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INVERSION OF THE RIEMANN-LIOUVILLE OPERATOR AND ITS DUAL USING WAVELETS

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Abstract. We define and study the generalized continuous wavelet transform associated with the Riemann-Liouville operator that we use to express the new inversion formulas of the Riemann-Liouville operator and its dual.

Keywords: inverse problem, Riemann-Liouville operator, Fourier transform, wavelets.

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1. INTRODUCTION

The mean operator is defined for a continuous function f on $\mathbb{R}^2$, even with respect to the first variable by

$$\mathscr{R}_0(f)(r,x) = \frac{1}{2\pi}\int_0^{2\pi} f\left(r\sin\theta, x + r\cos\theta\right) d\theta,$$

which means that $\mathscr{R}_0(f)(r,x)$ is the mean value of f on the circle centered at $(0,x)$ and radius r. The dual operator of $\mathscr{R}_0$ is defined by

$${}^t\mathscr{R}_0(f)(r,x) = \frac{1}{\pi}\int_{\mathbb{R}} f\left(\sqrt{r^2+(x-y)^2}, y\right) dy.$$

The operator $\mathscr{R}_0$ and its dual ${}^t\mathscr{R}_0$ plays an important role and has many applications, for example, in image processing of so-called synthetic aperture radar (SAR) data [3, 16], or in the linearized inverse scattering problem in acoustics [7, 10].

In [4], the authors have generalized $\mathscr{R}_0$ and ${}^t\mathscr{R}_0$ by introducing the so-called Riemann-Liouville operator defined on the space of continuous functions on $\mathbb{R}^2$, even with respect to the first variable, by

$$\mathscr{R}_\alpha(f)(r,x) = \begin{cases} \dfrac{\alpha}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1} f\left(rs\sqrt{1-t^2}, x+rt\right)\left(1-t^2\right)^{\alpha-\frac{1}{2}}\left(1-s^2\right)^{\alpha-1} dtds & \text{if } \alpha>0, \\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} f\left(r\sqrt{1-t^2}, x+rt\right)\dfrac{dt}{\sqrt{1-t^2}} & \text{if } \alpha=0, \end{cases}$$

and its dual transform ${}^t\mathscr{R}_\alpha$, defined on $\mathscr{S}_e(\mathbb{R}^2)$ (the space of infinitely differentiable functions f on $\mathbb{R}^2$, and rapidly decreasing together with all their derivatives even with respect to the first variable) by

$${}^t\mathscr{R}_\alpha(f)(r,x) = \begin{cases} \dfrac{2\alpha}{\pi}\displaystyle\int_{r}^{+\infty}\int_{\sqrt{u^2-r^2}}^{-\sqrt{u^2-r^2}} f(u, x+v)(u^2-v^2-r^2)^{\alpha-1} u\, du\, dv & \text{if } \alpha>0, \\ \dfrac{1}{\pi}\displaystyle\int_{\mathbb{R}} f(\sqrt{r^2+(x-y)^2}, y)dy & \text{if } \alpha=0. \end{cases}$$

Many harmonic analysis results related to the Riemann-Liouville operator have been established see for example [5, 6, 14, 25] and the references therein.

The description of the range and the problem of inverting the mean operator have been studied by many authors motivated by their applications in several contemporary domains, like mechanics, physics, medical imaging modalities using the thermoacoustic tomography technic (TCT) and the radio frequency energy (RF) (see [1–3, 10, 16, 17]). This problem was taken forward by the authors in [4] for the Riemann-Liouville operator and its dual. Indeed, they have proved the same results given by Ludwig, Helgason and Solmon for the classical Radon transform on $\mathbb{R}^2$ [15, 21, 26] and for the spherical mean operator in [23], more precisely they have established that the Riemann-Liouville operator and its dual are isomorphisms on some subspaces of $\mathscr{S}_e(\mathbb{R}^2)$ and they have provided their inversion formulas in terms of integro-differential operators. Herein, we invert $\mathscr{R}_\alpha$ and ${}^t\mathscr{R}_\alpha$ using generalized wavelets associated to the Riemann-Liouville operator and classical wavelets (see [24, 29]). These new expressions are advantageous because of the large choice of wavelets, that are recognized as a powerful new mathematical tool in many areas, for example signal and image processing, time series analysis, geophysics ([8, 11–13]).

This paper is arranged as follows.

In the second section, we recall some harmonic analysis results for the Fourier transform connected with the Riemann-Liouville operator, we also give the inversion formulas of $\mathscr{R}_\alpha$ and ${}^t\mathscr{R}_\alpha$ in terms of integro-differential operators and we establish some new results that will be useful later.

In the third section, we define and study the generalized continuous wavelet transform associated with the Riemann-Liouville operator. In particular, we prove Plancherel's and Parseval's formulas and provide an inversion formula.

In the last section, we provide relations between the generalized continuous wavelet transform associated to the Riemann-Liouville, the classical continuous wavelet transform, the Riemann-Liouville operator and its dual that we use it to prove the main result of this paper, that is, the expressions of $\mathscr{R}_\alpha^{-1}$ and ${}^t\mathscr{R}_\alpha^{-1}$ using wavelet transforms.

2. THE RIEMANN-LIOUVILLE OPERATOR AND ITS DUAL

In this section, we recall some harmonic analysis results related to the Fourier transform associated with the Riemann-Liouville operator, and we check out new results that will be useful hereafter.

In [4], Baccar *et al.* have considered the function $\varphi_{\mu,\lambda}$, where $(\mu,\lambda)\in\mathbb{C}^2$, given by

$$\varphi_{\mu,\lambda}(r,x)=\mathscr{R}_\alpha\left(\cos(\mu.)\exp(-i\lambda.)\right)(r,x),$$

and they proved that for $(\mu,\lambda)\in\mathbb{C}^2$

$$\varphi_{\mu,\lambda}(r,x)=j_\alpha\left(r\sqrt{\mu^2+\lambda^2}\right)\exp\left(-i\lambda x\right),$$

where j_α is the modified Bessel function of first kind and index α (see [9, 20, 30]) given by

$$j_\alpha(z)=\Gamma(\alpha+1)\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!\Gamma(\alpha+n+1)}\left(\frac{z}{2}\right)^{2n},\quad z\in\mathbb{C}.$$

The function $\varphi_{\mu,\lambda}$ is the unique infinitely differentiable function on $\mathbb{R}^2$ even with respect to the first variable satisfying

$$\begin{cases}\Delta_1u(r,x)=-i\lambda u(r,x),\\ \Delta_2u(r,x)=-\mu^2u(r,x),\\ u(0,0)=1,\quad \frac{\partial u}{\partial r}(0,x)=0\quad\text{for all}\quad x\in\mathbb{R},\end{cases}$$

where Δ_1 and Δ_2 are the singular partial differential operators, given by

$$\begin{aligned}\Delta_1&=\frac{\partial}{\partial x},\\ \Delta_2&=\frac{\partial^2}{\partial r^2}+\frac{2\alpha+1}{r}\frac{\partial}{\partial r}-\frac{\partial^2}{\partial x^2},\quad (r,x)\in(0,+\infty)\times\mathbb{R},\quad \alpha\geq 0.\end{aligned}$$

In addition, the function $\varphi_{\mu,\lambda}$ is bounded on $[0,+\infty)\times\mathbb{R}$ if and only if (μ,λ) belongs to the set

$$\Upsilon=\mathbb{R}^2\cup\left\{(i\mu,\lambda):(\mu,\lambda)\in\mathbb{R}^2,\ |\mu|\leq|\lambda|\right\}.$$

In this case

$$\sup_{(r,x)\in\mathbb{R}^2} |\varphi_{\mu,\lambda}(r,x)| = 1. \tag{2.1}$$

Define the measure ν_α on $[0,+\infty)\times\mathbb{R}$, by

$$d\nu_\alpha(r,x) = \frac{1}{\sqrt{2\pi}2^\alpha\Gamma(\alpha+1)} r^{2\alpha+1} dr\otimes dx.$$

