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# MAILLET TYPE THEOREM FOR SINGULAR FIRST ORDER NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF TOTALLY CHARACTERISTIC TYPE. PART II

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**Abstract.** In this paper, we study the following nonlinear first order partial differential equation:

$$f(t,x,u,\partial_t u,\partial_x u)=0 \quad \text{with} \quad u(0,x)\equiv 0.$$

The purpose of this paper is to determine the estimate of Gevrey order under the condition that the equation is singular of a totally characteristic type. The Gevrey order is indicated by the rate of divergence of a formal power series. This paper is a continuation of the previous papers [*Convergence of formal solutions of singular first order nonlinear partial differential equations of totally characteristic type*, Funkcial. Ekvac. 45 (2002), 187–208] and [*Maillet type theorem for singular first order nonlinear partial differential equations of totally characteristic type*, Surikaiseki Kenkyujo Kokyuroku, Kyoto University 1431 (2005), 94–106]. Especially the last-mentioned paper is regarded as part I of this paper.

**Keywords:** singular partial differential equations, totally characteristic type, nilpotent vector field, formal solution, Gevrey order, Maillet type theorem.

**Mathematics Subject Classification:** 35F20, 35A20, 35C10.

## 1. INTRODUCTION AND MAIN RESULTS

Let $\mathbb{C}$ be the set of complex numbers or a variable, $t=(t_1,\ldots,t_d)\in\mathbb{C}^d$ and $x=(x_1,\ldots,x_n)\in\mathbb{C}^n$. We consider the following first order nonlinear partial differential equation:

$$\begin{cases} f(t,x,u,\partial_t u,\partial_x u)=0, \\ u(0,x)\equiv 0, \end{cases} \tag{1.1}$$

where $u(t,x)$ denotes the unknown function, $\partial_t u = (\partial_{t_1} u, \dots, \partial_{t_d} u)$ and $\partial_x u$ is defined similarly. Here, we assume that the function $f(t,x,u,\tau,\xi)$ ($\tau = (\tau_1, \dots, \tau_d) \in \mathbb{C}^d$, $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$) is holomorphic in a neighborhood of the origin of $\mathbb{C}^d \times \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^d \times \mathbb{C}^n$, and is an entire function in $\tau$ variables for any fixed $t$, $x$, $u$ and $\xi$.

The purpose of this paper is to characterize the rate of divergence of formal solutions by using the "Gevrey order", and such a characterization theorem is called "Maillet type theorem". In order to study the Maillet type theorem for the above equation, we assume the following three assumptions.

**Assumption 1.1** (Singular equation)**.** The function $f(t,x,u,\tau,\xi)$ is singular in $t$ variables in the sense that

$$f(0,x,0,\tau,0) \equiv 0 \quad (\text{for all } x \in \mathbb{C}^n \text{ near } x = 0, \text{ and all } \tau \in \mathbb{C}^d). \tag{1.2}$$

**Assumption 1.2** (Existence of formal solutions)**.** The equation (1.1) has a formal solution of the form

$$u(t,x) = \sum_{j=1}^{d} \varphi_j(x) t_j + \sum_{|\alpha| \geq 2, |\beta| \geq 0} u_{\alpha,\beta} t^\alpha x^\beta \quad \text{for some} \quad \{\varphi_j(x)\}_{j=1}^d \in \mathbb{C}\{x\}^d, \tag{1.3}$$

where $\mathbb{C}\{x\}$ denotes the set of holomorphic functions at $x = 0$, for $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ ($\mathbb{N} = \{0, 1, 2, \dots\}$) we define $|\alpha| = \alpha_1 + \dots + \alpha_d$ and $t^\alpha = t_1^{\alpha_1} \dots t_d^{\alpha_d}$, and $|\beta|$ and $x^\beta$ are defined similarly for $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$.

**Assumption 1.3** (Totally characteristic type)**.** The equation (1.1) is of totally characteristic type with respect to $\{\varphi_j(x)\}$ in (1.3), which means that the following conditions hold:

$$\begin{cases} f_{\xi_k}(0,x,0,\{\varphi_j(x)\},0) \not\equiv 0 \\ f_{\xi_k}(0,0,0,\{\varphi_j(0)\},0) = 0 \end{cases} \quad \text{for} \quad k = 1, 2, \dots, n. \tag{1.4}$$

**Remark 1.4.** The functions $\{\varphi_j(x)\}$ in (1.3) are obtained as a solutions of the following $d$-system of equations:

$$\begin{aligned} &\frac{\partial}{\partial t_i} f(t,x,u(t,x),\partial_t u(t,x),\partial_x u(t,x))\Big|_{t=0} \\ &= f_{t_i}(0,x,0,\{\varphi_j(x)\},0) + f_u(0,x,0,\{\varphi_j(x)\},0)\varphi_i(x) \\ &\quad + \sum_{k=1}^{n} f_{\xi_k}(0,x,0,\{\varphi_j(x)\},0) \frac{\partial \varphi_i}{\partial x_k}(x) = 0, \quad i = 1, 2, \dots, d. \end{aligned} \tag{1.5}$$

In the case $d = 1$, a sufficient condition that the formal solution of (1.5) to be convergent is obtained by Miyake and Shirai ([6]). In the case $d \geq 2$, a sufficient condition obtained by Shirai ([13]).

Now we put $\boldsymbol{\varphi}(x) = (0, x, 0, \{\varphi_j(x)\}, 0)$ for simplicity of notation. We define functions $a_{i,j}(x)$ $(i, j = 1, \dots, d)$ and $b_k(x)$ $(k = 1, \dots, n)$ by

$$a_{i,j}(x) := f_{t_i,\tau_j}(\boldsymbol{\varphi}(x)) + f_{u,\tau_j}(\boldsymbol{\varphi}(x))\varphi_i(x) + \sum_{k=1}^{n} f_{\tau_j,\xi_k}(\boldsymbol{\varphi}(x))\frac{\partial \varphi_i}{\partial x_k}(x), \tag{1.6}$$

$$b_k(x) := f_{\xi_k}(\boldsymbol{\varphi}(x)). \tag{1.7}$$

**Remark 1.5.** By the assumption of totally characteristic type, $b_k(x)$ satisfies $b_k(x) \not\equiv 0$, $b_k(0) = 0$ for all $k = 1, 2, \dots, n$.

Let $M_1$ and $M_2$ be the Jordan canonical forms of $(a_{i,j}(0))$ and $J(b_1, \dots, b_n)(0)$ respectively, where $J(b_1, \dots, b_n)(x)$ denotes the Jacobi matrix of $(b_1(x), \dots, b_n(x))$, we denote them by

$$(a_{i,j}(0)) \sim M_1, \quad J(b_1, \dots, b_n)(0) \sim M_2.$$

Then the following two cases were already studied by the author's previous papers.

(a) $M_1$ and $M_2$ are regular matrices with Poincaré condition (see Theorem 2.1 in §2 or [13]).
(b) $M_1$ is a regular matrix with Poincaré condition and $M_2$ is a nilpotent matrix (see Theorem 2.2 in §2 or [15]).

In this paper we shall study the following two cases, and the main results are stated as Theorem 1.6 and Theorem 1.7.

(a) $M_1$ is a nilpotent matrix and $M_2$ is a regular matrix with Poincaré condition.
(b) $M_1$ and $M_2$ are nilpotent matrices.

In order to state our main theorems, we prepare some notations.

