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PARAMETRIC BOREL SUMMABILITY FOR SOME SEMILINEAR SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper we study the Borel summability of formal solutions with a parameter of first order semilinear system of partial differential equations with n independent variables. In [*Singular perturbation of linear systems with a regular singularity*, J. Dynam. Control. Syst. 8 (2002), 313–322], Balser and Kostov proved the Borel summability of formal solutions with respect to a singular perturbation parameter for a linear equation with one independent variable. We shall extend their results to a semilinear system of equations with general independent variables.

Keywords: Borel summability, singular perturbation, Euler type operator.

Mathematics Subject Classification: 35C10, 45E10, 35Q15.

1. INTRODUCTION

Since the pioneering works by Lutz-Miyake-Schäfke, Balser *et al.* the Borel summability of formal solutions of partial differential equations with respect to the independent variables has been studied extensively (cf. [3, 5, 8, 10, 11]). On the other hand, concerning the summability of formal solutions of a partial differential equation with a singular perturbation parameter we cite [2] and [4]. (See also [6, 7] and [9].)

In this paper we shall study the Borel summability of formal solutions of partial differential equations with a parameter. More precisely, we shall extend the results in [2] to a semilinear system of partial differential equations with general independent variables. We note that our system is not contained in the class of equations studied in the above, nor can be decomposed into first order single equations. We use the method of characteristics in order to prove our theorem which is different from that of [2]. We observe that our method also yields the summability when the independent variable moves in a given bounded open set.

This paper is organized as follows. In Section 2, we state the main theorem, Theorem 2.1 and give remarks to Theorem 2.1. In Section 3, we study formal solutions and the Gevrey estimate. In Section 4, we prove elementary properties of the convolution needed for the proof of Theorem 2.1. In Section 5, we reduce the proof of Theorem 2.1 to that of Theorem 5.1. After having prepared six lemmas we give the proofs of Theorems 2.1 and 5.1. In Section 6, we give an extension of Theorem 2.1 when the independent variable lies in some open set not containing the origin.

2. STATEMENT OF RESULTS

Let $x = (x_1, \ldots, x_n)$, $n \geq 1$, be the variable in $\mathbb{C}^n$. For $\lambda_j \in \mathbb{C}$, $\lambda_j \neq 0$ $(j = 1, 2, \ldots, n)$, define

$$\mathcal{L} := \sum_{j=1}^{n} \lambda_j x_j \frac{\partial}{\partial x_j}. \tag{2.1}$$

Let $N \geq 1$ be an integer and let $f(x,u) = (f_1(x,u), \ldots, f_N(x,u))$, $u = (u_1, \ldots, u_N) \in \mathbb{C}^N$ be a holomorphic vector function in the neighborhood of the origin of $x \in \mathbb{C}^n$ and $u \in \mathbb{C}^N$. We consider Borel summability of formal solutions of the semilinear system of equations

$$\eta \mathcal{L} u = f(x,u), \tag{2.2}$$

where $\eta \in \mathbb{C}$ is a complex parameter. We assume

$$f(0,0) = 0, \qquad \det(\nabla_u f(0,0)) \neq 0, \tag{2.3}$$

where $\nabla_u f(0,0)$ denotes the Jacobi matrix of $f(x,u)$ with respect to u at the point $x = 0, u = 0$.

We shall construct the formal power series solution $v(x,\eta)$ of (2.2) in the form

$$v(x,\eta) = \sum_{\nu=0}^{\infty} \eta^\nu v_\nu(x) = v_0(x) + \eta v_1(x) + \ldots, \tag{2.4}$$

where the series is a formal power series in η with the coefficient $v_\nu(x)$ being a holomorphic vector function of x in some open set independent of ν. We set $v_\nu(x) \equiv v_\nu = (v_\nu^{(1)}, \ldots, v_\nu^{(N)})$. We denote by Ω_0 the neighborhood of the origin on which every coefficient $v_\nu(x)$ is defined.

In order to state our results we recall some definitions (cf. [1] and [2]). The formal Borel transform of $v(x,\eta)$ is defined by

$$B(v)(x,y) := \sum_{\nu=0}^{\infty} v_\nu(x) \frac{y^\nu}{\Gamma(\nu+1)}, \tag{2.5}$$

where $\Gamma(z)$ is the Gamma function. For an opening $\theta > 0$ and the bisecting direction ξ, define the sector $S_{\theta,\xi}$ by

$$S_{\theta,\xi} = \left\{ z \in \mathbb{C};\ |\arg z - \xi| < \frac{\theta}{2} \right\}. \tag{2.6}$$

($S_{\theta,\xi}$ is illustrated in Figure 1.) We say that $v(x, \eta)$ is 1-summable in the direction ξ with respect to η if $B(v)(x, y)$ converges in the neighborhood of the origin of (x, y), and there exists the neighborhood U of the origin $x = 0$ and a $\theta > 0$ such that $B(v)(x, y)$ can be analytically continued to $(x, y) \in U \times S_{\theta,\xi}$ and of exponential growth of order 1 with respect to y in $S_{\theta,\xi}$. For the sake of simplicity, we denote the analytic continuation with the same notation $B(v)(x, y)$. The Borel sum $V(x, \eta)$ of $v(x, \eta)$ is then given by the Laplace transform

$$V(x,\eta) := \int_0^{\infty e^{i\xi}} y^{-1} e^{-y\eta^{-1}} B(v)(x,y)dy. \tag{2.7}$$

We assume

$$\nabla_u f(x,0) \text{ is a diagonal matrix.} \tag{2.8}$$

We set

$$\nabla_u f(0,0) = \mathrm{diag}\,(\mu_1, \dots, \mu_N). \tag{2.9}$$

Moreover, we assume

$$\lambda_j > 0,\ \mathrm{Re}\,\mu_k > 0 \quad (j = 1, \dots, n,\ k = 1, \dots, N). \tag{2.10}$$

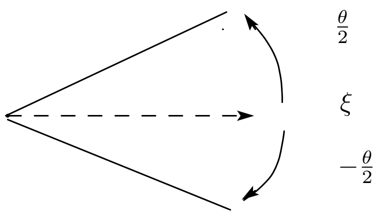

Fig. 1. $S_{\theta,\xi}$

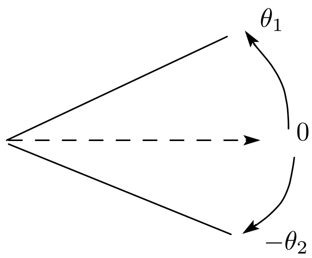

Fig. 2. C_0

Let C_0 be the convex closed positive cone with vertex at the origin containing λ_j $(j = 1, 2, \dots, n)$ and $(\mu_k)^{-1}$ $(j = 1, 2, \dots, n; k = 1, \dots, N)$. Write

$$C_0 = \{z \in \mathbb{C}; -\theta_2 \le \arg\, z \le \theta_1\} \tag{2.11}$$

for some $0 \le \theta_1 < \pi/2$ and $0 \le \theta_2 < \pi/2$ (Figure 2). Define $\xi = -\pi + \frac{\theta_1 - \theta_2}{2}$ and $\theta = \pi - \theta_1 - \theta_2$. We observe that $S_{\pi+\theta,\xi}$ is equal to $\mathbb{C} \setminus C_0$. Then we have the following theorem.

Theorem 2.1. *Suppose* (2.3), (2.8) *and* (2.10). *Then there exists the neighborhood* U *of* $x = 0$ *such that* $v(x, \eta)$ *is* 1*-summable in the direction* $\arg \eta$ *with* $\eta \in S_{\theta,\xi}$ *when* $x \in U$. *Moreover,* $V(x, \eta)$ *is holomorphic and satisfies* (2.2) *when* $(x, \eta) \in U \times S_{\pi+\theta,\xi}$.

Remark 2.2.

(a) In [2] the summability of formal solutions (2.4) was shown for (2.2) with $N = 1$ and $n = 1$ assuming that f is the polynomial of degree 1 with respect to u. In fact, in Theorem 5.1 of [2] the summability was proved under the condition equivalent to (2.10). It was also shown that (2.10) is necessary in general.

(b) An interesting phenomenon shown in [2] is that a certain Diophantine phenomenon appears in the summability, while it does not appear for an irregular singular equation (cf. [4]). In the case of general independent variables one can easily see that a similar multi-dimensional Diophantine condition enters in the analysis. Because we do not know how to generalize the proof in [2] to a semilinear multi-dimensional case, we use the method of characteristics in order to prove the summability. More precisely, the stable behavior of the characteristics in our proof corresponds to the Diophantine type condition in [2]. We note that our method also shows the summability in the case when the independent variable is outside the origin without assuming (2.10). We briefly mention the extension in the last section.

3. FORMAL POWER SERIES IN THE PERTURBATION PARAMETER

In this section we construct a formal solution of (2.2) and obtain some estimates of formal series.