The translation operators $\tau_{(r,x)}, (r,x) \in [0,+\infty)\times\mathbb{R}$, associated with the Riemann-Liouville operator are defined on $L^p(d\nu_\alpha)$, $p\in[1,+\infty]$, (the Lebesgue space on $[0,+\infty)\times\mathbb{R}$ with respect to the measure ν_α with the L^p-norm denoted by $\|\cdot\|_{p,\nu_\alpha}$), by

$$\tau_{(r,x)}(f)(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)} \int_0^\pi f\left(\sqrt{r^2+s^2+2rs\cos\theta}, x+y\right)\sin^{2\alpha}\theta d\theta.$$

Then, for every $f\in L^p(d\nu_\alpha)$, $1\le p\le+\infty$ and $(r,x)\in[0,+\infty)\times\mathbb{R}$, the function $\tau_{(r,x)}(f)$ belongs to $L^p(d\nu_\alpha)$ and we have

$$\left\|\tau_{(r,x)}(f)\right\|_{p,\nu_\alpha} \le \|f\|_{p,\nu_\alpha}. \tag{2.2}$$

The convolution product of $f,g\in L^1(d\nu_\alpha)$ associated with the Riemann-Liouville operator is given by

$$f\sharp g(r,x) = \int_0^{+\infty}\int_{\mathbb{R}} \tau_{(r,-x)}(\check{f})(s,y)g(s,y)d\nu_\alpha(s,y) \quad \text{for all} \quad (r,x)\in[0,+\infty)\times\mathbb{R},$$

where $\check{f}(s,y) = f(s,-y)$.

The Young inequality for the convolution product "$\sharp$" states that if $p,q,r\in[1,+\infty]$ are such that $1/p+1/q = 1+1/r$, then for every functions f in $L^p(d\nu_\alpha)$ and g in $L^q(d\nu_\alpha)$, $f\sharp g$ belongs to the space $L^r(d\nu_\alpha)$ and we have

$$\|f\sharp g\|_{r,\nu_\alpha} \le \|f\|_{p,\nu_\alpha}\|g\|_{q,\nu_\alpha}.$$

The Fourier transform $\mathscr{F}_\alpha$ associated with the Riemann-Liouville operator is defined for f in $L^1(d\nu_\alpha)$ by

$$\mathscr{F}_\alpha(f)(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\varphi_{\mu,\lambda}(r,x)d\nu_\alpha(r,x) \quad \text{for all} \quad (\mu,\lambda)\in\Upsilon.$$

Then, we have

$$\mathscr{F}_\alpha\left(\tau_{(r,-x)}(f)\right)(\mu,\lambda) = \varphi_{\mu,\lambda}(r,x)\mathscr{F}_\alpha(f)(\mu,\lambda) \quad \text{for all} \quad (\mu,\lambda)\in\Upsilon. \tag{2.3}$$

Let Υ^+ be the subset of Υ given by

$$\Upsilon^+ = ([0,+\infty)\times\mathbb{R}) \cup \{(i\mu,\lambda) : (\mu,\lambda)\in\mathbb{R}^2, 0\le\mu\le|\lambda|\}.$$

We define on it the σ-algebra $\mathcal{B}_{\Upsilon^+} = \theta^{-1}(\mathcal{B}_{[0,+\infty)\times\mathbb{R}})$, where θ is the bijective function defined on Υ^+ by

$$\theta(\mu,\lambda) = (\sqrt{\mu^2+\lambda^2},\lambda),$$

and the measure γ_α by

$$\gamma_\alpha(A) = \nu_\alpha(\theta(A)), \quad A\in\mathcal{B}_{\Upsilon^+}.$$

We denote by $L^p(d\gamma_\alpha)$, $p\in[1,+\infty]$, the Lebesgue space on Υ^+ with respect to the measure γ_α equipped with the L^p- norm denoted by $\|\cdot\|_{p,\gamma_\alpha}$. Then, for all non negative measurable functions f on Υ^+,

$$\iint_{\Upsilon^+} f(\mu,\lambda)d\gamma_\alpha(\mu,\lambda) = \frac{1}{2^\alpha\sqrt{2\pi}\Gamma(\alpha+1)}\left\{\int_0^{+\infty}\int_{\mathbb{R}} f(\mu,\lambda)(\mu^2+\lambda^2)^\alpha \mu d\mu d\lambda \right.$$
$$\left. + \int_0^{|\lambda|}\int_{\mathbb{R}} f(i\mu,\lambda)(\lambda^2-\mu^2)^\alpha \mu d\mu d\lambda\right\}.$$

If f is a measurable function on $[0,+\infty)\times\mathbb{R}$, then the function $f\circ\theta$ is measurable on Υ^+. Furthermore, if f is a non negative or an integrable function on $[0,+\infty)\times\mathbb{R}$ with respect to the measure ν_α, we have

$$\iint_{\Upsilon^+} (f\circ\theta)(\mu,\lambda)d\gamma_\alpha(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)d\nu_\alpha(r,x).$$

Moreover, the function f belongs to $L^p(d\nu_\alpha)$ if and only if $f\circ\theta$ belongs to $L^p(d\gamma_\alpha)$ and we have

$$\|f\circ\theta\|_{p,\gamma_\alpha} = \|f\|_{p,\nu_\alpha}. \tag{2.4}$$

According to these notations, the following facts hold.

– For $(\mu,\lambda)\in\Upsilon$, we have

$$\mathscr{F}_\alpha(f)(\mu,\lambda) = \widetilde{\mathscr{F}_\alpha}(f)\circ\theta(\mu,\lambda), \tag{2.5}$$

where $\widetilde{\mathscr{F}_\alpha}$ is the Fourier-Bessel transform defined on $L^1(d\nu_\alpha)$ (see [28, 29]) by

$$\widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)j_\alpha(r\mu)exp(-i\lambda x)d\nu_\alpha(r,x), \quad (\mu,\lambda)\in\mathbb{R}^2.$$

This shows that the Fourier transform $\mathscr{F}_\alpha$ is a continuous mapping from $\mathscr{S}_e(\mathbb{R}^2)$ into itself.

– (Inversion formula) For every function f in $L^1(d\nu_\alpha)$ such that the function $\mathscr{F}_\alpha(f)$ belongs to $L^1(d\gamma_\alpha)$, we have

$$f(r,x) = \iint_{\Upsilon^+} \mathscr{F}_\alpha(f)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}d\gamma_\alpha(\mu,\lambda) \quad \text{a.e.} \tag{2.6}$$

– (Plancherel's theorem) Since the mapping $\widetilde{\mathscr{F}_\alpha}$ is an isometric isomorphism from $L^2(d\nu_\alpha)$ onto itself, then the relations (2.4) and (2.5) show that the Fourier transform $\mathscr{F}_\alpha$ is an isometric isomorphism from $L^2(d\nu_\alpha)$ into $L^2(d\gamma_\alpha)$. Namely, for every $f \in L^2(d\nu_\alpha)$, the function $\mathscr{F}_\alpha(f)$ belongs to the space $L^2(d\gamma_\alpha)$ and we have

$$\|\mathscr{F}_\alpha(f)\|_{2,\gamma_\alpha} = \|f\|_{2,\nu_\alpha}.$$

As a corollary of Plancherel's theorem, we have the following Parseval's formula for $\mathscr{F}_\alpha$.

Corollary 2.1. *For all functions f and g in $L^2(d\nu_\alpha)$, we have*

$$\int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\overline{g(r,x)}d\nu_\alpha(r,x) = \iint_{\Upsilon^+} \mathscr{F}_\alpha(f)(\mu,\lambda)\overline{\mathscr{F}_\alpha(g)(\mu,\lambda)}d\gamma_\alpha(\mu,\lambda). \tag{2.7}$$

In addition, we state the following results that will be used in the next sections.

Proposition 2.2.