In case (c), we put $M_1$ and $M_2$ by

$$M_1 = \begin{pmatrix} N_1 & & & \\ & N_2 & & \\ & & \ddots & \\ & & & N_I \end{pmatrix}, \quad \text{where} \quad N_j = \begin{pmatrix} 0 & & & \\ \delta & 0 & & \\ & \ddots & \ddots & \\ & & \delta & 0 \end{pmatrix} \quad \text{of size } d_j (\geq 1),$$

$$M_2 = \begin{pmatrix} \mu_1 & & & \\ \nu_1 & \mu_2 & & \\ & \ddots & \ddots & \\ & & \nu_{n-1} & \mu_n \end{pmatrix}, \quad \delta = 1, \quad \nu_j = 0 \ \text{ or } \ 1.$$

We note that $d_1 + d_2 + \ldots + d_I = d$. If the size $d_j = 1$, then $N_j = (0)$.

**Theorem 1.6.** *We consider case* (c). *Let Assumptions 1.1, 1.2, 1.3 and* $f_u(\boldsymbol{\varphi}(0)) \neq 0$ *be satisfied, and* $M_1$ *and* $M_2$ *be as above. Moreover, we assume the nonresonance-Poincaré condition for* $M_2$*, that is,*

$$\left| \sum_{k=1}^{n} \mu_k \beta_k + f_u(\boldsymbol{\varphi}(0)) \right| \geq C(|\beta| + 1) \tag{1.8}$$

*by a positive constant $C$ independent $\beta \in \mathbb{N}^n$ for all $\beta$. Then the formal solution $u(t,x)$ belongs to the Gevrey class of order at most $(2d_0, d_0+1)$ by $d_0 := \max\{d_1, \ldots, d_I\}$ (which is greater or equal to 1). This means that for the formal solution*

$$u(t,x) = \sum_{|\alpha|\geq 1, |\beta|\geq 0} u_{\alpha,\beta} t^\alpha x^\beta,$$

*the power series*

$$\sum_{|\alpha|\geq 1, |\beta|\geq 0} \frac{u_{\alpha,\beta}}{|\alpha|!^{2d_0-1}|\beta|!^{d_0}} t^\alpha x^\beta$$

*is convergent in a neighborhood of the origin.*

In case (d), we have $M_1$ and $M_2$

$$M_1 = \begin{pmatrix} N_1 & & & \\ & N_2 & & \\ & & \ddots & \\ & & & N_I \end{pmatrix}, \quad \text{where} \quad N_j = \begin{pmatrix} 0 & & & \\ \delta & 0 & & \\ & \ddots & \ddots & \\ & & \delta & 0 \end{pmatrix} \quad \text{of size } d_j (\geq 1),$$

$$M_2 = \begin{pmatrix} \hat{N}_1 & & & \\ & \hat{N}_2 & & \\ & & \ddots & \\ & & & \hat{N}_J \end{pmatrix}, \quad \text{where} \quad \hat{N}_k = \begin{pmatrix} 0 & & & \\ \delta & 0 & & \\ & \ddots & \ddots & \\ & & \delta & 0 \end{pmatrix} \quad \text{of size } n_k (\geq 1).$$

We note that $d_1 + d_2 + \ldots + d_I = d$ and $n_1 + n_2 + \ldots + n_J = n$.

**Theorem 1.7.** *We consider case* (d). *Let Assumptions 1.1, 1.2, 1.3 and $f_u(\boldsymbol{\varphi}(0)) \neq 0$ be satisfied and $M_1$ and $M_2$ be as above. Then the formal solution $u(t,x)$ belongs to the Gevrey class of order at most $2n_0$ by $n_0 := \max\{d_1, \ldots, d_I, n_1, \ldots, n_J\} (\geq 1)$. This means that for the formal solution*

$$u(t,x) = \sum_{|\alpha|\geq 1, |\beta|\geq 0} u_{\alpha,\beta} t^\alpha x^\beta,$$

*the power series*

$$\sum_{|\alpha|\geq 1, |\beta|\geq 0} \frac{u_{\alpha,\beta}}{(|\alpha|+|\beta|)!^{2n_0-1}} t^\alpha x^\beta$$

*is convergent in a neighborhood of the origin.*

## 2. RELATED RESULTS

For the formal solution $u(t,x)$, we put $v(t,x) = u(t,x) - \sum_{j=1}^d \varphi_j(x) t_j = O(|t|^K)$ $(K \geq 2)$ as a new known function. By substituting $u = v + \sum_{j=1}^d \varphi_j(x) t_j$ into the equation (1.1), $v(t,x)$ satisfies the following singular first order nonlinear partial differential equation:

$$\begin{cases} \left( \displaystyle\sum_{i,j=1}^{d} a_{i,j}(x) t_i \partial_{t_j} + \sum_{k=1}^{n} b_k(x) \partial_{x_k} + c(x) \right) v(t,x) \\ \quad = \displaystyle\sum_{|\alpha|=K} d_\alpha(x) t^\alpha + f_{K+1}(t,x,v(t,x),\partial_t v(t,x),\partial_x v(t,x)), \\ v(t,x) = O(|t|^K), \end{cases} \tag{2.1}$$

where $c(x) = f_u(\varphi(x))$, $d_\alpha(x)$ is holomorphic in a neighborhood of the origin, and $f_{K+1}(t,x,v,\tau,\xi)$ is also holomorphic in a neighborhood of the origin with the Taylor expansion

$$f_{K+1}(t,x,v,\tau,\xi) = \sum_{V(\alpha,p,q,r)\geq K+1} f_{\alpha pqr}(x) t^\alpha v^p \tau^q \xi^r. \tag{2.2}$$

Here we used the following notation:

$$V(\alpha,p,q,r) = |\alpha| + Kp + (K-1)|q| + K|r|, \tag{2.3}$$

which denotes the order of zeros in $t$ for each terms $t^\alpha v(t,x)^p (\partial_t v(t,x))^q (\partial_x v(t,x))^r$.

For the equation (2.1), if $b_k(x) \equiv 0$ $(k = 1,2,\dots,n)$, (2.1) is written as follows:

$$\begin{cases} \left( \displaystyle\sum_{i,j=1}^{d} a_{i,j}(x) t_i \partial_{t_j} + c(x) \right) v(t,x) \\ \quad = \displaystyle\sum_{|\alpha|=K} d_\alpha(x) t^\alpha + f_{K+1}(t,x,v(t,x),\partial_t v(t,x),\partial_x v(t,x)), \\ v(t,x) = O(|t|^K). \end{cases}$$

This equation is called *the Fuchsian equation with respect to t variables.* To this equation, a lot of Maillet type theorems have been studied by many mathematicians. For example, Gérard-Tahara, and Miyake-Shirai study the nonlinear case, which is found in the book or papers [4,6,7] and [8]. They obtained the Maillet type theorem, which include the convergent case.

The case of $b_k(x) \not\equiv 0$ and $b_k(0) = 0$, which is the case of totally characteristic type, Chen-Tahara studied the convergence of a formal solution in the case when $(t,x) \in \mathbb{C}^2$ and $b_k(x) = O(x)$ with Poincaré condition ([3]). This result was extended to the case of several space variables by Chen-Luo ([1]). Moreover, these results were generalized by the author to the case of several time-space variables ([13]) (see Theorem 2.1).

On the other hand, Chen-Luo-Tahara studied the Maillet type theorem in the case of $(t,x) \in \mathbb{C}^2$ and $b_k(x) = O(x^K)$ $(K \geq 2)$ ([2]), and they obtained that the formal solution belongs to the Gevrey class of order $K/(K-1)$. Their Maillet type theorem was generalized by the author to the case of several time-space variables ([15]) (see Theorem 2.2).

The statements of [13] and [15] are written as follows.

