta(x) = 1$ for all $x \in \overline{\omega}$, and $\theta(x) \in [0,1]$ for all $x \in \Omega$.

The above piece of information and relation (4.9) lead us to

$$\begin{aligned}
T_\lambda(t\theta) &= J_V(t\theta) - \lambda I(t\theta) \\
&= \int\limits_\Omega \sum_{i=1}^N \Phi_i(t|\partial_i\theta|)dx + \int\limits_\Omega \frac{t^{m(x)}V(x)}{m(x)}|\theta|^{m(x)}dx \\
&\quad - \lambda \int\limits_\Omega \left(\frac{t^{q_1(x)}}{q_1(x)}|\theta|^{q_1(x)} + \frac{t^{q_2(x)}}{q_2(x)}|\theta|^{q_2(x)} \right) dx \\
&\leq t^{(P_0)-} \sum_{i=1}^N \int\limits_\Omega \Phi_i(|\partial_i\theta|)dx + \frac{t^{m^-}}{m^-} \int\limits_\Omega |V(x)||\theta|^{m(x)}dx \\
&\quad - \frac{\lambda t^{q_1^- + \epsilon}}{q_1^+} \int\limits_\omega \left(|\theta|^{q_1(x)} + |\theta|^{q_2(x)} \right) dx \\
&< t^{m^-} \left(\sum_{i=1}^N \int\limits_\Omega \Phi_i(|\partial_i\theta|)dx + \frac{1}{m^-} \int\limits_\Omega |V(x)||\theta|^{m(x)}dx \right) \\
&\quad - \frac{\lambda t^{q_1^- + \epsilon}}{q_1^+} \int\limits_\omega \left(|\theta|^{q_1(x)} + |\theta|^{q_2(x)} \right) dx.
\end{aligned}$$

Consequently,

$$T_\lambda(t\theta) < 0,$$

for $t < \rho^{1/(m^- - q_1^- - \epsilon)}$ with

$$0 < \delta < \min \left\{ 1, \frac{\frac{\lambda}{q_1^+} \int\limits_\omega \left(|\theta|^{q_1(x)} + |\theta|^{q_2(x)} \right) dx}{\sum_{i=1}^N \int\limits_\Omega \Phi_i(|\partial_i\theta|)dx + \frac{1}{m^-} \int\limits_\Omega |V(x)||\theta|^{m(x)}dx} \right\}.$$

The above fraction is meaningful if we have $\sum_{i=1}^N \int\limits_\Omega \Phi_i(|\partial_i\theta|)dx > 0$. Indeed, it is evident that

$$0 < |\omega| = \int\limits_\omega 1dx = \int\limits_\omega |\theta|^{q_1(x)}dx \leq \int\limits_\Omega |\theta|^{q_1(x)}dx \leq \int\limits_\Omega |\theta|^{q_1^-}dx = |\theta|_{q_1^-}^{q_1^-} \leq c^{q_1^-} \|\theta\|_{\vec{\Phi}}^{q_1^-},$$

where $c > 0$ is the constant given by the continuous embedding $E \subset L^{q_1^-}(\Omega)$. Hence, we obtain that $\|\theta\|_{\vec{\Phi}} > 0$.

Now, we focus our attention on the case when $\theta \in E$ so that $\|\theta\|_{\vec{\Phi}} < 1$, obtaining the fact that $\|\partial_i\theta\|_{\Phi_i} < 1$, for all $i \in \{1, \ldots, N\}$. Therefore, by relation (3.2) we arrive at $\sum_{i=1}^N \int_\Omega \Phi_i(|\partial_i\theta|)dx > 0$, which completes the proof. □

Proof of Theorem 2.2. Let $\lambda_* > 0$ be as in (4.8) and $\lambda \in (0, \lambda_*]$. By Lemma 4.2 we derive that there exists $\theta \in E$ such that $T_\lambda(t\theta) < 0$ for every $t > 0$ sufficiently small. At the same time, by Lemma 4.1 it results that on the ball $B_\rho(0)$ we have

$$\inf_{\partial B_\rho(0)} T_\lambda > 0.$$

These together with inequality (4.7) and hypothesis (2.2) imply

$$-\infty < \underline{c} := \inf_{\overline{B_\rho(0)}} T_\lambda < 0. \tag{4.10}$$