1. *For f and g in $\mathscr{S}_e(\mathbb{R}^2)$ (respectively f in $L^1(d\nu_\alpha)$ and g in $L^2(d\nu_\alpha)$), we have*

$$\mathscr{F}_\alpha(f\sharp g) = \mathscr{F}_\alpha(f).\mathscr{F}_\alpha(g). \tag{2.8}$$

2. *Let f and g be in $L^2(d\nu_\alpha)$. The function $f\sharp g$ belongs to $L^2(d\nu_\alpha)$ if and only if $\mathscr{F}_\alpha(f).\mathscr{F}_\alpha(g)$ belongs to $L^2(d\gamma_\alpha)$ and we have*

$$\mathscr{F}_\alpha(f\sharp g) = \mathscr{F}_\alpha(f).\mathscr{F}_\alpha(g). \tag{2.9}$$

In [4], the authors have showed that the dual transform ${}^t\mathscr{R}_\alpha$ maps continuously $\mathscr{S}_e(\mathbb{R}^2)$ into itself and that for all f in $\mathscr{S}_e(\mathbb{R}^2)$,

$$\mathscr{F}_\alpha(f)(\mu,\lambda) = \Lambda_\alpha \circ {}^t\mathscr{R}_\alpha(f)(\mu,\lambda) \quad \text{for all} \quad (\mu,\lambda) \in \mathbb{R}^2, \tag{2.10}$$

where Λ_α is the usual Fourier transform on $\mathbb{R}^2$ defined by

$$\Lambda_\alpha(f)(\mu,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\cos(r\mu)e^{-i\lambda x}dm_\alpha(r,x),$$

and m_α the measure defined on $[0,+\infty) \times \mathbb{R}$ by

$$dm_\alpha(r,x) = \frac{1}{\sqrt{2\pi}2^\alpha\Gamma(\alpha+1)}dr \otimes dx.$$

In the sequel, we use the following notations.

- We denote by $L^p(dm_\alpha)$, $1 \leq p \leq +\infty$, the Lebesgue space on $[0,+\infty) \times \mathbb{R}$ with respect to the measure m_α equipped with the L^p-norm denoted by $\| \cdot \|_{p,m_\alpha}$.
- For a function f defined on $\mathbb{R}^2$ even with respect to the first variable, the usual translation operators, $\sigma_{(r,x)}$, $(r,x) \in \mathbb{R}^2$ is defined by

$$\sigma_{(r,x)}(f)(s,y) = \frac{1}{2}(f(r+s, y-x) + f(r-s, y-x)), \quad (s,y) \in \mathbb{R}^2. \tag{2.11}$$

- Classical convolution product "$*$" is defined for functions f and g even with respect to the first variable on $\mathbb{R}^2$, in $L^1(dm_\alpha)$, by

$$f * g(r,x) = \int_0^{+\infty} \int_{\mathbb{R}} \sigma_{(r,x)}(\check{f})(s,y) g(s,y) dm_\alpha(s,y), \quad (r,x) \in \mathbb{R}^2,$$

with $\check{f}(s,y) = f(s,-y)$.

For all functions f and g in $\mathscr{S}_e(\mathbb{R}^2)$, the function $f * g$ belongs to $\mathscr{S}_e(\mathbb{R}^2)$ and we have

$$\Lambda_\alpha(f * g) = \Lambda_\alpha(f)\Lambda_\alpha(g). \tag{2.12}$$

Moreover, for all f and g in $L^2(dm_\alpha)$, the function $f * g$ belongs to $L^2(dm_\alpha)$ if and only if $\Lambda_\alpha(f)\Lambda_\alpha(g)$ belongs to $L^2(dm_\alpha)$ and the relation (2.12) holds.

Proposition 2.3. *For f and g in $\mathscr{S}_e(\mathbb{R}^2)$, we have*

$${}^t\mathscr{R}_\alpha(f \sharp g) = {}^t\mathscr{R}_\alpha(f) * {}^t\mathscr{R}_\alpha(g).$$

Proof. Since f and g are in $\mathscr{S}_e(\mathbb{R}^2)$, we get from relations (2.8), (2.10) and (2.12)

$$\begin{aligned}\Lambda_\alpha({}^t\mathscr{R}_\alpha(f \sharp g)) &= \mathscr{F}_\alpha(f \sharp g) = \mathscr{F}_\alpha(f)\mathscr{F}_\alpha(g) \\ &= \Lambda_\alpha({}^t\mathscr{R}_\alpha(f))\Lambda_\alpha({}^t\mathscr{R}_\alpha(g)) = \Lambda_\alpha({}^t\mathscr{R}_\alpha(f) * {}^t\mathscr{R}_\alpha(g)).\end{aligned}$$

The result follows from the fact that Λ_α is an isomorphism from $\mathscr{S}_e(\mathbb{R}^2)$ onto itself. □

We denote by

- $\mathcal{N}$ the subspace of $\mathscr{S}_e(\mathbb{R}^2)$ consisting of functions f satisfying

$$\forall k \in \mathbb{N}\, \forall x \in \mathbb{R}: \quad \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^k f(0,x) = 0,$$

- $\mathscr{S}_e^0(\mathbb{R}^2)$ the subspace of $\mathscr{S}_e(\mathbb{R}^2)$ consisting of functions f such that

$$\mathrm{Supp}\widetilde{\mathscr{F}_\alpha}(f) \subset \left\{(\mu,\lambda) \in \mathbb{R}^2 : |\mu| \geq |\lambda|\right\},$$

- $\mathscr{S}_{e,0}(\mathbb{R}^2)$ the subspace of $\mathscr{S}_e(\mathbb{R}^2)$ consisting of functions f such that

$$\forall k \in \mathbb{N}\, \forall x \in \mathbb{R}: \quad \int_0^{+\infty} f(r,x) r^{2k} dr = 0.$$

To palliate the fact that the dual transform ${}^t\mathscr{R}_\alpha$ is not injective when applied to $\mathscr{S}_e(\mathbb{R}^2)$, the authors, in [4], have proved the following results.

Theorem 2.4.

1. *The transform* ${}^t\mathscr{R}_\alpha$ *is an isomorphism from* $\mathscr{S}_e^0(\mathbb{R}^2)$ *into* $\mathscr{S}_{e,0}(\mathbb{R}^2)$.
2. *The Riemann-Liouville operator* $\mathscr{R}_\alpha$ *is an isomorphism from* $\mathscr{S}_{e,0}(\mathbb{R}^2)$ *into* $\mathscr{S}_e^0(\mathbb{R}^2)$.

Lemma 2.5.

1. *The mapping* Λ_α *is an isomorphism from* $\mathscr{S}_{e,0}(\mathbb{R}^2)$ *into* $\mathscr{N}$.
2. *The Fourier transform* $\mathscr{F}_\alpha$ *associated with the Riemann-Liouville is an isomorphism from* $\mathscr{S}_e^0(\mathbb{R}^2)$ *into* $\mathscr{N}$.

Theorem 2.6.

1. *The operator* $\mathscr{K}_\alpha^1$ *defined by*
$$\mathscr{K}_\alpha^1(f)(r,x) = \Lambda_\alpha^{-1}\left(\frac{\pi}{2^{2\alpha+1}(\Gamma(\alpha+1))^2}(\mu^2+\lambda^2)^\alpha|\mu|\Lambda_\alpha(f)\right)(r,x) \tag{2.13}$$
is an automorphism of $\mathscr{S}_{e,0}(\mathbb{R}^2)$.
2. *The operator* $\mathscr{K}_\alpha^2$ *defined by*
$$\mathscr{K}_\alpha^2(f)(r,x) = \mathscr{F}_\alpha^{-1}\left(\frac{\pi}{2^{2\alpha+1}(\Gamma(\alpha+1))^2}(\mu^2+\lambda^2)^\alpha|\mu|\mathscr{F}_\alpha(f)\right)(r,x) \tag{2.14}$$
is an automorphism of $\mathscr{S}_e^0(\mathbb{R}^2)$.

The inversion formulas for the Riemann-Liouville operator and its dual in terms of the integro-differential operators $\mathscr{K}_\alpha^1$ and $\mathscr{K}_\alpha^2$ are given by the following theorem.

Theorem 2.7.

1. *For* $f \in \mathscr{S}_{e,0}(\mathbb{R}^2)$ *and* $g \in \mathscr{S}_e^0(\mathbb{R}^2)$, *we have the inversion formula for* $\mathscr{R}_\alpha$
$$g = \mathscr{R}_\alpha\mathscr{K}_\alpha^1\,{}^t\mathscr{R}_\alpha(g),$$
$$f = \mathscr{K}_\alpha^1\,{}^t\mathscr{R}_\alpha\mathscr{R}_\alpha(f).$$
2. *For* $f \in \mathscr{S}_{e,0}(\mathbb{R}^2)$ *and* $g \in \mathscr{S}_e^0(\mathbb{R}^2)$, *we have the inversion formula for* ${}^t\mathscr{R}_\alpha$
$$f = {}^t\mathscr{R}_\alpha\mathscr{K}_\alpha^2\mathscr{R}_\alpha(f),$$
$$g = \mathscr{K}_\alpha^2\mathscr{R}_\alpha\,{}^t\mathscr{R}_\alpha(g).$$

The following result gives a connection between the maps $\mathscr{K}_\alpha^1$ and $\mathscr{K}_\alpha^2$.