**Theorem 2.1** ([13])**.** *If all eigenvalues* $\{\lambda_j\}_{j=1,2,\dots,d}$ *of* $(a_{i,j}(0))_{i,j=1,2,\dots,d}$ *and all eigenvalues* $\{\mu_k\}_{k=1,\dots,n}$ *of the Jacobi matrix* $J(b_1,\dots,b_n)(0)$ *satisfy the Poincaré condition* $\mathrm{Ch}(\{\lambda_j\},\{\mu_k\}) \not\ni 0$ *(convex hull of points* $\{\lambda_j\}$ *and* $\{\mu_k\}$*), then the formal solution converges in a neighborhood of the origin.*

**Theorem 2.2** ([15])**.** *If all eigenvalues* $\{\lambda_j\}_{j=1,\dots,d}$ *of* $(a_{i,j}(0))_{i,j=1,2,\dots,d}$ *satisfy the Poincaré condition* $\mathrm{Ch}(\{\lambda_j\}) \not\ni 0$*, and* $J(b_1,\dots,b_n)(0)$ *is nilpotent, then the formal solution belongs to the Gevrey class of order at most* $2d_0$ *in* $(t,x)$*, where* $d_0$ *denotes the maximum of size of nilpotent Jordan blocks of* $J(b_1,\dots,b_n)(0)$*.*

## 3. REFINEMENT OF THEOREM 1.6

In order to prove Theorem 1.6, we shall estimate the Gevrey order in each variables $(t_1,\dots,t_d,x_1,\dots,x_n)$ of formal solution of (2.1). To do so, we reduce (2.1) to a more exact form.

First, we set $\hat{a}_{i,j}(x) = a_{i,j}(0) - a_{i,j}(x) = O(|x|)$. Then the vector field with respect to $t$ variables is written by

$$\sum_{i,j=1}^{d} a_{i,j}(x) t_i \partial_{t_j} = (t_1,\dots,t_d) \begin{pmatrix} a_{1,1}(0) & \cdots & a_{1,d}(0) \\ \vdots & \ddots & \vdots \\ a_{d,1}(0) & \cdots & a_{d,d}(0) \end{pmatrix} \begin{pmatrix} \partial_{t_1} \\ \vdots \\ \partial_{t_d} \end{pmatrix} - \sum_{i,j=1}^{d} \hat{a}_{i,j}(x) t_i \partial_{t_j}.$$

Here we introduce new variables $\tau = (\tau^{(1)},\dots,\tau^{(I)}) \in \mathbb{C}^d$, $(\tau^{(j)} = (\tau_{j,1},\dots,\tau_{j,d_j}) \in \mathbb{C}^{d_j}, d = d_1 + \ldots + d_I)$ by

$$(\tau^{(1)},\dots,\tau^{(I)}) = (t_1,\dots,t_d)P, \qquad P^{-1}(a_{i,j}(0))P = M_1.$$

By this linear change of variables, the above vector field is reduced to

$$(\tau^{(1)},\dots,\tau^{(I)}) \begin{pmatrix} N_1 & & \\ & \ddots & \\ & & N_I \end{pmatrix} \begin{pmatrix} \partial_{\tau^{(1)}} \\ \vdots \\ \partial_{\tau^{(I)}} \end{pmatrix} - \sum_{i,j,k,l} \alpha_{ijkl}(x) \tau_{i,j} \partial_{\tau_{k,l}},$$

where $\sum_{i,j,k,l}$ is a summation taken over

$$1 \le i \le I,\ 1 \le j \le d_i,\ 1 \le k \le I,\ 1 \le l \le d_k.$$

Next, we write the differential operator with respect to $x$ variables by the following form:

$$\sum_{k=1}^{n} b_k(x) \partial_{x_k} = (x_1,\dots,x_n) J(b_1,\dots,b_n)(0) \begin{pmatrix} \partial_{x_1} \\ \vdots \\ \partial_{x_n} \end{pmatrix} - \sum_{k=1}^{n} \hat{b}_k(x) \partial_{x_k},$$

where $\hat{b}_k(x) = O(|x|^2)$ $(k = 1, \dots, n)$. Then we introduce new variables $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ by

$$(\xi_1, \dots, \xi_n) = (x_1, \dots, x_n)Q, \quad Q^{-1}J(b_1, \dots, b_n)(0)Q = M_2.$$

By this linear change of variables $x$, the above vector field is reduced to

$$\sum_{k=1}^{n} \mu_k \xi_k \partial_{\xi_k} + \sum_{k=1}^{n-1} \nu_k \xi_{k+1} \partial_{\xi_k} - \sum_{k=1}^{n} \beta_k(\xi) \partial_{\xi_k},$$

where $\beta_k(\xi) = O(|\xi|^2)$ $(k = 1, \dots, n)$.

Hereafter we rewrite $(\tau, \xi)$ by $(t, x)$ again. Then the equation (2.1) is reduced to the following one.

$$\begin{cases} (\mathcal{N} + \mathcal{D} + \Delta)v = \displaystyle\sum_{i,j,k,l} \alpha_{ijkl}(x) t_{i,j} \partial_{t_{k,l}} v + \sum_{k=1}^{n} \beta_k(x) \partial_{x_k} v \\ \qquad\qquad\qquad + \eta(x)v + \displaystyle\sum_{|\alpha|=K} \zeta_\alpha(x) t^\alpha + g_{K+1}(t, x, v, \partial_t v, \partial_x v), \\ v(t, x) = O(|t|^K), \end{cases} \tag{3.1}$$

where the operators $\mathcal{N}$, $\mathcal{D}$ and $\Delta$ are

$$\mathcal{N} = \sum_{j=1}^{I} \sum_{k=1}^{d_j - 1} \delta t_{j,k+1} \partial_{t_{j,k}}, \tag{3.2}$$

$$\mathcal{D} = \sum_{k=1}^{n} \mu_k x_k \partial_{x_k} + c(0), \quad (c(0) = f_u(\boldsymbol{\varphi}(0))), \tag{3.3}$$

$$\Delta = \sum_{k=1}^{n-1} \nu_k x_{k+1} \partial_{x_k}, \tag{3.4}$$

respectively. Moreover, $\eta(x) = c(0) - c(x) (= f_u(\boldsymbol{\varphi}(0)) - f_u(\boldsymbol{\varphi}(x))) = O(|x|)$, and $g_{K+1}(t, x, v, \tau, \xi)$ has the similar Taylor expansion with (2.2).

In order to give our refinement form, we prepare notations and definitions.

**Definition 3.1** (Borel transform)**.** Let $\mathbb{R}_{\geq 1} = \{x \in \mathbb{R} \,|\, x \geq 1\}$, and let $\boldsymbol{s} = (s_1, \dots, s_d) \in (\mathbb{R}_{\geq 1})^d$, $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n) \in (\mathbb{R}_{\geq 1})^n$. For a formal power series $u(t, x) = \sum u_{\alpha,\beta} t^\alpha x^\beta$, we define the $\boldsymbol{s}$-Borel transform in $t$, the $\boldsymbol{\sigma}$-Borel transform in $x$ and the $(\boldsymbol{s}, \boldsymbol{\sigma})$-Borel transform in $(t, x)$ as follows, respectively:

– $\boldsymbol{s}$-Borel transform in $t$: $\quad \mathcal{B}_t^{\boldsymbol{s}}(u)(t, x) = \sum \dfrac{u_{\alpha,\beta} |\alpha|!}{(\boldsymbol{s} \cdot \alpha)!} t^\alpha x^\beta,$

– $\boldsymbol{\sigma}$-Borel transform in $x$: $\quad \mathcal{B}_x^{\boldsymbol{\sigma}}(u)(t, x) = \sum \dfrac{u_{\alpha,\beta} |\beta|!}{(\boldsymbol{\sigma} \cdot \beta)!} t^\alpha x^\beta,$

– $(\boldsymbol{s}, \boldsymbol{\sigma})$-Borel transform in $(t, x)$: $\quad \mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}(u)(t, x) = (\mathcal{B}_t^{\boldsymbol{s}} \circ \mathcal{B}_x^{\boldsymbol{\sigma}})(u) = (\mathcal{B}_x^{\boldsymbol{\sigma}} \circ \mathcal{B}_t^{\boldsymbol{s}})(u),$

where $\boldsymbol{s} \cdot \alpha$ denotes $\boldsymbol{s} \cdot \alpha = s_1 \alpha_1 + \ldots + s_d \alpha_d$, ($\boldsymbol{\sigma} \cdot \beta$ is also a similar definition) and $a! = \Gamma(a + 1)$.