Now, we let $0 < \varepsilon < \inf_{\partial B_\rho(0)} T_\lambda - \inf_{B_\rho(0)} T_\lambda$ and apply Ekeland's variational principle (see [11]) to the functional $T_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ obtaining $u_\varepsilon \in \overline{B_\rho(0)}$ so that

$$T_\lambda(u_\varepsilon) \leq \inf_{\overline{B_\rho(0)}} T_\lambda + \varepsilon,$$

$$T_\lambda(u_\varepsilon) < T_\lambda(u) + \varepsilon\|u - u_\varepsilon\|_{\vec{\Phi}}, \quad u \neq u_\varepsilon.$$

Then, we infer the following inequalities:

$$T_\lambda(u_\varepsilon) \leq \inf_{\overline{B_\rho(0)}} T_\lambda + \varepsilon \leq \inf_{B_\rho(0)} T_\lambda + \varepsilon < \inf_{\partial B_\rho(0)} T_\lambda,$$

meaning that $u_\varepsilon \in B_\rho(0)$. Next, we define $\Lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by $\Lambda(u) = T_\lambda(u) + \varepsilon\|u - u_\varepsilon\|_{\vec{\Phi}}$. We have

$$\Lambda(u_\varepsilon) = T_\lambda(u_\varepsilon) < T_\lambda(u) + \varepsilon\|u - u_\varepsilon\|_{\vec{\Phi}} = \Lambda(u), \quad u \neq u_\varepsilon.$$

It results that u_ε is a minimum point of Λ, whence

$$\frac{\Lambda(u_\varepsilon + tv) - \Lambda(u_\varepsilon)}{t} \geq 0 \quad \text{for small } t > 0 \text{ and every } v \in B_\rho(0).$$

This shows that

$$\frac{T_\lambda(u_\varepsilon + tv) - T_\lambda(u_\varepsilon)}{t} + \varepsilon\|v\|_{\vec{\Phi}} \geq 0.$$

We let $t \to 0$, which means that $\frac{\langle T'_\lambda(u_\varepsilon), v\rangle}{\|v\|_{\vec{\Phi}}} \geq -\varepsilon$. Hence, we deduce that $\|T'_\lambda(u_\varepsilon)\| \leq \varepsilon$. Therefore, there is a sequence $(w_n) \subset B_\rho(0)$ so that

$$T_\lambda(w_n) \to \underline{c} \quad \text{and} \quad T'_\lambda(w_n) \to 0. \tag{4.11}$$

Evidently, (w_n) is bounded in E. Then, there is an element $w \in E$ so that, up to a subsequence, denoted again (w_n), $w_n \rightharpoonup w$ in E. In a similar fashion as in Theorem 2.1 we can prove that $w_n \to w$ in E. Consequently, using relation (4.11) we finally have

$$T_\lambda(w) = \underline{c} < 0 \quad \text{and} \quad T'_\lambda(w) = 0.$$

In other words, w is a nontrivial weak solution for problem (1.1), and the proof of theorem is complete. □

5. PROOF OF THEOREM 2.3

We first prove an auxiliary result.

Lemma 5.1. *Assume that the hypothesis of Theorem 2.3 is fulfilled. Then the functional T_λ is coercive on E.*

Proof. We focus attention on the elements $u \in E$ with $\|u\|_{\vec{\Phi}} > 1$. Taking into account hypothesis (2.3), and relation (3.4), similar arguments as those used in the proof of Lemma 3.1 show that

$$T_\lambda(u) \geq \frac{\|u\|_{\vec{\Phi}}^{(P_0)_-}}{N^{(P_0)_- -1}} - N - \left(B'_{11}\|u\|_{\vec{\Phi}}^{q_1^-} + B'_{12}\|u\|_{\vec{\Phi}}^{q_1^+} + B'_{21}\|u\|_{\vec{\Phi}}^{q_2^-} + B'_{22}\|u\|_{\vec{\Phi}}^{q_2^+}\right),$$

where $B_{ij},\ i,j = 1,2$, are positive constants. Hence, passing to the limit as $n \to \infty$, we obtain that $T_\lambda(u) \to \infty$, that is T_λ is coercive in E. □

Proof of Theorem 2.3. Lemma 5.1 ensures us that the functional T_λ is coercive on E. On the other hand, using Lemma 1 in [19], similar arguments as those used in the proof of Theorem 2 in [20] lead us to the fact that T_λ is weakly lower semicontinuous, as well. So, we have the necessary data to apply Theorem 1.2 in [32] to obtain the existence of an element $\underline{u} \in E$, global minimizer of T_λ and, consequently, the weak solution of problem (1.1).