Corollary 2.8. *For* f *in* $\mathscr{S}_e^0(\mathbb{R}^2)$, *we have*
$$\mathscr{K}_\alpha^2(f) = {}^t\mathscr{R}_\alpha^{-1} \circ \mathscr{K}_\alpha^1 \circ {}^t\mathscr{R}_\alpha(f).$$

Proof. From relations (2.10), (2.13) and (2.14) and using the fact that the function f belongs to $\mathscr{S}_e^0(\mathbb{R}^2)$, we get

$$\begin{aligned}\mathscr{K}_\alpha^2(f) &= \mathscr{F}_\alpha^{-1}\left(\frac{\pi}{2^{2\alpha+1}(\Gamma(\alpha+1))^2}(\mu^2+\lambda^2)^\alpha|\mu|\mathscr{F}_\alpha(f)\right)\\ &= {}^t\mathscr{R}_\alpha^{-1}\circ\Lambda_\alpha^{-1}\left(\frac{\pi}{2^{2\alpha+1}(\Gamma(\alpha+1))^2}(\mu^2+\lambda^2)^\alpha|\mu|\Lambda_\alpha\circ{}^t\mathscr{R}_\alpha(f)\right)\\ &= {}^t\mathscr{R}_\alpha^{-1}\circ\mathscr{K}_\alpha^1\circ{}^t\mathscr{R}_\alpha(f).\end{aligned}$$

□

Next, we provide some properties involving the convolution products "$*$" and "$\sharp$".

Proposition 2.9.

1. *For every f in $\mathscr{S}_{e,0}(\mathbb{R}^2)$ and g in $\mathscr{S}_e(\mathbb{R}^2)$, the function $f*g$ belongs to $\mathscr{S}_{e,0}(\mathbb{R}^2)$.*
2. *For every f in $\mathscr{S}_e^0(\mathbb{R}^2)$ and g in $\mathscr{S}_e(\mathbb{R}^2)$, the function $f\sharp g$ belongs to $\mathscr{S}_e^0(\mathbb{R}^2)$.*

Proof.

1. The result follows from the relation (2.12) and (1) of Lemma 2.5.
2. According to (1) of Theorem 2.4, the function ${}^t\mathscr{R}_\alpha(f)$ belongs to $\mathscr{S}_{e,0}(\mathbb{R}^2)$. Then from (1) the convolution product ${}^t\mathscr{R}_\alpha(f)*{}^t\mathscr{R}_\alpha(g)$ is also in $\mathscr{S}_{e,0}(\mathbb{R}^2)$. In virtue of Proposition 2.3, we have

$$ {}^t\mathscr{R}_\alpha(f\sharp g) = {}^t\mathscr{R}_\alpha(f)*{}^t\mathscr{R}_\alpha(g).$$

   We deduce the result using again (1) of Theorem 2.4. □

Proposition 2.10.

1. *For all $f\in\mathscr{S}_{e,0}(\mathbb{R}^2)$ and $g\in\mathscr{S}_e(\mathbb{R}^2)$, we have*

$$\mathscr{K}_\alpha^1(f*g) = \mathscr{K}_\alpha^1(f)*g. \tag{2.15}$$

2. *For all $f\in\mathscr{S}_e^0(\mathbb{R}^2)$ and $g\in\mathscr{S}_e(\mathbb{R}^2)$, we have*

$$\mathscr{K}_\alpha^2(f\sharp g) = \mathscr{K}_\alpha^2(f)\sharp g. \tag{2.16}$$

Proof.

1. From the expression of $\mathscr{K}_\alpha^1$ given by the relation (2.13) and (1) of Proposition 2.9, we have

$$\begin{aligned}\mathscr{K}_\alpha^1(f*g) &= \Lambda_\alpha^{-1}\left(\frac{\pi}{2^{2\alpha+1}(\Gamma(\alpha+1))^2}(\mu^2+\lambda^2)^\alpha|\mu|\Lambda_\alpha(f*g)\right)\\ &= \Lambda_\alpha^{-1}\left(\frac{\pi}{2^{2\alpha+1}(\Gamma(\alpha+1))^2}(\mu^2+\lambda^2)^\alpha|\mu|\Lambda_\alpha(f)\Lambda_\alpha(g)\right)\\ &= \Lambda_\alpha^{-1}\left(\frac{\pi}{2^{2\alpha+1}(\Gamma(\alpha+1))^2}(\mu^2+\lambda^2)^\alpha|\mu|\Lambda_\alpha(f)\right)*g = \mathscr{K}_\alpha^1(f)*g.\end{aligned}$$

2. The proof is the same as in (1). □

Corollary 2.11. *Let f and g be in $\mathscr{S}_{e,0}(\mathbb{R}^2)$. Then,*

$$\mathscr{R}_\alpha(f * g) = \mathscr{R}_\alpha(f)\sharp\ {}^t\mathscr{R}_\alpha^{-1}(g). \tag{2.17}$$

Proof. Since the function ${}^t\mathscr{R}_\alpha^{-1}(g)$ belongs to the subspace $\mathscr{S}_e^0(\mathbb{R}^2)$, then from (2) of Proposition 2.9 the function $\mathscr{R}_\alpha(f)\sharp\ {}^t\mathscr{R}_\alpha^{-1}(g)$ belongs to $\mathscr{S}_e^0(\mathbb{R}^2)$. Using the inversion formula for $\mathscr{R}_\alpha$ given in Theorem 2.7, Proposition 2.3 and the relation (2.15), we get

$$\begin{aligned}\mathscr{R}_\alpha^{-1}(\mathscr{R}_\alpha(f)\sharp^t\mathscr{R}_\alpha^{-1}(g)) &= \mathscr{K}_\alpha^1\,{}^t\mathscr{R}_\alpha(\mathscr{R}_\alpha(f)\sharp\ {}^t\mathscr{R}_\alpha^{-1}(g))\\ &= \mathscr{K}_\alpha^1({}^t\mathscr{R}_\alpha\mathscr{R}_\alpha(f) * g) = \mathscr{K}_\alpha^1\,{}^t\mathscr{R}_\alpha\mathscr{R}_\alpha(f) * g.\end{aligned}$$

Thus, from Theorem 2.7 we have

$$\mathscr{R}_\alpha^{-1}(\mathscr{R}_\alpha(f)\sharp\ \ {}^t\mathscr{R}_\alpha^{-1}(g)) = f * g,$$

and therefore

$$\mathscr{R}_\alpha(f)\sharp\ \ {}^t\mathscr{R}_\alpha^{-1}(g) = \mathscr{R}_\alpha(f * g). \qquad \square$$

3. CONTINUOUS WAVELET TRANSFORM ASSOCIATED WITH THE RIEMANN-LIOUVILLE OPERATOR

In this section, we define and study the wavelets and the continuous wavelet transforms connected with the operator $\mathscr{R}_\alpha$. Using the harmonic analysis results related to the Fourier transform, we establish in particular an inversion formula and the Plancherel theorem ([18, 19, 22]).

Let a be a positive real number. We define the dilation operator D_a of a function ψ by

$$D_a(\psi)(x,y) = \frac{1}{a^{\alpha+3/2}}\psi\Big(\frac{x}{a},\frac{y}{a}\Big), \quad (x,y)\in\mathbb{C}^2.$$

- For all $a, b > 0$, we have $D_a \circ D_b = D_{ab}$.
- For every $a > 0$, the operator D_a is an isometric isomorphism from $L^2(d\nu_\alpha)$ onto itself.

Property 3.1. *Denote by $\langle\cdot,\cdot\rangle_{\nu_\alpha}$ the inner product of $L^2(d\nu_\alpha)$. Let $a > 0$. Then for all ψ, φ in $L^2(d\nu_\alpha)$, we have:*

1. $\langle D_a(\psi),\varphi\rangle_{\nu_\alpha} = \langle\psi, D_{\frac{1}{a}}(\varphi)\rangle_{\nu_\alpha}$,
2. $D_a(\tau_{(r,x)}(\psi)) = \tau_{(ar,ax)}(D_a(\psi))$,
3. $\mathscr{F}_\alpha(D_a(\psi)) = D_{\frac{1}{a}}(\mathscr{F}_\alpha(\psi))$.