Construction of a formal solution. We substitute the expansion (2.4) into (2.2) with $u = v$. The left-hand side is given by

$$\eta \mathcal{L} v = \sum_{\nu=0}^{\infty} \mathcal{L} v_\nu(x) \eta^{\nu+1}. \tag{3.1}$$

By the partial Taylor expansion of f with respect to v the right-hand side of (2.2) is written as

$$\begin{aligned} f(x,v) &= f(x, v_0 + v_1\eta + v_2\eta^2 + \ldots) \\ &= f(x, v_0) + \eta(\nabla_u f)(x, v_0)v_1 + O(\eta^2). \end{aligned} \tag{3.2}$$

By comparing the coefficients of η, we obtain for $\eta^0 = 1$

$$f(x, v_0(x)) = 0 \tag{3.3}$$

and for η

$$\mathcal{L} v_0 = (\nabla_u f)(x, v_0)v_1. \tag{3.4}$$

We solve (3.3) with the condition $v_0(0) = 0$ by means of an implicit function theorem on some Ω_0 in view of the assumption $f(0,0) = 0$ in (2.3). Next, we solve v_1 from (3.4) on Ω_0, where we may assume $\det(\nabla_u f(x, v_0(x))) \neq 0$ on Ω_0, since $\det(\nabla_u f(0,0)) \neq 0$.

In order to determine $v_\nu(x)$ $(\nu \geq 2)$ we compare the coefficients of η^ν of (2.2). Indeed, we differentiate (3.2) $(\nu - 1)$-times with respect to η and put $\eta = 0$. Then we obtain

$$\mathcal{L}v_{\nu-1} = (\nabla_u f)(x, v_0)v_\nu + (\text{terms consisting of } v_k,\ k < \nu). \tag{3.5}$$

We observe that the second term in the right-hand side appears from products of terms in (3.2) of the form $v_{i_j}\eta^{i_j}$ such that

$$i_1 + i_2 + \ldots + i_\ell = \nu, \quad i_1 \geq 0, \ldots, i_\ell \geq 0, i_j \neq 0$$

for some $\ell \geq 2$ and $j \leq \ell$. It follows that all terms in the second term satisfy v_k, $k < \nu$. Therefore, one can write (3.5) in the following way

$$\nabla_u f(x, v_0)v_\nu = H_\nu(x, v_0, v_1, \ldots, v_{\nu-1}) \quad \text{for all } \nu \geq 2.$$

Since $\det(\nabla_u f(x, v_0(x))) \neq 0$ on Ω_0, one can inductively determine v_ν.

The next theorem gives the existence of a formal solution.

Proposition 3.1. *Assume* (2.3)*. Then every coefficient of* (2.4) *is uniquely determined as a holomorphic function on* Ω_0*.*

Proof. By (2.3) and an implicit function theorem, $v_0(x)$ is uniquely determined as the holomorphic function at the origin such that $v_0(x) = O(|x|)$. Suppose that $v_k(x)$ is determined up to some $\ell - 1$ in the neighborhood of the origin. Then, by an implicit function theorem one can determine $v_\ell(x)$ uniquely in the neighborhood of the origin depending on ℓ. Because $v_k(x)$ are determined recursively by differentiations and algebraic calculations, the recurrence formula for $v_\ell(x)$ implies that $v_\ell(x)$ is holomorphic on Ω_0. □

Gevrey estimate of order 1. We shall show the following proposition.

Proposition 3.2. *Assume that $f(x, u)$ be analytic with respect to x in the neighborhood of the origin $0 \in \mathbb{C}^n$ and an entire function of $u \in \mathbb{C}^N$. Let v in* (2.4) *be the formal series solution given by Proposition 3.1. Then there exist a neighborhood U of the origin, $x = 0$ and a neighborhood W of the origin $y = 0$ in $\mathbb{C}$ such that $B(v)(x, y)$ converges in $U \times W$.*

Proof. We use the majorant relation $u \ll v$. Namely, for $u = \sum_\alpha x^\alpha u_\alpha$ and $v = \sum_\alpha x^\alpha v_\alpha$ the relation $u \ll v$ holds if $|u_\alpha| \leq v_\alpha$ for every α. If u and v are vector functions, then $u \ll v$ means that for every j, the j-th component u_j of u and v_j of v satisfy $u_j \ll v_j$. If v is a scalar function, then $u \ll v$ means that $u_j \ll v$ for every j. For $\rho > 0$, define

$$\phi_\rho(x) := \left(1 - \frac{x_1 + \ldots + x_n}{\rho}\right)^{-1}. \tag{3.6}$$

The set of holomorphic functions at the origin such that $u \ll \phi_\rho C$ for some $C \geq 0$ forms a Banach space with the norm $\|u\|$ given by the infimum of C satisfying $u \ll \phi_\rho C$.

First we estimate the differentiation. For any integers $1 \leq j \leq n$ and $k \geq 1$, we have

$$\frac{\partial}{\partial x_j}\phi_\rho(x)^k = \frac{k}{\rho}\phi_\rho(x)^{k+1}. \tag{3.7}$$

On the other hand, because $x_j(\nabla_u f)(x, v_0)^{-1}$ is analytic at the origin for $1 \leq j \leq n$ we have, for sufficiently small $\rho > 0$

$$x_j(\nabla_u f)(x, v_0)^{-1} \ll K\phi_\rho \tag{3.8}$$

for some $K > 0$. Similarly, we have $v_0 \ll \|v_0\|\phi_\rho$.

We next estimate v_1. By virtue of (3.4) we have $v_1 = (\nabla_u f)(x, v_0)^{-1}\mathcal{L}v_0$. Hence, by (3.7) and (3.8), we have $v_1 \ll \|v_0\|C_0\phi_\rho^3$ for some $C_0 > 0$. We shall show that there exists $C \geq 1$ independent of $\nu \geq 1$ such that

$$v_m \ll C^{2m-1}m!\phi_\rho^{4m-1}, \quad m = 1, 2, \ldots \tag{3.9}$$

Suppose that (3.9) holds up to $m \leq \nu - 1$ and consider v_ν. In view of (3.5) we first consider $(\nabla_u f)(x, v_0)^{-1}\mathcal{L}v_{\nu-1}$.

$$(\nabla_u f)(x, v_0)^{-1}\mathcal{L}v_{\nu-1} \ll C^{2\nu-3}(\nu-1)!(4\nu-5)\phi_\rho^{4\nu-3}C_1 \leq 4C_1C^{2\nu-3}\nu!\phi_\rho^{4\nu-3} \tag{3.10}$$

for some $C_1 > 0$ depending only on K and $\mathcal{L}$. Hence, if $4C_1 \leq C$ and $C > 1$, then we have an estimate like (3.9) since $1 \ll \phi_\rho$.

Next, we estimate the nonlinear term. Set $v = v_0 + u$, $u = \eta v_1 + \eta^2 v_2 + \ldots$ and expand

$$f(x, v) = f(x, v_0) + \nabla_u f(x, v_0) \cdot u + \sum_{|\beta| \geq 2} r_\beta(x, v_0)u^\beta. \tag{3.11}$$

By inserting the expansion of u and by comparing the coefficients of η^ν of the right-hand side of (3.11) we see that the nonlinear term in (3.5) is given by

$$\sum_{|\beta| \geq 2}\sum_{\ell=2}^{|\beta|}\sum_{\nu_1+\ldots+\nu_\ell=\nu, \nu_j \geq 1} r_\beta(x, v_0)v_{\nu_1}\ldots v_{\nu_\ell}. \tag{3.12}$$

By inductive assumptions on v_m we have

$$\sum_{\ell=2}^{|\beta|}\sum_{\nu_1+\ldots+\nu_\ell=\nu, \nu_j \geq 1, \ell \geq 2} v_{\nu_1}\ldots v_{\nu_\ell} \ll \sum_{\ell=2}^{|\beta|}\sum \nu_1!\ldots\nu_\ell!C^{2\nu-\ell}\phi_\rho^{4\nu-\ell}. \tag{3.13}$$

We recall the inequality

$$\sum_{\nu_1+\ldots+\nu_\ell=\nu, \nu_j \geq 1, \ell \geq 2} \frac{\nu_1!\ldots\nu_\ell!}{\nu!} \leq 1. \tag{3.14}$$

Then the right-hand side of (3.13) is bounded by

$$\ll C^{2\nu-2}\nu!\sum_{\ell=2}^{|\beta|} C^{2-\ell}\phi_\rho^{4\nu-2} \ll C^{2\nu-2}C_2\nu!\phi_\rho^{4\nu-2}$$

for some $C_2 > 0$ independent of ν because $\sum_{\ell=2}^{\infty} C^{2-\ell} < \infty$ by $C > 1$.