We intend to show that $\underline{u}$ is not trivial for λ sufficiently large. To this end, let $t_0 > 1$ be a fixed real number and $\Omega_0 \subset \Omega$ be a nonempty open subset. Therefore, we infer that there is an element $\underline{v} \in C_0^\infty(\Omega) \subset E$ so that $\underline{v}(x) = t_0$ for every $x \in \overline{\Omega_0}$, and $v(x) \in [0, t_0]$ for every $x \in \Omega \setminus \Omega_0$. We have the following:

$$\begin{aligned}
T_\lambda(\underline{v}) &= \int_\Omega \sum_{i=1}^N \Phi_i(|\partial_i \underline{v}|)dx + \int_\Omega \frac{V(x)}{m(x)}|\underline{v}|^{m(x)}dx \\
&\quad - \lambda \int_\Omega \left(\frac{1}{q_1(x)}|\underline{v}|^{q_1(x)} + \frac{1}{q_2(x)}|\underline{v}|^{q_2(x)}\right)dx \\
&\leq L + \kappa - \frac{\lambda}{q_1^+}\int_\Omega \left(|\underline{v}|^{q_1(x)} + |\underline{v}|^{q_2(x)}\right)dx \\
&\leq L_0 - \frac{\lambda}{q_1^+}\int_{\Omega_0} \left(|\underline{v}|^{q_1(x)} + |\underline{v}|^{q_2(x)}\right)dx \\
&\leq L_0 - \frac{\lambda}{q_1^+}\int_{\Omega_0} \left(t_0^{q_1^-} + t_0^{q_2^-}\right)dx \\
&\leq L_0 - \frac{2\lambda|\Omega_0|t_0^{q_0^-}}{q_1^+},
\end{aligned}$$

where $L > 0$ is constant and κ is the constant given in the definition of E. Thus, there is $\lambda^* > 0$ so that $T_\lambda(\underline{v}) < 0$ for all $\lambda \in [\lambda^*, \infty)$. This, together with the fact that $\underline{u} \in E$

is global minimizer of T_λ, gives us $T_\lambda(\underline{u}) < 0$ for any $\lambda \in [\lambda^*, \infty)$. In other words, $\underline{u}$ is a nontrivial weak solution of our problem for λ sufficiently large, and this completes the proof of Theorem 2.3. □

Acknowledgments
The first author was supported by Grant PN-II-RU-TE-2014-4-1109 of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI.

REFERENCES

[1] E. Acerbi, G. Mingione, *Gradient estimates for the $p(x)$-Laplacean system*, J. Reine Angew. Math. **584** (2005), 117–148.

[2] R. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[3] D.R. Adams, L.I. Hedberg, *Function Spaces and Potential Theory*, [in:] *Grundlehren der Mathematischen Wissenschaften* (*Fundamental Principles of Mathematical Sciences*), vol. 314, Springer-Verlag, Berlin, 1996.

[4] A. Ambrosetti, P.H. Rabinowitz, *Dual variational methods in critical point theory*, J. Funct. Anal. **14** (1973), 349–381.

[5] J. Chabrowski, Y. Fu, *Existence of solutions for $p(x)$-Laplacian problems on a bounded domain*, J. Math. Anal. Appl. **306** (2005), 604–618.

[6] Y. Chen, S. Levine, R. Rao, *Functionals with p(x)-growth in image processing*, Duquesne University, Department of Mathematics and Computer Science Technical Report 2004-01, available at http://www.mathcs.duq.edu/∼sel/CLR05SIAPfinal.pdf.

[7] Ph. Clément, M. García-Huidobro, R. Manásevich, K. Schmitt, *Mountain pass type solutions for quasilinear elliptic equations*, Calc. Var. Partial Differential Equations **11** (2000), 33–62.

[8] Ph. Clément, B. de Pagter, G. Sweers, F. de Thélin, *Existence of solutions to a semilinear elliptic system through Orlicz-Sobolev spaces*, Mediterr. J. Math. **1** (2004), 241–267.