Definition 3.2. Let ψ be a measurable function on $[0,+\infty)\times\mathbb{R}$. We say that ψ is a generalized admissible wavelet associated to the Riemann-Liouville operator if for almost every $(\mu,\lambda)\in\Upsilon\backslash\{(0,0)\}$, we have

$$0 < C_\psi = \int_0^{+\infty}\Big|\mathscr{F}_\alpha(\psi)\Big(\frac{\mu}{a},\frac{\lambda}{a}\Big)\Big|^2\frac{da}{a} < +\infty.$$

Let ψ be an admissible wavelet in $L^p(d\nu_\alpha)$, $1 \le p \le +\infty$. For all $a > 0$ and $(r,x) \in [0,+\infty) \times \mathbb{R}$, we define the function $\psi_{a,r,x}$ by

$$\psi_{a,r,x}(s,y) = \tau_{(r,-x)}(D_a(\psi))(s,y) \quad \text{for all} \quad (s,y) \in [0,+\infty) \times \mathbb{R}. \tag{3.1}$$

Proposition 3.3.

1. *For all ψ in $L^p(d\nu_\alpha)$, $1 \le p \le +\infty$ and for all $(a,r,x) \in (0,+\infty) \times [0,+\infty) \times \mathbb{R}$, the function $\psi_{a,r,x}$ belongs to $L^p(d\nu_\alpha)$, and we have*
$$\|\psi_{a,r,x}\|_{p,\nu_\alpha} \le a^{\frac{2\alpha+3}{p} - (\alpha+\frac{3}{2})} \|\psi\|_{p,\nu_\alpha}. \tag{3.2}$$
2. *For all generalized admissible wavelets ψ in $L^2(d\nu_\alpha)$ and for all $(a,r,x) \in (0,+\infty) \times [0,+\infty) \times \mathbb{R}$, the function $\psi_{a,r,x}$ is a generalized admissible wavelet in $L^2(d\nu_\alpha)$ and we have*
$$C_{\psi_{a,r,x}} \le a^{2\alpha+3} C_\psi.$$

Proof.

1. The case $p = +\infty$ is trivial. Let $1 \le p < +\infty$. From the relation (2.2) and by a change of variables, we get
$$\|\psi_{a,r,x}\|^p_{p,\nu_\alpha} \le \int_0^{+\infty} \int_{\mathbb{R}} |D_a(\psi)(s,y)|^p d\nu_\alpha(s,y) \le a^{2\alpha+3-(\alpha+3/2)p} \|\psi\|^p_{p,\nu_\alpha}.$$
2. Using (3) of Property 3.1, for all $(\mu,\lambda) \in \Upsilon$, we have
$$\mathscr{F}_\alpha(\psi_{a,r,x})(\mu,\lambda) = \varphi_{\mu,\lambda}(r,x)\mathscr{F}_\alpha(D_a(\psi))(\mu,\lambda) = \varphi_{\mu,\lambda}(r,x) a^{\alpha+3/2} \mathscr{F}_\alpha(\psi)(a\mu, a\lambda).$$
Thus,
$$\begin{aligned}\int_0^{+\infty} \Big|\mathscr{F}_\alpha(\psi_{a,r,x})\Big(\frac{\mu}{b}, \frac{\lambda}{b}\Big)\Big|^2 \frac{db}{b} &= \int_0^{+\infty} a^{2\alpha+3} \Big|\varphi_{\frac{\mu}{b},\frac{\lambda}{b}}(r,x)|^2 |\mathscr{F}_\alpha(\psi)\Big(\frac{a}{b}\mu, \frac{a}{b}\lambda\Big)\Big|^2 \frac{db}{b}, \\ &= a^{2\alpha+3} \int_0^{+\infty} |\varphi_{\frac{c}{a}\mu,\frac{c}{a}\lambda}(r,x)|^2 |\mathscr{F}_\alpha(\psi)(c\mu, c\lambda)|^2 \frac{dc}{c}.\end{aligned}$$
Using the relation (2.1), we get
$$C_{\psi_{a,r,x}} \le a^{2\alpha+3} C_\psi. \qquad \square$$

Example 3.4. Let us consider the function

$$\psi(r,x) = \frac{1}{2}(x^2 - r^2 + 2\alpha + 1) \exp\Big(-\frac{r^2}{2} - \frac{x^2}{2}\Big).$$

By a simple calculus, we get

$$\mathscr{F}_\alpha(\psi)(\mu,\lambda) = \Big(\frac{\mu^2}{2} + \lambda^2\Big) \exp -\Big(\frac{\mu^2}{2} + \lambda^2\Big).$$

Then, for all $(\mu,\lambda)\in\Upsilon\setminus\{(0,0)\}$,

$$C_\psi=\frac{1}{4}.$$

The function ψ is a generalized admissible wavelet associated with the Riemann--Liouville operator in $\mathscr{S}_e(\mathbb{R}^2)$.

Definition 3.5. Let ψ be a generalized admissible wavelet in $L^2(d\nu_\alpha)$. The generalized continuous wavelet transform T_ψ associated with the Riemann-Liouville operator is defined for a function f in $L^p(d\nu_\alpha)$, $p=1,2$, and for all $(a,r,x)\in(0,+\infty)\times[0,+\infty)\times\mathbb{R}$, by

$$T_\psi(f)(a,r,x)=\int_0^{+\infty}\int_{\mathbb{R}} f(s,y)\overline{\psi_{a,r,x}(s,y)}d\nu_\alpha(s,y).$$

We have the following expressions of the transform T_ψ.

1. For f in $L^p(d\nu_\alpha)$, $p=1,2$,

$$T_\psi(f)(a,r,x)=f\sharp D_a(\overline{\check{\psi}})(r,x). \tag{3.3}$$

2. For f in $L^2(d\nu_\alpha)$,

$$T_\psi(f)(a,r,x)=\langle f,\psi_{a,r,x}\rangle_{\nu_\alpha}. \tag{3.4}$$

We denote by ρ_α the measure defined on $(0,+\infty)\times[0,+\infty)\times\mathbb{R}$, by

$$d\rho_\alpha(a,r,x)=\frac{1}{a^{2\alpha+4}}da\otimes d\nu_\alpha(r,x),$$

and $L^p(d\rho_\alpha)$, $p\in[2,+\infty]$, the Lebesgue space on $(0,+\infty)\times[0,+\infty)\times\mathbb{R}$ with respect to the measure ρ_α equipped with the L^p-norm denoted by $\|\cdot\|_{p,\rho_\alpha}$.

Theorem 3.6. *Let ψ be a generalized admissible wavelet in $L^2(d\nu_\alpha)$.*

1. (*Plancherel's formula for T_ψ*) *For every function f in $L^2(d\nu_\alpha)$, we have*

$$\int_0^{+\infty}\int_{\mathbb{R}}|f(r,x)|^2d\nu_\alpha(r,x)=\frac{1}{C_\psi}\int_0^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}}|T_\psi(f)(a,r,x)|^2d\rho_\alpha(a,r,x). \tag{3.5}$$

2. (*Parseval's formula for T_ψ*) *For all functions f and g in $L^2(d\nu_\alpha)$, we have*

$$\int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\overline{g(r,x)}d\nu_\alpha(r,x)=\frac{1}{C_\psi}\int_0^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}}T_\psi(f)(a,r,x)\overline{T_\psi(g)(a,r,x)}d\rho_\alpha(a,r,x). \tag{3.6}$$

Proof.