**Definition 3.2** (Gevrey class)**.** We define $u(t,x) \in \mathcal{G}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}$, which is called of $(\boldsymbol{s},\boldsymbol{\sigma})$ Gevrey class in $(t,x)$ variables, if $\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}(u)(t,x)$ is convergent in a neighborhood of the origin.

By an easy calculation, the following relation holds:

$$\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}(u)(t,x) \in \mathcal{G}_{t,x}^{(\boldsymbol{s}',\boldsymbol{\sigma}')} \implies u(t,x) \in \mathcal{G}_{t,x}^{(\boldsymbol{s}+\boldsymbol{s}'-\mathbf{1}_d,\boldsymbol{\sigma}+\boldsymbol{\sigma}'-\mathbf{1}_n)},$$

where $\mathbf{1}_d = (1,\ldots,1) \in \mathbb{N}^d$.

Now, we obtain the following refinement of Theorem 1.6.

**Theorem 3.3.** *Let $\boldsymbol{s}^j = (1,2,\ldots,d_j) \in \mathbb{N}^{d_j}$ $(j = 1,2,\ldots,I)$, and let $d_0 = \max\{d_1,d_2,\ldots,d_I\}$. Then under the assumptions 1.1, 1.2, 1.3 and the nonresonance-Poincaré condition* (1.8)*, the formal solution of* (7.1) *belongs to the Gevrey class of order at most $(\boldsymbol{s}',\boldsymbol{\sigma}')$ by*

$$(\boldsymbol{s}',\boldsymbol{\sigma}') = \begin{cases} (\boldsymbol{s}+\boldsymbol{d}_0^1, \boldsymbol{d}_0^2+\mathbf{1}_n) \in \mathbb{N}^{d+n} & \textit{if } \ \alpha_{ijkl}(x) \not\equiv 0 \ \textit{ for some } \ i,j,k,l, \\ (\boldsymbol{s}+\boldsymbol{d}', \mathbf{1}_n) \in \mathbb{N}^{d+n} & \textit{if } \ \alpha_{ijkl}(x) \equiv 0 \ \textit{ for all } \ i,j,k,l, \end{cases}$$

*where $\boldsymbol{s} = (\boldsymbol{s}^1,\ldots,\boldsymbol{s}^I) \in \mathbb{N}^d$, $\boldsymbol{d}_0^1 = (d_0,\ldots,d_0) \in \mathbb{N}^d$, $\boldsymbol{d}_0^2 = (d_0,\ldots,d_0) \in \mathbb{N}^n$ and $\boldsymbol{d}' = (d',\ldots,d') \in \mathbb{R}^d$, where*

$$d' = \max_{\alpha,p,q,r} \left\{ \frac{d_0}{V(\alpha,p,q,r)-K} \right\} (\leq d_0). \tag{3.5}$$

Theorem 1.6 is an immediate consequence of Theorem 3.3. Indeed, all the components of $(\boldsymbol{s}^1,\ldots,\boldsymbol{s}^I)$ are estimated by $d_0$. Therefore, all the components of $\boldsymbol{s}'$ are estimated by $2d_0$, which gives the conclusion of Theorem 1.6.

## 4. EXAMPLES FOR THEOREM 3.3

In this section, we give typical examples for Theorem 3.3.

**Example 4.1.** Let $t = (t_1,t_2) \in \mathbb{C}^2$ and $x \in \mathbb{C}$. We consider the equation

$$\begin{cases} (t_2\partial_{t_1} + x\partial_x + 1)u = x(t_1+t_2)^2 + xt_1\partial_{t_2}u + (t_2\partial_{t_2}u)(\partial_x u), \\ u(t,x) = O(|t|^2). \end{cases}$$

In the linear part of derivatives in the right hand side of the equation, there exists a derivative related to $t$. Hence, by Theorem 3.3, the formal solution belongs to the Gevrey class of order at most $(s_1,s_2,\sigma) = (3,4,3)$.

**Example 4.2.** Let $t = (t_1,t_2) \in \mathbb{C}$ and $x \in \mathbb{C}$. We consider the equation

$$\begin{cases} (t_2\partial_{t_1} + x\partial_x + 1)u = x(t_1+t_2)^2 + x^2t_1\partial_x u + (t_2\partial_{t_2}u)(\partial_x u), \\ u(t,x) = O(|t|^2). \end{cases}$$

In the linear part of derivatives in the right hand side of the equation, there does not exist a derivative related to $t$. Hence, by Theorem 3.3, the formal solution belongs to the Gevrey class of order at most $(s_1,s_2,\sigma) = (3,4,1)$.

5. PROOF OF THEOREM 3.3

We define the set of homogeneous polynomials of degree $L$ in $t$ and degree $M$ in $x$ by

$$\mathbb{C}[t]_L[x]_M = \left\{ \sum_{|\alpha|=L,|\beta|=M} u_{\alpha,\beta} t^\alpha x^\beta \;\middle|\; u_{\alpha\beta} \in \mathbb{C} \right\}. \tag{5.1}$$

First we give a following lemma.

**Lemma 5.1.**

(i) *The operator $P := \mathcal{N} + \mathcal{D} + \Delta$ is invertible on $\mathbb{C}[t]_L[x]_M$ for all $L \geq K$ and $M \geq 0$.*

(ii) *Let $\boldsymbol{s} = (\boldsymbol{s}^1, \ldots, \boldsymbol{s}^I) \in \mathbb{N}^d$ ($\boldsymbol{s}^j = (1, 2, \ldots, d_j) \in \mathbb{N}^{d_j}$), and $T = t_1 + \ldots + t_d \in \mathbb{C}$, $X = x_1 + \ldots + x_n \in \mathbb{C}$. For $u(t,x) \in \mathbb{C}[t]_L[x]_M$, if a majorant relation $\mathcal{B}_t^{\boldsymbol{s}}(u)(t,x) \ll W_{L,M} T^L X^M$ ($W_{L,M} \geq 0$) holds, then the following majorant relation holds by a positive constant $C_0$ independent of $L$ and $M$.*

$$\mathcal{B}_t^{\boldsymbol{s}}(P^{-1}u)(t,x) \ll \frac{C_0}{M+1} W_{L,M} T^L X^M = C_0 (X\partial_X + 1)^{-1} W_{L,M} T^L X^M. \tag{5.2}$$

We omit the proof of Lemma 5.1, since the similar proposition is already proved in [14, Lemma 1].