[9] G. Dankert, *Sobolev embedding theorems in Orlicz spaces*, PhD Thesis, University of Köln, 1966.

[10] T.K. Donaldson, N.S. Trudinger, *Orlicz-Sobolev spaces and imbedding theorems*, J. Funct. Anal. **8** (1971), 52–75.

[11] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl. **47** (1974), 324–353.

[12] X. Fan, *Solutions for $p(x)$-Laplacian Dirichlet problems with singular coefficients*, J. Math. Anal. Appl. **312** (2005), 464–477.

[13] X. Fan, Q. Zhang, D. Zhao, *Eigenvalues of $p(x)$-Laplacian Dirichlet problem*, J. Math. Anal. Appl. **302** (2005), 306–317.

[14] M. García-Huidobro, V.K. Le, R. Manásevich, K. Schmitt, *On principal eigenvalues for quasilinear elliptic differential operators: an Orlicz-Sobolev space setting*, NoDEA Nonlinear Differential Equations Appl. **6** (1999), 207–225.

[15] J.P. Gossez, *Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients*, Trans. Amer. Math. Soc. **190** (1974), 163–205.

[16] O. Kováčik, J. Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J. **41** (1991) 4, 592–618.

[17] M. Mihăilescu, G. Moroşanu, V. Rădulescu, *Eigenvalue problems for anisotropic elliptic equations: an Orlicz-Sobolev space setting*, Nonlinear Anal. **73** (2010), 3239–3252.

[18] M. Mihăilescu, V. Rădulescu, *A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids*, Proc. R. Soc. London Ser. A **462** (2006), 2625–2641.

[19] M. Mihăilescu, V. Rădulescu, *Eigenvalue problems associated to nonhomogeneous differential operators in Orlicz-Sobolev spaces*, Analysis and Applications **6** (2008) 1, 1–16.

[20] M. Mihăilescu, V. Rădulescu, *Existence and multiplicity of solutions for quasilinear nonhomogeneous problems: an Orlicz-Sobolev space setting*, J. Math. Anal. Appl. **330** (2007) 1, 416–432.

[21] M. Mihăilescu, V. Rădulescu, *Neumann problems associated to nonhomogeneous differential operators in Orlicz-Sobolev spaces*, Ann. Inst. Fourier **58** (2008) 6, 2087–2111.

[22] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, vol. 1034, Springer, Berlin, 1983.

[23] H. Nakano, *Modulared Semi-ordered Linear Spaces*, Maruzen Co., Ltd, Tokyo, 1950.

[24] R. O'Neill, *Fractional integration in Orlicz spaces*, Trans. Amer. Math. Soc. **115** (1965), 300–328.

[25] W. Orlicz, *Über konjugierte Exponentenfolgen*, Studia Math. **3** (1931), 200–212.

[26] M.M. Rao, Z.D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, Inc., New York, 1991.

[27] V. Rădulescu, *Nonlinear elliptic equations with variable exponent: old and new*, Nonlinear Anal., in press, DOI 10.1016/j.na.2014.11.007.

[28] V. Rădulescu, D. Repovš, *Partial Differential Equations with Variable Exponents: Variational methods and Quantitative Analysis*, CRC Press, Taylor & Francis Group, Boca Raton FL, 2015.

[29] V. Rădulescu, I. Stăncuţ, *Combined concave-convex effects in anisotropic elliptic equations with variable exponent*, NoDEA Nonlinear Differential Equations Appl., in press, DOI 10.1007/s00030-014-0288-8.

[30] V. Rădulescu, B. Zhang, *Morse theory and local linking for a nonlinear degenerate problem arising in the theory of electrorheological fluids*, Nonlinear Anal. Real World Appl. **17** (2014), 311–321.

[31] M. Ružička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Springer, Berlin, 2000.

[32] M. Struwe, *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer, Heidelberg, 1996.

[33] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.

Ionela-Loredana Stăncuţ
stancutloredana@yahoo.com

University of Craiova
Department of Mathematics
200585 Craiova, Romania

Iulia Dorotheea Stîrcu
iulia.stircu@hotmail.com

University of Craiova
Department of Mathematics
200585 Craiova, Romania

Received: January 9, 2015.
Accepted: March 23, 2015.