1. Using relations (2.9), (3.3) and applying Plancherel's theorem for the Fourier transform $\mathscr{F}_\alpha$, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}} |T_\psi(f)(a,r,x)|^2 d\rho_\alpha(a,r,x) \\
&= \int_0^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}} |f\sharp D_a(\overline{\check{\psi}})(r,x)|^2 d\rho_\alpha(a,r,x) \\
&= \int_0^{+\infty}\int\int_{\Upsilon_+} |\mathscr{F}_\alpha(f)(\mu,\lambda)|^2 |\mathscr{F}_\alpha(D_a(\psi))(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda)\frac{da}{a^{2\alpha+4}} \\
&= \int\int_{\Upsilon_+} |\mathscr{F}_\alpha(f)(\mu,\lambda)|^2 \left(\int_0^{+\infty} |a^{\alpha+\frac{3}{2}}\mathscr{F}_\alpha(\psi)(\mu a,\lambda a)|^2 \frac{da}{a^{2\alpha+4}}\right) d\gamma_\alpha(\mu,\lambda) \\
&= C_\psi \int_0^{+\infty}\int_{\mathbb{R}} |f(r,x)|^2 d\nu_\alpha(r,x).
\end{aligned}$$

2. We deduce the result from (1) and from the polarization identity. □

Theorem 3.7. *Let ψ be a generalized admissible wavelet in $L^2(d\nu_\alpha)$. For every $f \in L^2(d\nu_\alpha)$, the function $T_\psi(f)$ belongs to $L^p(d\rho_\alpha)$, $p \in [2,+\infty]$, and we have*

$$\|T_\psi(f)\|_{p,\rho_\alpha} \leq N(\psi)\|f\|_{2,\nu_\alpha},$$

where $N(\psi) = (\|\psi\|^2_{2,\nu_\alpha} + C_\psi)^{\frac{1}{2}}$.

Proof. For $p = 2$, the Plancherel's formula for the generalized continuous wavelet transform (3.5) gives

$$\|T_\psi(f)\|_{2,\rho_\alpha} = C_\psi^{\frac{1}{2}}\|f\|_{2,\nu_\alpha} \leq N(\psi)\|f\|_{2,\nu_\alpha}.$$

For $p = +\infty$, from relations (3.2) and (3.4) we have

$$|T_\psi(f)(a,r,x)| \leq \|\psi_{a,r,x}\|_{2,\nu_\alpha}\|f\|_{2,\nu_\alpha} \leq \|\psi\|_{2,\nu_\alpha}\|f\|_{2,\nu_\alpha},$$

so

$$\|T_\psi(f)\|_{\infty,\rho_\alpha} \leq N(\psi)\|f\|_{2,\nu_\alpha}.$$

We get the result from the Riesz-Thorin theorem ([27]). □

In the following, we establish a reconstruction and an inversion formula for T_ψ.

Theorem 3.8 (Reconstruction formula)**.** *Let ψ be a positive generalized admissible wavelet in $L^2(d\nu_\alpha)$. Then, for all f in $L^2(d\nu_\alpha)$, we have*

$$f(\cdot,\cdot)=\frac{1}{C_\psi}\int_0^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}} T_\psi(f)(a,r,x)\psi_{a,r,x}(\cdot,\cdot)d\rho_\alpha(a,r,x)$$

weakly in $L^2(d\nu_\alpha)$.

Proof. From the relation (3.6) and Fubini's theorem we have for all $g\in L^2(d\nu_\alpha)$,

$$\begin{aligned}&\int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\overline{g(r,x)}d\nu_\alpha(r,x)\\&=\frac{1}{C_\psi}\int_0^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}} T_\psi(f)(a,r,x)\left(\int_0^{+\infty}\int_{\mathbb{R}}\overline{g(t,y)}\psi_{a,r,x}(t,y)d\nu_\alpha(t,y)\right)d\rho_\alpha(a,r,x)\\&=\int_0^{+\infty}\int_{\mathbb{R}}\left(\frac{1}{C_\psi}\int_0^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}} T_\psi(f)(a,r,x)\psi_{a,r,x}(t,y)d\rho_\alpha(a,r,x)\right)\overline{g(t,y)}d\nu_\alpha(t,y),\end{aligned}$$

which gives the result. □

Theorem 3.9. *Let ψ be a generalized admissible wavelet in $L^2(d\nu_\alpha)$. For all $f\in L^1(d\nu_\alpha)$ (respectively $f\in L^2(d\nu_\alpha)$) such that $\mathscr{F}_\alpha(f)\in L^1(d\gamma_\alpha)$ (respectively $\mathscr{F}_\alpha(f)\in L^1(d\gamma_\alpha)\cap L^\infty(d\gamma_\alpha)$), we have*

$$f(s,y)=\frac{1}{C_\psi}\int_0^{+\infty}\left(\int_0^{+\infty}\int_{\mathbb{R}} T_\psi(f)(a,r,x)\psi_{a,r,x}(s,y)d\nu_\alpha(r,x)\right)\frac{da}{a^{2\alpha+4}}\quad a.e.,$$

where both the inner and the outer integrals are absolutely convergent but possibly not the double integral.

Proof.

1. Suppose that $f\in L^1(d\nu_\alpha)$ is such that $\mathscr{F}_\alpha(f)\in L^1(d\gamma_\alpha)$. From the relation (3.1) we have

$$T_\psi(f)(a,r,x)\psi_{a,r,x}(s,y)=(f\sharp D_a(\overline{\check{\psi}}))(r,x)\tau_{(s,-y)}D_a(\check{\psi})(r,x).$$

   Then, for all $a>0$ and $(s,y)\in[0,+\infty)\times\mathbb{R}$, the function

$$(r,x)\longmapsto T_\psi(f)(a,r,x)\psi_{a,r,x}(s,y)$$

belongs to $L^1(d\nu_\alpha)$, because the functions $f\sharp D_a(\overline{\check{\psi}})$ and $\tau_{(s,-y)}\left(D_a(\check{\psi})\right)$ clearly belong to $L^2(d\nu_\alpha)$. On the other hand, we have from the relations (2.3) and (2.8)

$$\mathscr{F}_\alpha\left(\tau_{(s,-y)}D_a(\check{\psi})\right)(\mu,\lambda) = \varphi_{\mu,\lambda}(s,y)\mathscr{F}_\alpha(D_a(\check{\psi}))(\mu,\lambda)$$

and

$$\begin{aligned}\mathscr{F}_\alpha(f\sharp D_a(\overline{\check{\psi}}))(\mu,\lambda) &= \mathscr{F}_\alpha(f)(\mu,\lambda)\mathscr{F}_\alpha(D_a(\overline{\check{\psi}}))(\mu,\lambda)\\ &= \mathscr{F}_\alpha(f)(\mu,\lambda)\overline{\mathscr{F}_\alpha(D_a(\psi))}(\mu,\lambda).\end{aligned}$$

Thus, applying Parseval's formula for the Fourier transform $\mathscr{F}_\alpha$ given by the relation (2.7), we get

$$\begin{aligned}&\int_0^{+\infty}\int_{\mathbb{R}} T_\psi(f)(a,r,x)\psi_{a,r,x}(s,y)d\nu_\alpha(r,x)\\ &\quad= \int_0^{+\infty}\int_{\mathbb{R}}(f\sharp D_a(\overline{\check{\psi}}))(r,x)\overline{\tau_{(s,-y)}(D_a(\overline{\check{\psi}}))}(r,x)d\nu_\alpha(r,x)\\ &\quad= \int\int_{\Upsilon^+}\overline{\varphi_{\mu,\lambda}(s,y)}\mathscr{F}_\alpha(f)(\mu,\lambda)\left|\mathscr{F}_\alpha(D_a(\psi))(\mu,\lambda)\right|^2 d\gamma_\alpha(\mu,\lambda).\end{aligned}$$

The relation (2.1) yields

$$\begin{aligned}&\frac{1}{C_\psi}\int_0^{+\infty}\left|\int_0^{+\infty}\int_{\mathbb{R}} T_\psi(f)(a,r,x)\psi_{a,r,x}(s,y)d\nu_\alpha(r,x)\right|\frac{da}{a^{2\alpha+4}}\\ &\quad\le \int\int_{\Upsilon^+}\left|\mathscr{F}_\alpha(f)(\mu,\lambda)\right|\left(\frac{1}{C_\psi}\int_0^{+\infty}\left|\mathscr{F}_\alpha(\psi)\Big(\frac{\mu}{a},\frac{\lambda}{a}\Big)\right|^2\frac{da}{a}\right)d\gamma_\alpha(\mu,\lambda) = \|\mathscr{F}_\alpha(f)\|_{1,\gamma_\alpha}.\end{aligned}$$