In order to estimate $(\nabla_u f)(x, v_0)^{-1}$ times (3.12) we consider

$$(\nabla_u f)(x, v_0)^{-1} \sum_{|\beta|\geq 2} r_\beta(x, v_0). \tag{3.15}$$

By virtue of (3.11) we have

$$\sum_{|\beta|\geq 2} r_\beta(x, v_0) = f(x, v_0 + e) - f(x, v_0) - \nabla_u f(x, v_0) \cdot e, \tag{3.16}$$

where $e = (1, \ldots, 1)$. By using the scale change of variables $u \mapsto \varepsilon u$, $\varepsilon > 0$, one may assume that $f(x, v_0 + e)$ is analytic at $x = 0$, if necessary. Therefore, one can estimate (3.15) like $\ll K\phi_\rho$ for some $K > 0$.

Therefore, $(\nabla_u f)(x, v_0)^{-1}$ times (3.12) can be estimated by $C^{2\nu-2}C_2 K\nu!\phi_\rho^{4\nu-1}$. By inserting this estimate and (3.10) into (3.5) we obtain (3.9) for $m = \nu$. By (3.9) and the definition of majorant relations, we obtain the convergence of the formal Borel transform in $U \times W$. This ends the proof. □

4. CONVOLUTION ESTIMATE

Let Ω be the smallest open set containing the sector $S_{\theta,\pi}$ in (2.6) with $0 < \theta < \pi$ and the disk $\{|z| < r_0\}$ for small $r_0 > 0$ such that $z \in \Omega$ implies $z + t \in \Omega$ for every real number $t \leq 0$. For $c > 0$, we define the space $\mathcal{H}_c(\Omega)$ as the set of those $h \in H(\Omega)$ such that there exists $K \geq 0$ for which

$$|h(z)| \leq Ke^{-c\operatorname{Re} z}(1 + |z|)^{-2} \quad \text{for all } z \in \Omega, \tag{4.1}$$

where $H(\Omega)$ is the set of holomorphic functions in Ω. Obviously, $\mathcal{H}_c(\Omega)$ is the Banach space with the norm

$$\|h\|_{\Omega,c} := \sup_{z\in\Omega} |h(z)|(1 + |z|)^2 e^{c\operatorname{Re} z}. \tag{4.2}$$

The convolution $f * g$ $(f, g \in \mathcal{H}_c(\Omega))$ is defined by

$$(f * g)(z) := \frac{d}{dz}\int_0^z f(z - t)g(t)dt = \frac{d}{dz}\int_0^z f(t)g(z - t)dt. \tag{4.3}$$

Remark 4.1. The above definition (4.3) seems different from the usual one of the convolution. In the summability theory developed in [1] or [2], the operation $*$ in (4.3) plays the role of the usual convolution. Indeed, for nonnegative integers i and j the formal Borel transform $\mathcal{B}(\eta^{i+j})$ of $\eta^{i+j} = \eta^i\eta^j$ is given by $\zeta^{i+j}/(i + j)!$ with ζ being the dual variable of η, which might be equal to $\mathcal{B}(\eta^i) * \mathcal{B}(\eta^j) = \zeta^i * \zeta^j/(i!j!)$, where $*$ denotes a "convolution". If we use the definition of the operator $*$ as in the above, then one can verify that $\zeta^i * \zeta^j/(i!j!)$ coincides with $\zeta^{i+j}/(i + j)!$. For more details we refer to [1].

Write $f'(z) = (df/dz)(z)$. Then we have the following proposition.

Proposition 4.2. *For every $f, g \in \mathcal{H}_c(\Omega)$ such that $f(0) = g(0) = 0$ and $f', g' \in \mathcal{H}_c(\Omega)$ we have $f * g \in \mathcal{H}_c(\Omega)$ with the estimates*

$$\|f * g\|_{\Omega,c} \leq 8\|f'\|_{\Omega,c}\|g\|_{\Omega,c}, \quad \|f * g\|_{\Omega,c} \leq 8\|f\|_{\Omega,c}\|g'\|_{\Omega,c}. \tag{4.4}$$

Proof. Because $f * g = g * f$ we shall prove the first inequality of (4.4). We have

$$f * g(z) = \frac{d}{dz}\int_0^z f(z-t)g(t)dt = f(0)g(z) + \int_0^z f'(z-t)g(t)dt$$
$$= \int_0^z f'(z-t)g(t)dt.$$

By (4.2) and by taking the path of integration from 0 to z, we have

$$\begin{aligned}\left|\int_0^z f'(z-t)g(t)dt\right| &\leq \|f'\|_{\Omega,c}\|g\|_{\Omega,c}e^{-c\operatorname{Re} z}\int_0^z (1+|z-t|)^{-2}(1+|t|)^{-2}|dt| \\ &\leq \|f'\|_{\Omega,c}\|g\|_{\Omega,c}e^{-c\operatorname{Re} z}\int_0^{|z|} (1+|z|-s)^{-2}(1+s)^{-2}ds.\end{aligned} \tag{4.5}$$

We divide the integral in the right-hand side into two parts, $s \leq \frac{|z|}{2}$ and $s > \frac{|z|}{2}$. If $s \leq \frac{|z|}{2}$, then we have $(1+|z|-s)^{-2} \leq 4(1+|z|)^{-2}$, while in case $s > \frac{|z|}{2}$ we have $(1+s)^{-2} \leq 4(1+|z|)^{-2}$. Hence we have

$$\int_0^{|z|/2} \frac{1}{(1+|z|-s)^2(1+s)^2}ds \leq \frac{4}{(1+|z|)^2}\int_0^{|z|/2}(1+s)^{-2}ds \leq \frac{4}{(1+|z|)^2}. \tag{4.6}$$

One can similarly estimate the other part like

$$\int_{|z|/2}^{|z|} (1+|z|-s)^{-2}(1+s)^{-2}ds \leq 4(1+|z|)^{-2}.$$

Therefore, we see that the left-hand side term of (4.5) can be estimated by $8\|f'\|_{\Omega,c}\|g\|_{\Omega,c}e^{-c\operatorname{Re} z}(1+|z|)^{-2}$. This ends the proof. □

5. PROOF OF THEOREM 2.1

First we define a function space. Let D and Ω be the open connected set in the neighborhood of the origin of $\mathbb{C}^n$ and the set given in (4.1), respectively. Let $H(D, \Omega)$

be the set of holomorphic functions in $(x, y) \in D \times \Omega$. Then we define $\mathcal{H}_c(D, \Omega)$ as the set of those $h \equiv h(x, y) \in H(D, \Omega)$ such that there exists $K_0 \geq 0$ for which

$$\sup_{x \in D} |h(x, y)| \leq K_0 e^{-c \mathrm{Re}\, y} (1 + |y|)^{-2} \quad \text{for all } y \in \Omega. \tag{5.1}$$

The space $\mathcal{H}_c(D, \Omega)$ is a Banach space with the norm $\|h\|_c = \inf K_0$, where K_0 is given in (5.1).

Proof of Theorem 2.1. We first show the summability of $v(x, \eta)$ in the direction $\arg \eta = \pi$ when $x \in U$, where U is given in Proposition 2. One may assume $\lambda_n = 1$ without loss of generality by dividing the equation with $\lambda_n \neq 0$. In terms of (2.2) with u replaced by $v_0 + u$, (3.11) and $f(x, v_0) = 0$ we obtain

$$\mathcal{L}u = -\mathcal{L}v_0 + \eta^{-1} \nabla_u f(x, v_0) u + \eta^{-1} \sum_{|\beta| \geq 2} r_\beta(x, v_0) u^\beta. \tag{5.2}$$

Let $\hat{u}(y) := \mathcal{B}(u)$ be the formal Borel transform of u with respect to η, where y is the dual variable of η. By the formal Borel transform of (5.2) and by recalling that η^{-1} corresponds to $\partial/\partial y$, we obtain

$$\mathcal{L}\hat{u} = -\mathcal{L}v_0 + \nabla_u f(x, v_0) \frac{\partial \hat{u}}{\partial y} + \frac{\partial}{\partial y} \sum_{|\beta| \geq 2} r_\beta(x, v_0) (\hat{u})^{*\beta}, \tag{5.3}$$

where $(\hat{u})^{*\beta} = (\hat{u}_1)^{*\beta_1} \ldots (\hat{u}_N)^{*\beta_N}$, $\beta = (\beta_1, \ldots, \beta_N)$, and $(\hat{u}_j)^{*\beta_j}$ is the β_j-convolution product, $(\hat{u}_j)^{*\beta_j} = \hat{u}_j * \ldots * \hat{u}_j$.