By Lemma 5.1, the operator $P$ is invertible on $\mathbb{C}[[t,x]]_K$ by

$$\mathbb{C}[[t,x]]_K = \bigcup_{L \geq K, M \geq 0} \mathbb{C}[t]_L[x]_M.$$

Here we put $U(t,x) = Pv(t,x)$ as a new unknown function. Then $U(t,x)$ satisfies the following:

$$\begin{cases} U(t,x) = \displaystyle\sum_{i,j,k,l} \alpha_{ijkl}(x) t_{i,j} \partial_{t_{k,l}} P^{-1}U + \sum_{k=1}^{n} \beta_k(x) \partial_{x_k} P^{-1}U \\ \qquad\qquad + \eta(x) P^{-1}U + \displaystyle\sum_{|\alpha|=K} \zeta_\alpha(x) t^\alpha \\ \qquad\qquad + g_{K+1}(t, x, P^{-1}U, \partial_t P^{-1}U, \partial_x P^{-1}U), \\ U(t,x) = O(|t|^K). \end{cases} \tag{5.3}$$

For the equation (5.3), we apply the Borel transform of order $\boldsymbol{s} = (\boldsymbol{s}^1, \ldots, \boldsymbol{s}^I) \in \mathbb{N}^d$ in $t$, then the equation is reduced to the following:

$$\begin{aligned} \mathcal{B}_t^{\boldsymbol{s}}(U)(t,x) = \mathcal{B}_t^{\boldsymbol{s}} &\left( \sum_{i,j,k,l} \alpha_{ijkl}(x) t_{i,j} \partial_{t_{k,l}} P^{-1}U \right) + \mathcal{B}_t^{\boldsymbol{s}} \left( \sum_{k=1}^{n} \beta_k(x) \partial_{x_k} P^{-1}U \right) \\ &+ \mathcal{B}_t^{\boldsymbol{s}} \Big( \eta(x) P^{-1}U \Big) + \sum_{|\alpha|=K} \frac{\zeta_\alpha(x) |\alpha|!}{(\boldsymbol{s} \cdot \alpha)!} t^\alpha \\ &+ \mathcal{B}_t^{\boldsymbol{s}} \Big\{ g_{K+1}(t, x, P^{-1}U, \partial_t P^{-1}U, \partial_x P^{-1}U) \Big\}. \end{aligned} \tag{5.4}$$

In order to estimate the Borel transforms of products and derivatives with respect to $t$ and $x$, we give the following lemma.

**Lemma 5.2.**

(i) *For two formal power series $u(t,x), v(t,x) \in \mathbb{C}[[t,x]]$, there exists a positive constant $C_1$ depending only on $\boldsymbol{s}$ such that the following majorant relation holds.*

$$\mathcal{B}_t^{\boldsymbol{s}}(uv)(t,x) \ll C_1 \mathcal{B}_t^{\boldsymbol{s}}(|u|)(t,x) \times \mathcal{B}_t^{\boldsymbol{s}}(|v|)(t,x), \tag{5.5}$$

*where for $u(t,x) = \sum u_{\alpha\beta} t^\alpha x^\beta$, $|u|(t,x)$ is defined by $|u|(t,x) = \sum |u_{\alpha\beta}| t^\alpha x^\beta$.*

(ii) *We put $T = t_1 + \ldots + t_d$ and $X = x_1 + \ldots + x_n$. Let $W(T,X)$ be a formal power series in $T$ and $X$. If $\mathcal{B}_t^{\boldsymbol{s}}(u)(t,x) \ll W(T,X)$, then the following majorant relations hold by a positive constant $C_2$.*

$$\begin{aligned}\mathcal{B}_t^{\boldsymbol{s}}\Big(\partial_{t_{i,j}} P^{-1} u\Big)(t,x) &\ll C_2 \partial_T (T\partial_T)^{j-1} (X\partial_X + 1)^{-1} W(T,X) \\ &\ll C_2 \partial_T (T\partial_T)^{j-1} W(T,X),\end{aligned} \tag{5.6}$$

$$\mathcal{B}_t^{\boldsymbol{s}}\Big(\partial_{x_k} P^{-1} u\Big)(t,x) \ll C_2 \partial_X (X\partial_X + 1)^{-1} W(T,X) \ll C_2 \times S(W)(T,X), \tag{5.7}$$

*where $S(W)(T,X)$ is the shift function in $X$ defined by*

$$S(W)(T,X) = \frac{W(T,X) - W(T,0)}{X}. \tag{5.8}$$

We omit the proof of Lemma 5.2, since the similar proposition is already proved in [14, Lemma 2].

By Lemma 5.2, if a majorant relation $\mathcal{B}_t^{\boldsymbol{s}}(U)(t,x) \ll W(T,X)$ holds, then there exists a positive constant $C_3 > 0$ such that the following majorant relations hold.

$$\mathcal{B}_t^{\boldsymbol{s}}\Big(\alpha_{ijkl}(x) t_{ij} \partial_{t_{kl}} P^{-1} U\Big)(t,x) \ll C_3 |\alpha_{ijkl}|(X) (T\partial_T)^l W(T,X), \tag{5.9}$$

$$\mathcal{B}_t^{\boldsymbol{s}}\Big(\beta_k(x) \partial_{x_k} P^{-1} U\Big)(t,x) \ll C_3 |\beta_k|(X) S(W)(T,X), \tag{5.10}$$

$$\mathcal{B}_t^{\boldsymbol{s}}\Big(\eta(x) P^{-1} U\Big)(t,x) \ll C_3 |\eta|(X) W(T,X), \tag{5.11}$$

$$\sum_{|\alpha|=K} \frac{\zeta_\alpha(x)|\alpha|!}{(\boldsymbol{s}\cdot\alpha)!} t^\alpha \ll \left(\sum_{|\alpha|=K} |\zeta_\alpha|(X)\right) T^K =: \zeta(X) T^K, \tag{5.12}$$

$$\begin{aligned}&\mathcal{B}_t^{\boldsymbol{s}}\Big(g_{K+1}(t,x,P^{-1}U, \partial_t P^{-1}U, \partial_x P^{-1}U)\Big) \\ &\quad \ll |g_{K+1}|(T,X,C_3 W, \{C_3 \partial_T (T\partial_T)^{j-1} W\}_{i,j}, \{C_3 S(W)\}_k).\end{aligned} \tag{5.13}$$

We remark that $1 \le j \le d_0$ for $j$ in (5.13). Moreover, since $W(T,X)$ is a majorant series of $\mathcal{B}_t^{\boldsymbol{s}}(u)(t,x)$, we have $W(T,X) \gg 0$. Therefore, we obtain the following majorant relation.

$$XS(W)(T,X) = W(T,X) - W(T,0) \ll W(T,X). \tag{5.14}$$

Since $|\beta_k|(X) = O(X^2)$, we put a holomorphic function $|\hat{\beta}_k|(X)$ by $|\hat{\beta}_k|(X) := |\beta_k|(X)/X = O(X)$. Then the following majorant relation holds.

$$|\beta_k|(X)S(W) = \frac{|\beta_k|(X)}{X} \cdot XS(W) \ll \frac{|\beta_k|(X)}{X} W = |\hat{\beta_k}|(X)W.$$

We consider the following equation.