Then, by Fubini's theorem and the inversion formula for $\mathscr{F}_\alpha$ (2.6), we get

$$\begin{aligned}&\frac{1}{C_\psi}\int_0^{+\infty}\left(\int_0^{+\infty}\int_{\mathbb{R}} T_\psi(f)(a,r,x)\psi_{a,r,x}(s,y)d\nu_\alpha(r,x)\right)\frac{da}{a^{2\alpha+4}}\\ &= \int\int_{\Upsilon^+}\mathscr{F}_\alpha(f)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(s,y)}\left(\frac{1}{C_\psi}\int_0^{+\infty}\left|\mathscr{F}_\alpha(\psi)\Big(\frac{\mu}{a},\frac{\lambda}{a}\Big)\right|^2\frac{da}{a}\right)d\gamma_\alpha(\mu,\lambda)\\ &= \int\int_{\Upsilon^+}\mathscr{F}_\alpha(f)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(s,y)}d\gamma_\alpha(\mu,\lambda) = f(s,y) \quad \text{a.e.}\end{aligned}$$

2. Let $f \in L^2(d\nu_\alpha)$ be such that $\mathscr{F}_\alpha(f) \in L^1(d\gamma_\alpha) \cap L^\infty(d\gamma_\alpha)$, the function $\mathscr{F}_\alpha(f)\mathscr{F}_\alpha(D_a(\overline{\check{\psi}}))$ belongs to $L^2\,(d\gamma_\alpha)$. Then from the relation (2.9) the function $f\sharp D_a(\overline{\check{\psi}})$ belongs to $L^2\,(d\nu_\alpha)$ and we have

$$\mathscr{F}_\alpha(f\sharp D_a(\overline{\check{\psi}}))(\mu,\lambda) = \mathscr{F}_\alpha(f)(\mu,\lambda)\overline{\mathscr{F}_\alpha(D_a(\psi))}(\mu,\lambda).$$

The remainder of the proof is the same as in (1). □

4. INVERSION OF $\mathscr{R}_\alpha$ AND ${}^t\mathscr{R}_\alpha$ USING WAVELETS

In this section, we will give the inversion formulas for the Riemann-Liouville operator and its dual in terms of continuous wavelet transforms ([18]). We recall first some facts for the classical wavelet transforms.

A measurable function ψ on $\mathbb{R}^2$ is said to be a classical admissible wavelet if for almost every $(\mu,\lambda) \in \mathbb{R}^2 \setminus \{(0,0)\}$, we have

$$0 < A_\psi = \int_0^{+\infty} \left|\Lambda_\alpha(\psi)\Big(\frac{\mu}{a},\frac{\lambda}{a}\Big)\right|^2 \frac{da}{a} < +\infty. \tag{4.1}$$

For a classical admissible wavelet ψ in $L^2(dm_\alpha)$, the classical continuous wavelet transform S_ψ is defined for a function $f \in L^p(dm_\alpha)$, $p = 1,2$, and for all $(a,r,x) \in (0,+\infty) \times [0,+\infty) \times \mathbb{R}$ by

$$S_\psi(f)(a,r,x) = \int_0^{+\infty}\int_{\mathbb{R}} f(s,y)\overline{\widetilde{\psi}_{a,r,x}(s,y)}dm_\alpha(s,y),$$

with

$$\widetilde{\psi}_{a,r,x}(s,y) = \sigma_{(r,x)}(H_a(\psi))(s,y),$$

where $\sigma_{(r,x)}$ are the translation operators given by the relation (2.11) and H_a $(a > 0)$, is the dilation operator defined by

$$H_a(\psi)(r,x) = \frac{1}{a}\psi\Big(\frac{r}{a},\frac{x}{a}\Big).$$

This transform can also be written in the form

$$S_\psi(f)(a,r,x) = f * H_a(\overline{\check{\psi}})(r,x),$$

and have the following inversion formula

$$f(s,y) = \frac{1}{A_\psi}\int_0^{+\infty}\left(\int_0^{+\infty}\int_{\mathbb{R}} S_\psi(f)(a,r,x)\widetilde{\psi}_{a,r,x}(s,y)dm_\alpha(r,x)\right)\frac{da}{a^3} \quad a.e. \tag{4.2}$$

when f and $\Lambda_\alpha(f)$ are integrable on $[0,+\infty) \times \mathbb{R}$ with respect to the measure m_α.

Proposition 4.1.

1. *For ψ in $\mathscr{S}_e(\mathbb{R}^2)$, we have*

$$ {}^t\mathscr{R}_\alpha D_a(\psi) = a^{2\alpha+1} D_a \, {}^t\mathscr{R}_\alpha(\psi). $$

2. *For ψ in $\mathscr{S}_{e,0}(\mathbb{R}^2)$, we have*

$$ H_a \mathscr{K}_\alpha^1(\psi) = a^{2\alpha+1} \mathscr{K}_\alpha^1 H_a(\psi). $$

3. *For ψ in $\mathscr{S}_e^0(\mathbb{R}^2)$, we have*

$$ D_a \mathscr{K}_\alpha^2(\psi) = a^{2\alpha+1} \mathscr{K}_\alpha^2 D_a(\psi). $$

Proof.

1. From the relation (2.10) we have

$$ \Lambda_\alpha \, {}^t\mathscr{R}_\alpha D_a(\psi) = \mathscr{F}_\alpha D_a(\psi) = D_{\frac{1}{a}} \mathscr{F}_\alpha(\psi) = D_{\frac{1}{a}} \Lambda_\alpha \, {}^t\mathscr{R}_\alpha(\psi) = a^{2\alpha+1} \Lambda_\alpha D_a \, {}^t\mathscr{R}_\alpha(\psi). $$

   The result follows from the fact that Λ_α is an automorphism on $\mathscr{S}_e(\mathbb{R}^2)$.
2. From the expression of $\mathscr{K}_\alpha^1$ given by the relation (2.13) we have

$$ \begin{aligned} \mathscr{K}_\alpha^1 H_a(\psi) &= \Lambda_\alpha^{-1} \left(\frac{\pi}{2^{2\alpha+1}(\Gamma(\alpha+1))^2} (\mu^2+\lambda^2)^\alpha |\mu| H_{\frac{1}{a}} \Lambda_\alpha(\psi) \right) \\ &= a^{-(2\alpha+1)} \Lambda_\alpha^{-1} H_{\frac{1}{a}} \left(\frac{\pi}{2^{2\alpha+1}(\Gamma(\alpha+1))^2} (\mu^2+\lambda^2)^\alpha |\mu| \Lambda_\alpha(\psi) \right) \\ &= a^{-(2\alpha+1)} H_a \mathscr{K}_\alpha^1(\psi). \end{aligned} $$

3. The proof is the same as in (2). □

Remark 4.2. According to relations (2.10), (3.2) and (4.1) we have:

1. if ψ is a generalized admissible wavelet in $\mathscr{S}_e(\mathbb{R}^2)$, then ${}^t\mathscr{R}_\alpha(\psi)$ is a classical admissible wavelet in $\mathscr{S}_e(\mathbb{R}^2)$ and

$$ C_\psi = A_{{}^t\mathscr{R}_\alpha(\psi)}, $$

2. if ψ belongs to $\mathscr{S}_{e,0}(\mathbb{R}^2)$ such that ${}^t\mathscr{R}_\alpha^{-1}(\psi)$ is a generalized admissible wavelet, then ψ is a classical admissible wavelet and

$$ C_{{}^t\mathscr{R}_\alpha^{-1}(\psi)} = A_\psi. $$

In the following, we provide relations between the generalized and the classical continuous wavelet transform via the Riemann-Liouville operator and its dual.