Let v be the formal solution given by Proposition 3.1 and consider the formal Borel transform $B(v)$. Define $\hat{u}(x, y) := B(v) - v_0$. Then $\hat{u}(x, y)$ is analytic when $(x, y) \in U \times W$, and $\hat{u}$ is the solution of (5.3) in the neighborhood of $y = 0$ such that $\hat{u}(x, 0) \equiv 0$ in x. We show that every solution of (5.3) analytic at $y = 0$ and satisfying $\hat{u}(x, 0) \equiv 0$ is uniquely determined. Indeed, by definition the convolution product of $y^i/i!$ and $y^j/j!$ is equal to $y^{i+j}/(i + j)!$. Hence, if we expand $\hat{u}$ in the power series of y and insert (5.3), then every coefficient of the expansion can be uniquely determined from the recurrence relation because $\nabla_u f(x, v_0)$ is invertible. Therefore, if we can show the existence of the solution of (5.3) being analytic in $(x, y) \in U \times W$ which is of exponential growth with respect to y in Ω, then we have the analytic continuation of the formal Borel transform of v with exponential growth in $y \in \Omega$. Hence we have the summability of v. □

Therefore, it is sufficient to prove the following theorem.

Theorem 5.1. *There exist $c > 0$, a neighborhood of the origin $x = 0$, D and Ω as in* (4.2) *such that* (5.3) *has a solution $\hat{u}$ in $\mathcal{H}_c(D, \Omega)$.*

The proof of Theorem 5.1 is given after having prepared six lemmas.

Let $c > 0$, D and Ω be given. We may assume that D is contained in an open ball centered at the origin. In order to prove the solvability of (5.3) when x is in the neighborhood of the origin and $y \in \Omega$ we shall study

$$\mathcal{L}w - (\nabla_u f)(x, 0) \frac{\partial w}{\partial y} = g(x, y), \tag{5.4}$$

where $w = (w_1, \dots, w_N)$ and $g = g(x, y) = (g_1, \dots, g_N)$, $g_j \in \mathcal{H}_c(D, \Omega)$ is a given function.

By the assumption (2.8), for a given $j, 1 \le j \le N$ we denote the j-th diagonal component of $(\nabla_u f)(x, 0)$ by $(\nabla_u f)_j(x, 0)$. We use the method of characteristics in order to solve (5.4). Namely, we consider

$$\frac{d\zeta}{\zeta} = \frac{dx_k}{\lambda_k x_k} = -\frac{dy}{(\nabla_u f)_j(x, 0)}, \quad k = 1, 2, \dots, n-1. \tag{5.5}$$

Let $b \in \mathbb{C}$, $b \neq 0$ be sufficiently small and $y_0 \in \Omega$ be given. By integrating (5.5) we have

$$x_k = c_k \zeta^{\lambda_k} \ (k = 1, 2, \dots, n-1), \ y = y_0 - \Phi_j(\zeta, b), \tag{5.6}$$

where

$$\Phi_j(\zeta, b) = \int_b^\zeta (\nabla_u f)_j(s^{\lambda_1} c_1, \dots, s^{\lambda_{n-1}} c_{n-1}, s; 0) s^{-1} ds, \tag{5.7}$$

and the integral is taken along the non self-intersecting curve which does not encircle the origin. Then we make analytic continuation around the origin. Here $y_0 := y(b) \in \Omega$ is the initial value of $y = y(\zeta)$ at $\zeta = b$ and c_k's are chosen so that the initial point $x^{(0)} := (x_1(b), \dots, x_{n-1}(b), b)$ lies in D. Define $\Phi(\zeta, b) := (\Phi_1(\zeta, b), \dots, \Phi_N(\zeta, b))$. Then we have the following lemma.

Lemma 5.2. *Let $\zeta_0 \in D \setminus \{0\}$. Then, for every j, $1 \le j \le N$ there exists a curve $\gamma_{\zeta_0, j}$ which passes ζ_0 and tends to the origin such that $Im\, \Phi_j(\zeta, b) = Im\, \Phi_j(\zeta_0, b)$ for every $\zeta \in \gamma_{\zeta_0, j}$.*

Proof. The condition $\operatorname{Im} \Phi_j(\zeta, b) = \operatorname{Im} \Phi_j(\zeta_0, b)$ is equivalent to $\operatorname{Im} \Phi_j(\zeta, \zeta_0) = 0$. We shall look for the curve $\gamma_{\zeta_0, j}$ satisfying the latter condition. We first observe that there exist $R(\zeta)$ and $\rho > 0$ such that

$$\Phi_j(\zeta, \zeta_0) = \mu_j \log\left(\frac{\zeta}{\zeta_0}\right) + R(\zeta), \tag{5.8}$$

where $R(\zeta) = O(\zeta^\rho)$ when $\zeta \to 0$. Indeed, by assumption (2.9) we have that $(\nabla_u f)_j(x, 0) = \mu_j + O(|x|)$ ($\operatorname{Re} \mu_j > 0$) when $x \in \Omega_0$. Because $\lambda_k > 0$, the integral $\Phi_j(\zeta, \zeta_0) = \int_{\zeta_0}^{\zeta} t^{-1} (\nabla_u f)_j(x, 0) dt$ with $x_k = c_k t^{\lambda_k}$ has the expression (5.8).

Set $\mu_j = \alpha_j + i\beta_j$ with $\alpha_j > 0$. Then we have

$$\begin{aligned}(\alpha_j + i\beta_j) \log(\zeta/\zeta_0) &= (\alpha_j + i\beta_j)(\log(|\zeta|/|\zeta_0|) + i \arg(\zeta/\zeta_0)) \\ &= (\alpha_j \log(|\zeta|/|\zeta_0|) - \beta_j \arg(\zeta/\zeta_0)) \\ &\quad + i\left(\alpha_j \arg(\zeta/\zeta_0) + \beta_j \log(|\zeta|/|\zeta_0|)\right).\end{aligned} \tag{5.9}$$

Hence the relation $\operatorname{Im} \Phi_j(\zeta, \zeta_0) = 0$ is written as

$$\alpha_j \arg(\zeta/\zeta_0) + \beta_j \log(|\zeta|/|\zeta_0|) + \operatorname{Im} R(\zeta) = 0. \tag{5.10}$$

We define $\theta := \arg(\zeta/\zeta_0)$, $r := |\zeta|/|\zeta_0|$ and $m(r,\theta) := \operatorname{Im} R(\zeta)$. Then (5.10) can be written in

$$\alpha_j \theta + \beta_j \log r + m(r,\theta) = 0. \tag{5.11}$$

By (5.8), we have $m(r,\theta) = O(r^\rho)$ as $r \to 0$. Let $\ell \geq 1$ be an integer such that $\rho\ell > 1$. Set $r = \tilde{r}^\ell$. Then (5.11) can be written as $\alpha_j\theta + \beta_j \ell \log \tilde{r} + m(\tilde{r}^\ell, \theta) = 0$. Assume that $\beta_j \neq 0$. Then it follows that

$$\log \tilde{r} = -(\beta_j \ell)^{-1}(\alpha_j \theta + m(\tilde{r}^\ell, \theta)). \tag{5.12}$$

Hence we have

$$\tilde{r} = \exp\left(-\frac{\alpha_j \theta}{\beta_j \ell} - \frac{m(\tilde{r}^\ell, \theta)}{\beta_j \ell}\right). \tag{5.13}$$

Clearly, $m(\tilde{r}^\ell, \theta)$ is continuously differentiable with respect to $\tilde{r}$ and the derivative is small if $\tilde{r}$ is sufficiently small. By an implicit function theorem we see that (5.13) can be solved as $\tilde{r} = \tilde{r}(\theta)$. Clearly, $\tilde{r}(\theta)$ asymptotically equals $\exp\left(-(\alpha_j\theta)/(\beta_j\ell)\right)$. We define the curve $\gamma_{\zeta_0,j} = \gamma_{\zeta_0,j}(\zeta)$ by the relation

$$r = \tilde{r}(\theta)^\ell, \;\; \theta := \arg(\zeta/\zeta_0), \;\; r := |\zeta|/|\zeta_0|, \tag{5.14}$$

which passes ζ_0 and tends to zero. In order that they tend to the origin we require the following conditions.

(i) If $\beta_j > 0$, then we have $\alpha_j/\beta_j > 0$. We define $\gamma_{\zeta_0,j}$ by (5.14) with $\theta \geq 0$. Hence the curve encircles around the origin counterclockwise and tends to the origin. (See Figure 3.)
(ii) If $\beta_j < 0$, then we have $\alpha_j/\beta_j < 0$. We define $\gamma_{\zeta_0,j}$ by (5.14) with $\theta \leq 0$. Then the curve encircles around the origin clockwise and tends to the origin. (See Figure 4.)
(iii) If $\beta_j = 0$, then by (5.11) we have $\alpha_j\theta + m(r,\theta) = 0$. In order to solve the relation with respect to θ we study the derivative of $R(\zeta)$ with respect to θ. By definition we have

$$R(\zeta) = \int_{\zeta_0}^{\zeta_0 r e^{i\theta}} \left((\nabla_u f)_j(s^{\lambda_1}c_1, \dots, s^{\lambda_{n-1}}c_{n-1}, s; 0) - \mu_j\right) s^{-1} ds. \tag{5.15}$$

Differentiating the right-hand side of (5.15) with respect to θ we see that it is continuous with respect to θ. Therefore, by an implicit function theorem, (5.11) can be solved as $\theta = \theta(r)$. We define $\gamma_{\zeta_0,j}$ by (5.14) with $r = \tilde{r}(\theta)^\ell$ replaced by $\theta = \theta(r)$, $0 < r \leq 1$, $\theta(1) = 0$. (See Figure 5.) Moreover, we have

$$|\theta(r)| \leq \alpha_j^{-1}|m(r,\theta)| \leq Cr^\rho/\alpha_j$$

for some $C > 0$ independent of r. This proves that the curve $\gamma_{\zeta_0,j}$ tends to the origin. This ends the proof.