$$\begin{cases} W = \sum_{i,j,k,l} \tilde{\alpha}_{ijkl}(X)(T\partial_T)^l W + \sum_{k=1}^{n} \tilde{\beta}_k(X)W + \tilde{\eta}(X)W + \zeta(X)T^K \\ \quad +|g_{K+1}|(T, X, C_3 W, \{C_3\partial_T(T\partial_T)^{j-1}W\}_{i,j}, \{C_3 S(W)\}_k), \\ W = O(T^K), \end{cases} \tag{5.15}$$

where $\tilde{\alpha}_{ijkl}(X) = C_3|\alpha_{ijkl}|(X)$, $\tilde{\beta}_k(X) = C_3|\hat{\beta}_k|(X)$, $\tilde{\eta}(X) = C_3|\eta|(X)$. These are all holomorphic functions in a neighborhood of $X = 0$ and vanish at $X = 0$. By the construction of this equation, it is easily seen that

$$\mathcal{B}_t^{\boldsymbol{s}}(U)(t, x) \ll W(T, X).$$

Here we put $F(X) = 1 - \sum_{k=1}^n \tilde{\beta}_k(X) - \tilde{\eta}(X)$. Since $F(0) = 1 \neq 0$, $1/F(X)$ is holomorphic in a neighborhood of $X = 0$. Therefore, by multiplying $1/F(X)$ for both sides, the equation (5.15) is reduced to the following.

$$\begin{aligned} W = &\sum_{i,j,k,l} \hat{\alpha}_{ijkl}(X)(T\partial_T)^l W + \hat{\zeta}(X)T^K \\ &+ G_{K+1}(T, X, C_3 W, \{C_3\partial_T(T\partial_T)^{j-1}W\}_{i,j}, \{C_3 S(W)\}_k), \end{aligned}$$

where

$$\hat{\alpha}_{ijkl}(X) = \tilde{\alpha}_{ijkl}(X)/F(X) = O(X), \quad \hat{\zeta}(X) = \zeta(X)/F(X)$$

and

$$G_{K+1}(T, X, u, \tau, \xi) = |g_{K+1}|(T, X, u, \tau, \xi)/F(X).$$

For the equation (5.16), the following lemma holds.

**Lemma 5.3.** *If $\hat{\alpha}_{ijkl}(X) \not\equiv 0$ for some $i, j, k, l$, then the formal solution $W(T, X)$ belongs to the Gevrey class $\mathcal{G}_{T,X}^{(d_0+1,d_0+1)}$. If $\hat{\alpha}_{ijkl}(X) \equiv 0$ for all $i, j, k, l$, then the formal solution $W(T, X)$ belongs to the Gevrey class $\mathcal{G}_{T,X}^{(d'+1,1)}$ where $d'$ is the constant defined by* (3.5).

By Lemma 5.3, which will be proved in the next section, $W(T, X) \in \mathcal{G}_{T,X}^{(d_0+1,d_0+1)}$ if $\hat{\alpha}_{ijkl}(X) \not\equiv 0$ for some $i, j, k, l$ or $W(T, X) \in \mathcal{G}_{T,X}^{(d'+1,1)}$ if $\hat{\alpha}_{ijkl}(X) \equiv 0$ for all $i, j, k, l$. On the other hand, for $\boldsymbol{s} = (\boldsymbol{s}^1, \dots, \boldsymbol{s}^I)$ ($\boldsymbol{s}^i = (1, 2, \dots, d_i)$), the majorant relation $\mathcal{B}_t^{\boldsymbol{s}}(U)(t, x) \ll W(T, X)$ holds. By combining these properties, we have

$$\mathcal{B}_t^{\boldsymbol{s}}(U)(t, x) = \mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{1}_n)}(U)(t, x) \in \mathcal{G}_{T,X}^{(d_0+1,d_0+1)} \text{ or } \mathcal{G}_{T,X}^{(d'+1,1)}.$$

Therefore, the Gevrey order $(\boldsymbol{s}', \boldsymbol{\sigma}')$ of $U(t,x)$ is obtained by

$$\boldsymbol{s}' = \begin{cases} \boldsymbol{s} + (d_0+1, \dots, d_0+1) - \mathbf{1}_d = \boldsymbol{s} + (d_0, \dots, d_0), \\ \qquad\qquad\qquad \text{if } \hat{\alpha}_{ijkl}(X) \not\equiv 0 \text{ for some } i,j,k,l, \\ \boldsymbol{s} + (d'+1, \dots, d'+1) - \mathbf{1}_d = \boldsymbol{s} + (d', \dots, d'), \\ \qquad\qquad\qquad \text{if } \hat{\alpha}_{ijkl}(X) \equiv 0 \text{ for all } i,j,k,l, \end{cases}$$

$$\boldsymbol{\sigma}' = \begin{cases} \mathbf{1}_n + (d_0+1, \dots, d_0+1) - \mathbf{1}_n = (d_0+1, \dots, d_0+1), \\ \qquad\qquad\qquad \text{if } \hat{\alpha}_{ijkl}(X) \not\equiv 0 \text{ for some } i,j,k,l, \\ \mathbf{1}_n + \mathbf{1}_n - \mathbf{1}_n = \mathbf{1}_n, \qquad \text{if } \hat{\alpha}_{ijkl}(X) \equiv 0 \text{ for all } i,j,k,l, \end{cases}$$

which proves Theorem 3.3.

## 6. PROOF OF LEMMA 5.3

First we consider the case when $\hat{\alpha}_{ijkl}(X) \not\equiv 0$ for some $i,j,k,l$. Let $\hat{\alpha}_{ijkl}(X) = \sum_{M\ge 1} \alpha_{ijklM} X^M$ be the Taylor expansion of $\hat{\alpha}_{ijkl}(X)$. For the sake of simplicity of notation, we put $C_3 = 1$. By putting $W(T,X) = \sum_{L\ge K} W_L(X) T^L$ and by substituting this into (5.16), we obtain the following recurrence formula for $\{W_L(X)\}_{L\ge K}$.

$$W_K(X) = \sum_{i,j,k,l} \hat{\alpha}_{ijkl}(X) K^l W_K(X) + \hat{\zeta}(X), \tag{6.1}$$

$$\begin{aligned} W_L(X) = &\sum_{i,j,k,l} \hat{\alpha}_{ijkl}(X) L^l W_L(X) \\ &+ {\sum}' G_{\alpha pqr}(X) \prod_{k=1}^{p} W_{L_k}(X) \\ &\times \prod_{i=1}^{I} \prod_{j=1}^{d_i} \prod_{l=1}^{q_{ij}} (L_{ijl})^j W_{L_{ijl}}(X) \prod_{k=1}^{n} \prod_{l=1}^{r_k} S(W_{L_{kl}})(X), \end{aligned} \tag{6.2}$$

where the summation ${\sum}'$ is taken over $V(\alpha,p,q,r) \ge K+1$ and

$$|\alpha| + \sum_{k=1}^{p} L_k + \sum_{i=1}^{I} \sum_{j=1}^{d_i} \sum_{l=1}^{q_{ij}} (L_{ijl} - 1) + \sum_{k=1}^{n} \sum_{l=1}^{r_k} L_{kl} = L. \tag{6.3}$$

The first recurrence formula (6.1) is an easier situation than (6.2). Therefore, in the following, we consider the case of (6.2).