Theorem 4.3. *Let ψ be a generalized admissible wavelet in $\mathscr{S}_e(\mathbb{R}^2)$ (respectively in $\mathscr{S}_e^0(\mathbb{R}^2)$). Then for every function f in $\mathscr{S}_e^0(\mathbb{R}^2)$ (respectively in $\mathscr{S}_e(\mathbb{R}^2)$), we have*

$$ T_\psi(f)(a,r,x) = a^{\alpha+\frac{1}{2}} \, {}^t\mathscr{R}_\alpha^{-1} \left(S_{{}^t\mathscr{R}_\alpha(\psi)}({}^t\mathscr{R}_\alpha(f))(a,\cdot,\cdot) \right)(r,x). $$

Proof. Let ψ be a generalized admissible wavelet in $\mathscr{S}_e(\mathbb{R}^2)$ (respectively in $\mathscr{S}_e^0(\mathbb{R}^2)$) and f a function in $\mathscr{S}_e^0(\mathbb{R}^2)$ (respectively in $\mathscr{S}_e(\mathbb{R}^2)$). Then from (2) of Proposition 2.9 we deduce that for all $a > 0$, the function

$$(r,x) \mapsto T_\psi(f)(a,r,x) = f \sharp D_a(\overline{\check{\psi}})(r,x)$$

belongs to $\mathscr{S}_e^0(\mathbb{R}^2)$. Using the fact that ${}^t\mathscr{R}_\alpha$ is an isomorphism from $\mathscr{S}_e^0(\mathbb{R}^2)$ into $\mathscr{S}_{e,0}(\mathbb{R}^2)$, (1) of Remark 4.2, Proposition 4.1 and Proposition 2.3, we get

$$\begin{aligned} T_\psi(f)(a,r,x) &= {}^t\mathscr{R}_\alpha^{-1}({}^t\mathscr{R}_\alpha(f \sharp D_a(\overline{\check{\psi}})))(r,x) \\ &= {}^t\mathscr{R}_\alpha^{-1}({}^t\mathscr{R}_\alpha(f) * {}^t\mathscr{R}_\alpha(D_a(\overline{\check{\psi}})))(r,x) \\ &= {}^t\mathscr{R}_\alpha^{-1}({}^t\mathscr{R}_\alpha(f) * a^{\alpha+\frac{1}{2}} H_a({}^t\mathscr{R}_\alpha(\overline{\check{\psi}})))(r,x) \\ &= a^{\alpha+\frac{1}{2}}\, {}^t\mathscr{R}_\alpha^{-1}\left(S_{{}^t\mathscr{R}_\alpha(\psi)}({}^t\mathscr{R}_\alpha(f))(a,\cdot,\cdot)\right)(r,x). \end{aligned}$$ ☐

Corollary 4.4. *Let ψ be a generalized admissible wavelet in $\mathscr{S}_e^0(\mathbb{R}^2)$. For every f in $\mathscr{S}_{e,0}(\mathbb{R}^2)$, we have*

$$S_{{}^t\mathscr{R}_\alpha(\psi)}(f)(a,r,x) = a^{-(\alpha+\frac{1}{2})}\mathscr{R}_\alpha^{-1}\left(T_\psi(\mathscr{R}_\alpha(f))(a,\cdot,\cdot)\right)(r,x). \tag{4.3}$$

Proof. In the previous theorem, replacing f by $\mathscr{R}_\alpha(f)$, using the relation (2.15) and the inversion formula for ${}^t\mathscr{R}_\alpha$ given in Theorem 2.7, we get the result. ☐

Remark 4.5. The proof of the previous corollary can be established by direct calculation as follows. Let ψ be a generalized wavelet in $\mathscr{S}_e^0(\mathbb{R}^2)$ then, from Remark 4.2, ${}^t\mathscr{R}_\alpha(\psi)$ is a classical admissible wavelet in $\mathscr{S}_{e,0}(\mathbb{R}^2)$. Using the relation (2.17) and Proposition 4.1, for all f in $\mathscr{S}_{e,0}(\mathbb{R}^2)$, we get

$$\begin{aligned} S_{{}^t\mathscr{R}_\alpha(\psi)}(f)(a,r,x) &= f * H_a({}^t\mathscr{R}_\alpha(\overline{\check{\psi}}))(r,x) \\ &= \mathscr{R}_\alpha^{-1}(\mathscr{R}_\alpha(f) \sharp\, a^{-(\alpha+\frac{1}{2})} D_a(\overline{\check{\psi}}))(r,x) \\ &= a^{-(\alpha+\frac{1}{2})}\mathscr{R}_\alpha^{-1}\left(T_\psi(\mathscr{R}_\alpha(f))(a,\cdot,\cdot)\right)(r,x). \end{aligned}$$

In the following theorem, we state the main results that is inversion formulas of the Riemann-Liouville operator $\mathscr{R}_\alpha$ and its dual using continuous wavelet transforms.

Theorem 4.6.

1. *Let ψ be a generalized admissible wavelet in $\mathscr{S}_e(\mathbb{R}^2)$, then for all f in $\mathscr{S}_{e,0}(\mathbb{R}^2)$, we have a.e.*

$$\begin{aligned} &{}^t\mathscr{R}_\alpha^{-1}(f)(s,y) \\ &= \frac{1}{C_\psi}\int_0^{+\infty}\left(\int_0^{+\infty}\int_{\mathbb{R}} \mathscr{R}_\alpha\left(S_{{}^t\mathscr{R}_\alpha(\psi)}(\mathscr{K}_\alpha^1 f)(a,\cdot,\cdot)\right)(r,x)\psi_{a,r,x}(s,y)d\nu_\alpha(r,x)\right)\frac{da}{a^{\alpha+\frac{7}{2}}}. \end{aligned}$$

2. *Let ψ be a generalized admissible wavelet in $\mathscr{S}_e^0(\mathbb{R}^2)$, then for all f in $\mathscr{S}_e^0(\mathbb{R}^2)$, we have a.e.*

$$\mathscr{R}_\alpha^{-1}(f)(s,y)$$
$$=\frac{1}{A_{{}^t\mathscr{R}_\alpha(\psi)}}\int_0^{+\infty}\left(\int_0^{+\infty}\int_{\mathbb{R}}^{t}\mathscr{R}_\alpha(T_\psi(\mathscr{K}_\alpha^2 f)(a,\cdot,\cdot))(r,x)(\widetilde{{}^t\mathscr{R}_\alpha(\psi)})_{a,r,x}(s,y)dm_\alpha(r,x)\right)\frac{da}{a^{\alpha+\frac{7}{2}}}.$$

Proof.

1. We get the result form the inversion formula of ${}^t\mathscr{R}_\alpha$ given in Theorem 2.7, Theorem 3.9, Theorem 4.3 and the relation (2.15).
2. We get the result form the inversion formula of $\mathscr{R}_\alpha$ given in Theorem 2.7, relations (4.2), (2.16) and Corollary 4.4. □

Remark 4.7.

1. If ψ is a generalized admissible wavelet in $\mathscr{S}_e^0(\mathbb{R}^2)$ and f in $\mathscr{S}_{e,0}(\mathbb{R}^2)$, then, in virtue of Proposition 4.1, the inversion formula of the dual transform, ${}^t\mathscr{R}_\alpha^{-1}(f)$ can be written as

$${}^t\mathscr{R}_\alpha^{-1}(f)(s,y)$$
$$=\frac{1}{C_\psi}\int_0^{+\infty}\left(\int_0^{+\infty}\int_{\mathbb{R}}\mathscr{R}_\alpha\left(S_{\mathscr{K}_\alpha^1\ {}^t\mathscr{R}_\alpha(\psi)}(f)(a,\cdot,\cdot)\right)(r,x)\psi_{a,r,x}(s,y)d\nu_\alpha(r,x)\right)\frac{da}{a^{3\alpha+\frac{9}{2}}},$$

noting that $\mathscr{K}_\alpha^1\ {}^t\mathscr{R}_\alpha(\psi)$ is not necessarily a classical admissible wavelet.
2. Using Proposition 4.1, the expression of $\mathscr{R}_\alpha^{-1}(f)$ given in (2) of the previous theorem can be written as follows:

$$\mathscr{R}_\alpha^{-1}(f)(s,y)$$
$$=\frac{1}{A_{{}^t\mathscr{R}_\alpha(\psi)}}\int_0^{+\infty}\left(\int_0^{+\infty}\int_{\mathbb{R}}{}^t\mathscr{R}_\alpha(T_{\mathscr{K}_\alpha^2(\psi)}(f)(a,\cdot,\cdot))(\widetilde{{}^t\mathscr{R}_\alpha(\psi)})_{a,r,x}(s,y)dm_\alpha(r,x)\right)\frac{da}{a^{3\alpha+\frac{9}{2}}}.$$

The function $\mathscr{K}_\alpha^2(\psi)$ is not necessarily a generalized admissible wavelet.

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