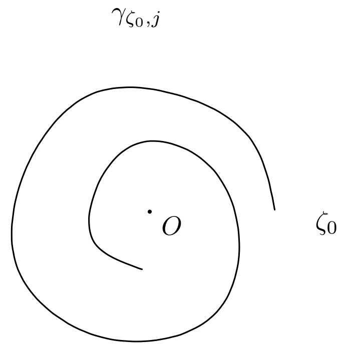

Fig. 3. $\beta_j > 0$

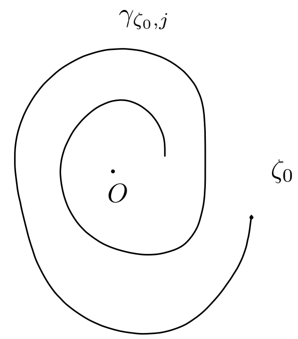

Fig. 4. $\beta_j < 0$

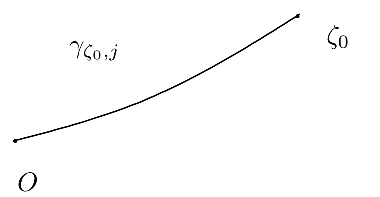

Fig. 5. $\beta_j = 0$

□

Lemma 5.3. *Let $c \neq 0$ and $\zeta_0 \neq 0$ be given complex constants. Then, for every j, $1 \le j \le N$, $\operatorname{Re}\Phi_j(\zeta, c)$ is monotone decreasing when ζ approaches the origin along the curve $\gamma_{\zeta_0,j}$.*

Proof. By (5.9), we have

$$\operatorname{Re}\Phi_j(\zeta,\zeta_0) = \alpha_j \log r - \beta_j\theta + \tilde{m}(r,\theta), \tag{5.16}$$

where $\tilde{m}(r,\theta) := \operatorname{Re} R(\zeta)$, $\zeta/\zeta_0 = re^{i\theta}$. First we consider the case $\beta_j > 0$. In view of the definition of $\gamma_{\zeta_0,j}$ the parameter of the curve is $\theta \ge 0$. It is sufficient to show that the right-hand side of (5.16) is a monotone decreasing function of θ in $\theta \ge 0$. Because $-\beta_j\theta$ trivially has the property, we consider $\alpha_j \log r + \tilde{m}(r,\theta)$. Let $\rho > 0$ the number given in (5.8). Let ℓ satisfy $\ell\rho > 1$. We set $r = \tilde{r}^\ell$. Then, in view of (5.12) we shall show that

$$\alpha_j \ell \log \tilde{r} + \tilde{m}(\tilde{r}^\ell,\theta) = -\alpha_j^2\theta/\beta_j - \alpha_j m(\tilde{r}^\ell,\theta)/\beta_j + \tilde{m}(\tilde{r}^\ell,\theta) \tag{5.17}$$

is a decreasing function of θ. We shall show that the derivatives of $m(\tilde{r}^\ell,\theta)$ and $\tilde{m}(\tilde{r}^\ell,\theta)$ with respect to θ are small if r is small. We consider $m(\tilde{r}^\ell,\theta) = \operatorname{Im} R(\zeta)$. Since $\zeta = \zeta_0 re^{i\theta}$, we will estimate $(\partial/\partial\theta)R(\zeta)$. In view of (5.15) we have

$$\frac{\partial}{\partial\theta}R(\zeta) = i(\nabla_u f)_j(\zeta^{\lambda_1}c_1,\ldots,\zeta^{\lambda_{n-1}}c_{n-1},\zeta;0) - i\mu_j. \tag{5.18}$$

By the assumption $\lambda_j > 0$ this quantity is bounded when $|\zeta|$ is sufficiently small uniformly in θ. This proves the assertion. The smallness of the derivative of $\tilde{m}(\tilde{r}^\ell, \theta)$ with respect to θ is proved similarly. Hence, by (5.16) and (5.17) we see that $\operatorname{Re}\Phi_j(\zeta, \zeta_0)$ is a decreasing function when ζ tends to the origin. Next we consider the case $\beta_j < 0$. We take $\theta \leq 0$ and we make the same argument as in the case $\beta_j > 0$ by using (5.16). Hence we have the same assertion.

We study the case $\beta_j = 0$. By a similar argument as in (5.16), we have

$$\operatorname{Re}\Phi_j(\zeta, \zeta_0) = \alpha_j \ell \log \tilde{r} + \tilde{m}(\tilde{r}^\ell, \theta). \tag{5.19}$$

By Lemma 5.2, the parameter of $\gamma_{\zeta,j}$ is $\tilde{r}$. The point ζ on the curve tends to the origin as $\tilde{r} \to 0$. We calculate $(\partial/\partial\tilde{r})\tilde{m}(\tilde{r}^\ell, \theta)$. By the same calculation as in (5.18) we may consider the following quantity

$$\zeta_0 e^{i\theta} \ell \tilde{r}^{\ell-1} \left((\nabla_u f)_j(\zeta^{\lambda_1} c_1, \ldots, \zeta^{\lambda_{n-1}} c_{n-1}, \zeta; 0) - \mu_j \right) \zeta^{-1}. \tag{5.20}$$

Since $(\nabla_u f)_j - \mu_j = O(|\zeta|^\rho)$ and $|\zeta| = \tilde{r}^\ell$, the quantity in (5.20) is bounded by $\tilde{r}^{-1}|\zeta|^\rho = \tilde{r}^{\rho\ell-1}$. Because $\rho\ell > 1$, the quantity is arbitrarily small if $\tilde{r}$ is sufficiently small. In terms of (5.19) this implies that $\operatorname{Re}\Phi_j(\zeta, \zeta_0)$ is monotone decreasing as $\tilde{r} \to 0$. This completes the proof. □

Lemma 5.4. *Let $g = g(x, y) = (g_1, \ldots, g_N)$, $g_j \in \mathcal{H}_c(D, \Omega)$ be such that $g(0, y) \equiv 0$ for every $y \in \Omega$. Then the solution of* (5.4) *is given by*

$$w = P_0 g := (P_{0,1} g_1, \ldots, P_{0,N} g_N). \tag{5.21}$$

Here, for every j, $1 \leq j \leq N$ and $\zeta \neq 0$ in a neighbourhood of the origin we take ζ_0 such that $\zeta \in \gamma_{\zeta_0,j}$ and $P_{0,j}$ is given by

$$P_{0,j} g_j := \int_{\zeta_0}^{\zeta} g_j(s^{\lambda_1} c_1, \ldots, s^{\lambda_{n-1}} c_{n-1}, s; y_0 - \Phi_j(s, b)) s^{-1} ds, \tag{5.22}$$

where the integral is taken along the curve $\gamma_{\zeta_0,j}$ from ζ_0 to $\zeta \in \gamma_{\zeta_0,j}$. The independent variables in (5.22) *satisfy the relation* (5.6).

Proof. We show that the integrand in (5.22) is well defined. By (5.6) and (5.7), we have

$$y_0 - \Phi_j(s, b) = y - \Phi_j(s, b) + \Phi_j(\zeta, b) = y + \Phi_j(\zeta, s). \tag{5.23}$$

By Lemma 5.2, we have that $\operatorname{Im}\Phi_j(\zeta, s) = 0$ if $s \in \gamma_{\zeta_0,j}$ because $\zeta \in \gamma_{\zeta_0,j}$. On the other hand, by Lemma 5.3, we have that $\operatorname{Re}\Phi_j(\zeta, s)$ is a monotone decreasing function of $\zeta \in \gamma_{\zeta_0,j}$ when ζ approaches the origin. Hence we have $\operatorname{Re}\Phi_j(\zeta, s) \leq 0$ on $\gamma_{\zeta_0,j}$. In view of the assumption on Ω we have $y + \operatorname{Re}\Phi_j(\zeta, s) \in \Omega$ for every $y \in \Omega$.