We put $W_L(X) = \sum_{M\ge 0} W_{L,M} X^M$. By substituting this in the formula (6.2), we get the following recurrence formula for $\{W_{L,M}\}_{L\ge K, M\ge 0}$.

$$\begin{aligned} W_{L,M} = &\sum_{i,j,k,l} \sum_{M_1=1}^{M} L^l \alpha_{ijklM_1} W_{L,M-M_1} \\ &+ {\sum}' {\sum}'' G_{\alpha pqrM'} \prod_{k=1}^{p} W_{L_k,M_k} \prod_{i=1}^{I} \prod_{j=1}^{d_i} \prod_{l=1}^{q_{ij}} (L_{ijl})^j W_{L_{ijl},M_{ijl}} \prod_{k=1}^{n} \prod_{l=1}^{r_k} W_{L_{kl},M_{kl}+1}, \end{aligned} \tag{6.4}$$

where the summation $\sum''$ is taken over

$$M' + \sum_{k=1}^{p} M_k + \sum_{i=1}^{I}\sum_{j=1}^{d_i}\sum_{l=1}^{q_{ij}} M_{ijl} + \sum_{k=1}^{n}\sum_{l=1}^{r_k} M_{kl} = M. \tag{6.5}$$

We set $Y_{L,M} = W_{L,M}/(L+M)!^{d_0}$. Then $\{Y_{L,M}\}$ satisfies the following recurrence formula.

$$\begin{aligned} Y_{L,M} = &\sum_{i,j,k,l}\sum_{M_1=1}^{M} \hat{C}_1 \alpha_{ijklM_1} Y_{L,M-M_1} \\ &+ \sum{}'\sum{}'' \hat{C}_2 G_{\alpha pqrM'} \prod_{k=1}^{p} Y_{L_k,M_k} \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}} Y_{L_{ijl},M_{ijl}} \prod_{k=1}^{n}\prod_{l=1}^{r_k} Y_{L_{kl},M_{kl}+1}, \end{aligned} \tag{6.6}$$

where $\hat{C}_1$, $\hat{C}_2 = \hat{C}_2(\alpha, p, q, r)$ and

$$\begin{aligned} \hat{C}_1 &= \frac{L^l (L+M-M_1)!^{d_0}}{(L+M)!^{d_0}}, \\ \hat{C}_2 &= \frac{1}{(L+M)!^{d_0}} \times \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}} (L_{ijl})^j \\ &\quad \times \prod_{k=1}^{p} (L_k + M_k)!^{d_0} \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}} (L_{ijl} + M_{ijl})!^{d_0} \prod_{k=1}^{n}\prod_{l=1}^{r_k} (L_{kl} + M_{kl} + 1)!^{d_0}. \end{aligned}$$

Since $l \le d_0$ and $M_1 \ge 1$, we get the following estimate for $\hat{C}_1$.

$$\hat{C}_1 = \frac{L^l (L+M-M_1)!^{d_0}}{(L+M)!^{d_0}} \le \frac{L^l (L+M-1)!^{d_0}}{(L+M)!^{d_0}} = \frac{L^l}{(L+M)^{d_0}} \le 1.$$

For the estimate of $\hat{C}_2$, we need the following lemma which is proved in [12, Lemma 6].

**Lemma 6.1.** *Let $L$ and $m_j$ be nonnegative integers such that $m_j \ge L$ for all $j = 1, 2, \ldots, n$. Then the following inequality holds:*

$$m_1! \ldots m_n! \le (L!)^{n-1} (m_1 + \ldots + m_n - (n-1)L)!. \tag{6.7}$$

By using Lemma 6.1, we can estimate $\hat{C}_2 = \hat{C}_2(\alpha, p, q, r)$ as follows.

$$\begin{aligned}
\hat{C}_2 &= \frac{\prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}} {L_{ijl}}^{j}}{(L+M)!^{d_0}} \\
&\quad \times \prod_{k=1}^{p}(L_k+M_k)!^{d_0} \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}}(L_{ijl}+M_{ijl})!^{d_0} \prod_{k=1}^{n}\prod_{l=1}^{r_k}(L_{kl}+M_{kl}+1)!^{d_0} \\
&\le \frac{\prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}} {L_{ijl}}^{j-d_0}}{(L+M)!^{d_0}} \\
&\quad \times \left\{ \prod_{k=1}^{p}(L_k+M_k+1)! \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}}(L_{ijl}+M_{ijl}+1)! \prod_{k=1}^{n}\prod_{l=1}^{r_k}(L_{kl}+M_{kl}+1)! \right\}^{d_0} \\
&\le \frac{(K+1)!^{d_0(p+|q|+|r|)}}{(L+M)!^{d_0}} \\
&\quad \times \left( \sum_{k=1}^{p}(L_k+M_k+1) + \sum_{i=1}^{I}\sum_{j=1}^{d_i}\sum_{l=1}^{q_{ij}}(L_{ijl}+M_{ijl}+1) \right. \\
&\qquad\qquad \left. + \sum_{k=1}^{n}\sum_{l=1}^{r_k}(L_{kl}+M_{kl}+1) - (K+1)(p+|q|+|r|-1) \right)!^{d_0} \\
&= \frac{\hat{C}_3^{p+|q|+|r|}(L+M-|\alpha|-M'+p+2|q|+|r|-(K+1)(p+|q|+|r|-1))!^{d_0}}{(L+M)!^{d_0}} \\
&\quad (\text{we put } \hat{C}_3 := (K+1)!^{d_0}) \\
&= \frac{\hat{C}_3^{p+|q|+|r|}(L+M+K+1-V(\alpha,p,q,r))!^{d_0}}{(L+M)!^{d_0}} \le \hat{C}_3^{p+|q|+|r|},
\end{aligned}$$

where

$$V(\alpha, p, q, r) = |\alpha| + Kp + (K-1)|q| + K|r|.$$

By these observations, (6.6) is estimated by

$$\begin{aligned}
Y_{L,M} &\le \sum_{i,j,k,l}\sum_{M_1=1}^{M} \alpha_{ijklM_1} Y_{L,M-M_1} \\
&\quad + \sum{}' \sum{}'' G_{\alpha pqrM'} \prod_{k=1}^{p} \hat{C}_3 Y_{L_k,M_k} \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}} \hat{C}_3 Y_{L_{ijl},M_{ijl}} \prod_{k=1}^{n}\prod_{l=1}^{r_k} \hat{C}_3 Y_{L_{kl},M_{kl}+1}.
\end{aligned}$$

Let us consider the following recurrence formula for $\{Z_{L,M}\}_{L\geq K,M\geq 0}$:

$$
\begin{aligned}
Z_{K,M} &= \sum_{i,j,k,l}\sum_{M_1=1}^{M}\alpha_{ijklM_1}Z_{K,M-M_1}+\hat{\zeta}_M,\\
Z_{L,M} &= \sum_{i,j,k,l}\sum_{M_1=1}^{M}\alpha_{ijklM_1}Z_{L,M-M_1}\\
&\quad+\sum{}'\sum{}''G_{\alpha pqrM'}\prod_{k=1}^{p}\hat{C}_3Z_{L_k,M_k}\prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}}\hat{C}_3Z_{L_{ijl},M_{ijl}}\prod_{k=1}^{n}\prod_{l=1}^{r_k}\hat{C}_3Z_{L_{kl},M_{kl}+1}.
\end{aligned}
$$

By this construction of the recurrence formula, we have $Y_{L,M}\leq Z_{L,M}$.

We remark that these recurrence formulas are obtained by the following equation:

$$
\begin{aligned}
Z(T,X) = \sum_{i,j,k,l}&\hat{\alpha}_{ijkl}(X)Z+\hat{\zeta}(X)T^K\\
&+G_{K+1}\left(T,X,\hat{C}_3Z,\left\{\frac{\hat{C}_3Z}{T}\right\},\left\{\hat{C}_3S(Z)\right\}\right)
\end{aligned}
\tag{6.8}
$$

with $Z(T,X)=O(T^K)$. By dividing both sides of the equation by $1-\sum_{i,j,k,l}\hat{\alpha}_{ijkl}(X)$, the equation is reduced to the following one:

$$
Z(T,X)=\bar{\zeta}(X)T^K+H_{K+1}\left(T,X,\hat{C}_3Z,\left\{\frac{\hat{C}_3Z}{T}\right\},\left\{\hat{C}_3S(Z)\right\}\right),\tag{6.9}
$$

where

$$
\bar{\zeta}(X)=\hat{\zeta}(X)/\Big(1-\sum_{i,j,k,l}\hat{\alpha}_{ijkl}(X)\Big)\in\mathbb{C}\{X\}
$$

and

$$
H_{K+1}=G_{K+1}/\Big(1-\sum_{i,j,k,l}\hat{\alpha}_{ijkl}(X)\Big)
$$

is holomorphic in a neighborhood of the origin.