Next, we take the neighborhood U_0 of the origin such that the formal solution is holomorphic in U_0. We want to show that substitution $x_k = s^{\lambda_k} c_k$ into the integrand of (5.22) is possible for s which is on the segment of $\gamma_{\zeta_0,j}$ between ζ_0 and ζ. For the purpose of this we shall show that $s^{\lambda_k}c_k$ is sufficiently small by taking c_k sufficiently small. We observe that

$$s^{\lambda_j} = \exp\left(\lambda_j(\log|s| + i \arg s)\right). \tag{5.24}$$

Because $\lambda_j > 0$, the absolute value of the right-hand side of (5.24) is monotone decreasing when $r = |s|$ tends to zero, namely s tends to the origin along $\gamma_{\zeta_0,j}$. This proves the assertion. Hence the right-hand side of (5.22) is well defined. We note that the integrand is integrable at the origin in view of the assumption $g_j(0, y) \equiv 0$ for every $y \in \Omega$.

Next, we shall show that $w_j := P_{0,j}g_j$ $(j = 1, 2, \dots, N)$ satisfies the equation (5.4), namely

$$\mathcal{L}_j w_j - (\nabla_u f)_j(x, 0)\frac{\partial w_j}{\partial y} = g_j(x, y). \tag{5.25}$$

Indeed, by (5.5) and (5.6), we have

$$\begin{aligned} g_j(x, y)x_n^{-1} = \frac{dw_j}{d\zeta} &= \sum_{k=1}^{n} \frac{\partial x_k}{\partial \zeta}\frac{\partial w_j}{\partial x_k} + \frac{\partial y}{\partial \zeta}\frac{\partial w_j}{\partial y} \\ &= \sum_{k=1}^{n} \frac{\lambda_k x_k}{\zeta}\frac{\partial w_j}{\partial x_k} - \frac{(\nabla_u f)_j(x, 0)}{\zeta}\frac{\partial w_j}{\partial y}. \end{aligned} \tag{5.26}$$

Multiplying both sides with ζ and setting $\zeta = x_n$ we have (5.25). This completes the proof. □

Let ζ_0 satisfy $|\zeta_0| = r_0 > 0$. In the following we assume that there exists an $\varepsilon_0 > 0$ such that $|\zeta|/|\zeta_0| \geq \varepsilon_0$ for ζ corresponding to D, where we recall relation (5.6).

Lemma 5.5. *There exists a constant* c_1 *such that, for every* $1 \leq j \leq N$, $g_j \in \mathcal{H}(D, \Omega)$, *we have*

$$\|P_{0,j}g_j\|_c \leq c_1\|g_j\|_c, \quad \left\|\frac{\partial}{\partial y}(P_{0,j}g_j)\right\|_c \leq c_1\|g_j\|_c. \tag{5.27}$$

The constant c_1 *is independent of* ζ_0, $|\zeta_0| = r_0 > 0$.

Proof. We first show that the integral (5.22) converges when $\zeta \in \gamma_{\zeta_0,j}$. Noting that $y_0 - \Phi_j(s, b) = y + \Phi_j(\zeta, s)$ we make the change of variable $\sigma = y + \Phi_j(\zeta, s)$ in (5.22) from s to σ. We have $d\sigma = -\frac{(\nabla_u f)_j}{s}ds$. Observe that the right-hand side is independent of y. We have $\sigma = y$ for $s = \zeta$ and $\sigma = y + \tilde{\zeta_0}$ for $s = \zeta_0$, where $\tilde{\zeta_0} = \Phi_j(\zeta, \zeta_0)$. Clearly, $s \in \gamma_{\zeta_0,j}$ is expressed as $\sigma \in y + \gamma_{\tilde{\zeta_0},j}$, where $\gamma_{\tilde{\zeta_0},j}$ is the straight line connecting 0 and $\tilde{\zeta_0}$. Then (5.22) is written in

$$w = -\int_{\gamma_{\tilde{\zeta_0},j}} g(s^{\lambda_1}c_1, \dots, s^{\lambda_{n-1}}c_{n-1}, s; \sigma)\frac{d\sigma}{(\nabla_u f)_j}, \tag{5.28}$$

where $(\nabla_u f)_j$ is bounded from below by the assumption (2.3).

We next estimate the growth of $y_0 - \Phi_j(s,b)$. In terms of (5.23) we have

$$\exp\left(-c\mathrm{Re}\left(y_0 - \Phi_j(s,b)\right)\right) = \exp\left(-c\mathrm{Re}\left(y + \Phi_j(\zeta,s)\right)\right). \tag{5.29}$$

By Lemma 5.3, $\mathrm{Re}\,\Phi_j(\zeta,s)$ is decreasing in ζ as ζ tends to the origin along $\gamma_{\zeta_0,j}$. It follows that $\mathrm{Re}\,\Phi_j(\zeta,s) \le \mathrm{Re}\,\Phi_j(s,s) = 0$. Hence we need to estimate $e^{-c\mathrm{Re}\,\Phi_j(\zeta,s)}$. We have that $\Phi_j(\zeta,s)$ is asymptotically equal to $\mu_j \log(\zeta/s)$. Set $\log(\zeta/s) = x+iy$ and $\mu_j = \alpha + i\beta$ with $\alpha > 0$. Then we have $\mathrm{Re}\left(\mu_j \log(\zeta/s)\right) = \alpha x - \beta y$. On the other hand, by definition we have $\beta x + \alpha y = c$ for some c. Hence $\alpha x - \beta y = (\alpha + \beta^2\alpha^{-1})x - c\beta\alpha^{-1}$. Noting that $x = \log(|\zeta|/|s|) > \log(|\zeta|/|\zeta_0|) > \log\varepsilon_0$, we have

$$\begin{aligned}\exp(-c(\alpha x - \beta y)) &= \exp(-(\alpha + \beta^2\alpha^{-1})cx + c^2\beta\alpha^{-1}) \\ &\le \exp\left((\alpha + \beta^2\alpha^{-1})c\log\varepsilon_0^{-1} + c^2\beta\alpha^{-1}\right) =: K_0.\end{aligned}$$

This proves

$$\exp\left(-c\mathrm{Re}\left(y_0 - \Phi_j(s,b)\right)\right) \le K_0 \exp\left(-c\mathrm{Re}\, y\right). \tag{5.30}$$

We shall estimate $|y_0 - \Phi_j(s,b)| = |y + \Phi_j(\zeta,s)|$ from the below. Because $\mathrm{Im}\,\Phi_j(\zeta,s) = 0$ and $\mathrm{Re}\,\Phi_j(\zeta,s) \le 0$ on $\gamma_{\zeta_0,j}$, there exists $C_1 > 0$ independent of ζ and s such that

$$(1 + |y_0 - \Phi_j(s,b)|)^{-2} \le C_1(1+|y|)^{-2} \quad \text{for all } y \in \Omega. \tag{5.31}$$

Therefore, we get from (5.30) and (5.31) that

$$\begin{aligned}\|w_j\|_c &\le \sup\left((1+|y|)^2 \exp(c\mathrm{Re}\, y) \int \|g_j\|_c \frac{\exp\left(-c_{\mathrm{Re}}\left(y_0 - \Phi_j(s,b)\right)\right)}{(1 + |y_0 - \Phi_j(s,b)|)^2} |d\sigma|\right) \\ &\le C_2 \|g_j\|_c \int |d\sigma| \le C_3 \|g_j\|_c\end{aligned} \tag{5.32}$$

for some $C_2 > 0$ and $C_3 > 0$.

We shall show the latter inequality of (5.27). We have

$$\begin{aligned}w_y = &-g(\zeta_0^{\lambda_1}c_1, \ldots, \zeta_0^{\lambda_{n-1}}c_{n-1}, \zeta_0; y + \tilde{\zeta}_0)\frac{1}{(\nabla_u f)_j} \\ &+ g(\zeta^{\lambda_1}c_1, \ldots, \zeta^{\lambda_{n-1}}c_{n-1}, \zeta; y)\frac{1}{(\nabla_u f)_j}.\end{aligned} \tag{5.33}$$

Using (5.33) we have the latter inequality of (5.27) by the same argument as $\|w\|_c$ since $(\nabla_u f)_j$ is bounded. □

We shall solve (5.3) in $\mathcal{H}_c(D,\Omega)$. First we note

$$\nabla_u f(x,v_0)\frac{\partial \hat{u}}{\partial y} = \nabla_u f(x,0)\frac{\partial \hat{u}}{\partial y} + (\nabla_u f(x,v_0) - \nabla_u f(x,0))\frac{\partial \hat{u}}{\partial y}. \tag{5.34}$$

We note $\|\nabla_u f(x,v_0) - \nabla_u f(x,0)\| = O(\|v_0\|)$ when $\|v_0\| \to 0$. Note that these terms are also estimated by $K_4\varepsilon\|w_y\|_c$, where ε is small and K_4 is some constant.

We define the approximate sequence $\hat{u}_k$ $(k = 0, 1, 2, \ldots)$ by $\hat{u}_0 = 0$ and

$$\hat{u}_1 = -P_0 \mathcal{L} v_0 \tag{5.35}$$

$$\begin{aligned}\hat{u}_2 = P_0 \sum_{|\beta|\geq 2} r_\beta(x, v_0) \frac{\partial}{\partial y} (\hat{u}_1)_*^\beta - P_0 \mathcal{L} v_0 + P_0 R(x) \frac{\partial}{\partial y} \hat{u}_1 \\ + P_0(\nabla_u f(x, v_0) - \nabla_u f(x, 0)) \frac{\partial \hat{u}_1}{\partial y},\end{aligned} \tag{5.36}$$

$$\vdots$$

$$\begin{aligned}\hat{u}_{k+1} = P_0 \sum_{|\beta|\geq 2} r_\beta(x, v_0) \frac{\partial}{\partial y} (\hat{u}_k)_*^\beta - P_0 \mathcal{L} v_0 + P_0 R(x) \frac{\partial}{\partial y} \hat{u}_k \\ + P_0(\nabla_u f(x, v_0) - \nabla_u f(x, 0)) \frac{\partial \hat{u}_k}{\partial y},\end{aligned} \tag{5.37}$$

where $k = 1, 2, \ldots$ Then we have the following lemma.

Lemma 5.6. *Let D be as in Lemma 5.5. Then there exists a constant $K_3 > 0$ independent of k such that*

$$\|\hat{u}_k\|_c \leq C\varepsilon K_3, \quad \|(\hat{u}_k)_y\|_c \leq C\varepsilon K_3, \quad k = 0, 1, 2, \ldots \tag{5.38}$$

Proof. In order to show that the sequence is well defined we make an a priori estimate. Given $\varepsilon > 0$. We take $|\zeta_0|$ sufficiently small such that $\|\mathcal{L} v_0\|_c \leq \varepsilon$. By (5.27), we have

$$\|\hat{u}_1\|_c \leq \|P_0 \mathcal{L} v_0\|_c \leq C \|\mathcal{L} v_0\|_c \leq C\varepsilon. \tag{5.39}$$

Similarly, by using (5.27) we have $\|(\hat{u}_1)_y\|_c \leq C\varepsilon$.

Next, we estimate $\|\hat{u}_2\|_c$ and $\|(\hat{u}_2)_y\|_c$. Because the argument is similar, we consider $\|\hat{u}_2\|_c$. Because $v_0(x) = O(|x|)$, there exist $K_5 > 0$ and $K_6 > 0$ such that for every $\varepsilon > 0$ we have

$$|r_\beta|_\infty := \sup_{x \in D} |r_\beta(x, v_0(x))| \leq \varepsilon K_5 K_6^{|\beta|}$$

for all $|\beta| \geq 2$ if D is sufficiently small. By (5.36), (5.39), (4.4) and the elementary property of convolution, we have

$$\begin{aligned}\|\hat{u}_2\|_c &\leq C \|\mathcal{L} v_0\|_c + C \sum_{|\beta|\geq 2} \left\| r_\beta \frac{\partial}{\partial y} (\hat{u}_1)^\beta \right\| + 2C^2 \varepsilon^2 K_4 \\ &\leq C\varepsilon + C \sum_\beta |r_\beta|_\infty (C\varepsilon)^{|\beta|} + 2C^2 \varepsilon^2 K_4 \\ &\leq C\varepsilon \left(1 + C\varepsilon K_5 \sum_{|\beta|\geq 2} K_6^{|\beta|} (C\varepsilon)^{|\beta|-1} \right) + 2C^2 \varepsilon^2 K_4.\end{aligned} \tag{5.40}$$

If we take $C\varepsilon K_6 < 1$, then there exists $K_7 > 0$ such that the right-hand side of (5.40) can be estimated by $C\varepsilon(1 + 2C\varepsilon K_4 + C^2 K_5 K_6^2 K_7 \varepsilon^2)$. Hence, if we take ε so that $C^2 K_5 K_6^2 K_7 \varepsilon \leq 1$, then we have $\|\hat{u}_2\|_c \leq C\varepsilon K_3$ for some $K_3 > 0$ independent of ε. Similarly, we have $\|(\hat{u}_2)_y\|_c \leq C\varepsilon K_3$.

We continue to estimate $\|\hat{u}_3\|_c$ and $\|(\hat{u}_3)_y\|_c$. Clearly, we see that the same argument works if we replace K_6 with some constant K_8. By induction we have an a priori estimate. □

Lemma 5.7. *Under the same assumptions as in Lemma 5.6 we have that* $\hat{u}_k$ $(k = 1, 2, \dots)$ *converges in* $\mathcal{H}_c(D, \Omega)$.

Proof. Let $l > m$ and write $\hat{u}_l - \hat{u}_m = \sum_{j=m}^{l-1} (\hat{u}_{j+1} - \hat{u}_j)$. By (5.37), we have

$$
\begin{aligned}
&\hat{u}_{j+1} - \hat{u}_j \\
&= P_0 \sum_{|\beta| \geq 2} r_\beta \frac{\partial}{\partial y} \left((\hat{u}_j)^{*\beta} - (\hat{u}_{j-1})^{*\beta} \right) \\
&\quad - P_0 R(x) \frac{\partial}{\partial y} (\hat{u}_j - \hat{u}_{j-1}) + P_0(\nabla_u f(x, v_0) - \nabla_u f(x, 0)) \frac{\partial}{\partial y} (\hat{u}_j - \hat{u}_{j-1}) \\
&= P_0 \sum_\beta r_\beta \frac{\partial}{\partial y} \left(\sum_{\nu=1}^n (\hat{u}_{j,\nu} - \hat{u}_{j-1,\nu}) * R_\nu(\hat{u}_j, \hat{u}_{j-1}) \right) \\
&\quad - P_0 R(x) \frac{\partial}{\partial y} (\hat{u}_j - \hat{u}_{j-1}) + P_0(\nabla_u f(x, v_0) - \nabla_u f(x, 0)) \frac{\partial}{\partial y} (\hat{u}_j - \hat{u}_{j-1}),
\end{aligned}
\tag{5.41}
$$

where $R_\nu(\hat{u}_j, \hat{u}_{j-1})$ is the polynomial of $\hat{u}_j$ and $\hat{u}_{j-1}$ with degree greater than or equal to $|\beta| - 1 \geq 1$ with respect to the convolution product.

We shall show that

$$
\|\hat{u}_{j+1} - \hat{u}_j\|_c \leq 2^{-1} \|(\hat{u}_j - \hat{u}_{j-1})_y\|_c, \quad \|(\hat{u}_{j+1} - \hat{u}_j)_y\|_c \leq 2^{-1} \|(\hat{u}_j - \hat{u}_{j-1})_y\|_c, \tag{5.42}
$$

if ε is sufficiently small. Because the proof is similar, we shall show the latter one. In order to estimate $\|(\hat{u}_{j+1} - \hat{u}_j)_y\|_c$ we apply $\partial/\partial y$ to both sides of (5.41). Then we estimate the right-hand side. In view of Lemma 5.5 we may consider the following terms

$$
\begin{aligned}
&\sum \left\| r_\beta \left(\sum (\hat{u}_{j,\nu} - \hat{u}_{j-1,\nu})_y * R_\nu(\hat{u}_j, \hat{u}_{j-1}) \right) \right\|_c \\
&+ \left\| R(x)(\hat{u}_j - \hat{u}_{j-1})_y \right\|_c + \left\| (\nabla_u f(x, v_0) - \nabla_u f(x, 0))(\hat{u}_j - \hat{u}_{j-1})_y \right\|_c .
\end{aligned}
$$

The first term is estimated by using the estimate of the convolution in §4. Because $\|R_\nu(\hat{u}_j, \hat{u}_{j-1})\|_c = O(\epsilon)$ by virtue of (5.38), we can estimate the first term by a constant times $\epsilon \|(\hat{u}_j - \hat{u}_{j-1})_y\|_c$. The second and the third terms can be estimated by a constant times $\epsilon \|(\hat{u}_j - \hat{u}_{j-1})_y\|_c$, because $R(x) = O(|x|)$ and $\nabla_u f(x, v_0) - \nabla_u f(x, 0) = O(|x|)$. Hence, by taking ϵ sufficiently small, we have the second inequality of (5.42). Finally, the estimate (5.42) shows that $\hat{u}_k$ is a Cauchy sequence in $\mathcal{H}_c(D, \Omega)$ and it converges to some $\hat{u} \in \mathcal{H}_c(D, \Omega)$. Hence, we obtain the solution $\hat{u}$. □

We observe that Lemma 5.7 implies the solvability of (5.3) in $\mathcal{H}_c(D,\Omega)$.

Proof of Theorem 5.1. First we show the solvability of (5.3) on D corresponding to the annulus $T_0 = \{\varepsilon_0 r_0 < |\zeta| < r_0\}$ in terms of (5.6). Take an open set A_0 in the annulus and solve (5.3) for D corresponding to A_0. By Lemma 5.7, we have the solvability of (5.3) on D as in the above, which is equivalent to the summability on D. Next, take an open set A_1 in the annulus such that $A_0 \cap A_1 \neq \emptyset$. Then we have the summability on some D_1 corresponding to A_1. By virtue of the uniqueness of the Borel sum two sums corresponding to A_0 and A_1 coincide on the set $A_0 \cap A_1$. Hence, we have an analytic continuation of the solution of (5.3) to the domain corresponding to $A_0 \cup A_1$. By repeating the argument we have the solvability of (5.3) for D corresponding to ζ such that $\varepsilon_0 r_0 < |\zeta| < r_0$.

Next, we take annulus T_1 with r_0 replaced by r_1 such that $T_0 \cap T_1 \neq \emptyset$. Then we have the summability on the domain corresponding to T_1. Moreover, in the proof of Lemma 5.5 the constant in the estimate in (5.28) depends on an integral like $\int_a^b s^{-1} ds = \log(a/b)$. Hence we have the solvability of (5.3) in the same domain in the sense that we have the summability in $\mathcal{H}_c(D_1,\Omega)$ for the same c and Ω. By the uniqueness of the Borel sum we can make analytic continuation with respect ζ. Therefore, we have the solution $\hat{u}$ in the small neighborhood of the origin such that $\zeta \neq 0$.

Let u be the Laplace transform of $\hat{u}$. Then u is the Borel sum of the formal solution with respect to η when $x \in D$. Note that u and $\hat{u}$ are analytic with respect to x in D. We denote u and $\hat{u}$, respectively, by u_D and $\hat{u}_D$. Let D' be any domain such that $D \cap D' \neq \emptyset$ and let u_D and $u_{D'}$ be the corresponding Borel sum in D and D', respectively. Because the Borel sum with respect to η is unique for every x, we have that $u_D = u_{D'}$ on $D \cap D'$, from which we have an analytic continuation of u_D to $D \cup D'$. By choosing the sequence of open sets D we make an analytic continuation of u_D to the set $(\mathbb{C} \setminus 0)^n \cap B_0$, where B_0 is a small open ball centered at the origin. By the uniqueness of the Borel sum the analytic continuation of $\hat{u}_D(x,y)$ with respect to x to the set $(\mathbb{C} \setminus 0)^n \cap B_0$, $y \in \Omega$ is single-valued. We also note that in view of the construction of $\hat{u}_D$ the growth estimate with respect to y of $\hat{u}_D(x,y)$ is uniform for $x \in (\mathbb{C} \setminus 0)^n \cap B_0$. Therefore, we can define $\hat{u}(x,y) := \hat{u}_D(x,y)$ on $x \in (\mathbb{C} \setminus 0)^n \cap B_0$ and $y \in \Omega$ by taking $x \in D$.

The function $\hat{u}(x,y)$ may have singularity on $x \in (\mathbb{C}^n \setminus (\mathbb{C} \setminus 0)^n) \cap B_0$, $y \in \Omega$. We shall prove that the singularity is removable. First consider the singularity with codimension 1. For simplicity, let us take $y_0 \in \Omega$, $x_0' = (x_2^0, \ldots, x_n^0)$ with $x_j^0 \neq 0$ and consider the expansion

$$\hat{u}(x,y) = \sum_{\nu \geq 0, j \geq 0} \hat{u}_{\nu,j}(x_1)(x' - x_0')^\nu (y - y_0)^j. \tag{5.43}$$

By what we have proved in the above, the right-hand side is convergent if $x' - x_0'$ and $y - y_0$ are sufficiently small and $x_1 \neq 0$. Moreover, by the boundedness of $\hat{u}(x,y)$ when $x_1 \to 0$ and Cauchy's integral formula we have that $\hat{u}_{\nu,j}(x_1)$ is holomorphic and single-valued and bounded in the neighborhood of the origin except for $x_1 = 0$.

Hence, its singularity is removable. In the same way one can show that the singularity of codimension 1 is removable.

Next, we consider the singularity of codimension 2. For the sake of simplicity, consider the one $x_1 = x_2 = 0$, $x_0'' = (x_3^0, \dots, x_n^0)$ with $x_j^0 \neq 0$. By arguing in the same way as in the codimension-one case we have an expansion similar to (5.43) where $x' - x_0'$ and $\hat{u}_{\nu,j}(x_1)$ are replaced by $x'' - x_0''$ and $\hat{u}_{\nu,j}(x_1, x_2)$, respectively. Because $\hat{u}_{\nu,j}(x_1, x_2)$ is holomorphic and single-valued except for $x_1 = x_2 = 0$, we see that the singularity is removable by Hartogs' theorem. As for the singularity of higher codimension ≥ 3 we can argue in the same way by using Hartogs' theorem. We see that $\hat{u}(x, y)$ is holomorphic and single-valued on $x \in \mathbb{C}^n \cap B_0$, $y \in \Omega$.

The exponential growth of $\hat{u}(x, y)$ when $y \to \infty$ in $y \in \Omega$ for $x \in \mathbb{C}^n \cap B_0$ can be proved by putting some c_k to be equal to zero when constructing $\hat{u}_D(x, y)$. Indeed, we have already proved the fact in the above argument. Hence, we have proved the solvability of (5.3), and the summability of our solution as desired. If we choose the neighborhood of $x = 0$ sufficiently small, then we have the summability of every component of the formal solution. This completes the proof of Theorem 5.1. □

End of the proof of Theorem 2.1. We shall prove the summability in the direction $\eta \in S_{\theta,\xi}$. By multiplying the equation (2.2) with $e^{-i\theta}$ we see that η, λ_k, μ_j are replaced by $\eta e^{-i\theta}$, λ_k and $\mu_j e^{-i\theta}$, respectively. Noting that the conditions (2.10) are satisfied for $0 \leq \theta < \pi/2 - \theta_1$, the summability holds for $\eta = e^{i(\pi-\theta)}$ with $0 \leq \theta < \pi/2 - \theta_1$. Hence, the summability holds for $-3\pi/2 + \theta_1 < \arg \eta \leq -\pi$. On the other hand, we see that (2.10) is satisfied for $-\pi/2 + \theta_2 < \theta \leq 0$. It follows that the summability holds for $-\pi < \arg \eta \leq -\pi/2 - \theta_2$. Therefore, the summability holds for $-3\pi/2 + \theta_1 < \arg \eta < -\pi/2 - \theta_2$. Hence, we have the latter half in view of the definition of Borel sum. This ends the proof of Theorem 2.1. □

6. SOME REMARKS

In Theorem 2.1 we proved Borel summability of $v(x, \eta)$ when $x \in U$. We study the summability in the case $x \neq 0$. Instead of (2.3) we assume that there exists $a \in \mathbb{C}^n$ and $b \in \mathbb{C}^N$ such that

$$f(a, b) = 0, \qquad \det(\nabla_u f(a, b)) \neq 0. \tag{6.1}$$

By an implicit function theorem one can construct $v_0(x)$ analytic at $x = a$ such that $v_0(a) = b$ and $f(x, v_0(x)) \equiv 0$ in the neighborhood of a. Define Σ_0 by

$$\Sigma_0 := \{x;\ \det\left((\nabla_u f)(x, v_0(x))\right) = 0,\ f(x, v_0(x)) = 0\}. \tag{6.2}$$

Observe that $a \notin \Sigma_0$. Let $\Omega_1 \subset \mathbb{C}^n \setminus \Sigma_0$ be the maximal domain containing a and not containing the origin on which v_0 is holomorphic. One can construct the formal solution $v(x, \eta)$ in (2.4). By a similar proof like Proposition 3.2 the formal Borel transform of $v(x, \eta)$ converges for x in some domain $\Omega' \subset \Omega_1$ with compact closure. For the sake of simplicity we assume $\Omega' = \Omega_1$ in the following. We study Borel summability of $v(x,\eta)$ with respect to η when $x \in \Omega_1$. In fact, we have the following theorem.

Theorem 6.1. *Assume that $f(x,u)$ is an entire function of $x \in \mathbb{C}^n$ and $u \in \mathbb{C}^N$ such that $\nabla_u f(x, v_0(x))$ is a diagonal matrix for every $x \in \Omega_1$. Then $v(x,\eta)$ is 1-summable in the direction ξ, $\frac{\pi}{2} < \arg \xi < \frac{3\pi}{2}$ with respect to η for any $x \in \Omega_1$.*

We observe that the condition (2.10) is not necessary in the above theorem. The proof of Theorem 6.1 is done by modifying the proof of Theorem 2.1. We omit the details.

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