By this construction of the equation (6.9), we have the following majorant relation between $Z$ and $Y$.

$$
Z(T,X)\gg Y(T,X)=\mathcal{B}_{T,X}^{(d_0+1,d_0+1)}(W)(T,X).
$$

We put $\varphi(T,X)=Z(T,X)/T$ as a new unknown function. Then $\varphi(T,X)$ satisfies the following.

$$
\varphi(T,X)=\bar{\zeta}(X)T^{K-1}+\frac{1}{T}H_{K+1}(T,X,\hat{C}_3T\varphi,\{\hat{C}_3\varphi\},\{\hat{C}_3TS(\varphi)\})\tag{6.10}
$$

with $\varphi(T,X)=O(T^{K-1})$.

We decompose the formal solution $\varphi(T,X)$ as follows.

$$\varphi(T,X)=\varphi_1(X)T^{K-1}+\varphi_2(X)T^K+T^K\psi(T,X),\quad \psi(0,X)\equiv 0.$$

By an easy calculation, $\varphi_1(X)$ and $\varphi_2(X)$ are given by

$$\varphi_1(X)=\bar{\zeta}(X),$$

$$\varphi_2(X)=\sum_{|\alpha|+Kp+(K-1)|q|+K|r|=K+1} H_{\alpha pqr}(X)\hat{C}_3^{p+|q|+|r|}\varphi_1(X)^{p+|q|}S(\varphi_1)(X)^{|r|}.$$

We remark that these are holomorphic in a neighborhood of $X=0$.

Moreover, $\psi(T,X)$ satisfies the following equation:

$$\begin{cases}\psi(T,X)=H(T,X,T\psi,TS(\psi)),\\ \psi(0,X)\equiv 0,\end{cases}\tag{6.11}$$

where

$$\begin{aligned}H(T,X,\eta_1,\eta_2)=\frac{1}{T^{K+1}}\Big[&H_{K+1}\Big(T,X,\hat{C}_3\varphi_1(X)T^K+\hat{C}_3\varphi_2(X)T^{K+1}+\hat{C}_3T^K\eta_1,\\ &\{\hat{C}_3\varphi_1(X)T^{K-1}+\hat{C}_3\varphi_2(X)T^K+\hat{C}_3T^K\eta_1\},\\ &\{\hat{C}_3S(\varphi_1)(X)T^K+\hat{C}_3S(\varphi_2)(X)T^{K+1}+\hat{C}_3T^K\eta_2\}\Big)\Big]\\ &-\sum_{|\alpha|+Kp+(K-1)|q|+K|r|=K+1} H_{\alpha pqr}(X)\hat{C}_3^{p+|q|+|r|}\varphi_1(X)^{p+|q|}S(\varphi_1)(X)^{|r|}.\end{aligned}$$

We remark that the order of zeros in $T$ of $H(T,X,T\psi(T,X),TS(\psi)(T,X))$ is greater than or equal to 1.

In order to prove the convergence of $\psi(T,X)$, it is sufficient to show the following:

**Lemma 6.2.** *There exists a small positive constant $\varepsilon>0$ such that $\psi_\varepsilon(\rho):=\psi(\varepsilon\rho,\rho)$ is convergent in a neighborhood of $\rho=0$.*

The proof of Lemma 6.2 can be found in [13], so we omit it.

Lemma 6.2 implies that $\varphi(T,X)$ is convergent. Therefore, the following majorant relations hold: In the case $\hat{\alpha}_{ijkl}(X)\not\equiv 0$ for some $i,j,k,l$,

$$\mathbb{C}\{T,X\}\ni T\varphi(T,X)=Z(T,X)\gg Y(T,X)=\mathcal{B}_{T,X}^{(d_0+1,d_0+1)}(W)(T,X).$$

Next we consider the case when $\hat{\alpha}_{ijkl}(X)\equiv 0$ for all $i,j,k,l$. In this case, the argument follows analogously from above by removing the term $\sum_{i,j,k,l}\hat{\alpha}_{ijkl}(X)(T\partial_T)^lW$. Therefore, by putting $W(T,X)=\sum_{L\ge K}\sum_{M\ge 0}W_{L,M}T^LX^M$, we get the following recurrence formula for $\{W_{L,M}\}_{L\ge K,M\ge 0}$

$$W_{L,M}=\sum{}'\sum{}''G_{\alpha pqrM'}\prod_{k=1}^{p}W_{L_k,M_k}\prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}}(L_{ijl})^jW_{L_{ijl},M_{ijl}}\prod_{k=1}^{n}\prod_{l=1}^{r_k}W_{L_{kl},M_{kl}+1},\tag{6.12}$$

where the summation $\sum'$ and $\sum''$ are taken by in the same way as (6.3) and (6.5), respectively. In this case we put $Y_{L,M} = W_{L,M}/L!^{d'}$, where $d'$ is defined by (3.5). Then $\{Y_{L,M}\}$ satisfies the following recurrence formula.

$$Y_{L,M} = \sum{}'\sum{}'' \tilde{C}_2 G_{\alpha pqrM'} \prod_{k=1}^{p} Y_{L_k,M_k} \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}} Y_{L_{ijl},M_{ijl}} \prod_{k=1}^{n}\prod_{l=1}^{r_k} Y_{L_{kl},M_{kl}+1}, \tag{6.13}$$

where $\tilde{C}_2 = \tilde{C}_2(\alpha,p,q,r)$ and

$$\tilde{C}_2 = \frac{\prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}}(L_{ijl})^j \times \prod_{k=1}^{p} L_k!^{d'} \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}} L_{ijl}!^{d'} \prod_{k=1}^{n}\prod_{l=1}^{r_k} L_{kl}!^{d'}}{L!^{d'}}.$$

By using Lemma 6.1, for an arbitrary $\mathcal{L} \in \mathbb{N}$, $\tilde{C}_2$ is estimated by

$$\begin{aligned}
\tilde{C}_2 &= \frac{\prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}}(L_{ijl})^j \times \{\prod_{k=1}^{p} L_k! \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}} L_{ijl}! \prod_{k=1}^{n}\prod_{l=1}^{r_k} L_{kl}!\}^{d'}}{L!^{d'}} \\
&\le \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}}(L_{ijl})^{d_0 - d'\mathcal{L}} \\
&\quad \times \frac{\{\prod_{k=1}^{p}(L_k+\mathcal{L})! \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}}(L_{ijl}+\mathcal{L})! \prod_{k=1}^{n}\prod_{l=1}^{r_k}(L_{kl}+\mathcal{L})!\}^{d'}}{L!^{d'}} \\
&\le (K+\mathcal{L})!^{d'(p+|q|+|r|-1)} \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}}(L_{ij})^{d_0-d'\mathcal{L}} \frac{(L - V(\alpha,p,q,r)+K+\mathcal{L})!^{d'}}{L!^{d'}}.
\end{aligned}$$

Here we set $\Omega_1 = \left\{(\alpha,p,q,r)\,;\, \dfrac{d_0}{V(\alpha,p,q,r)-K} = d'\right\}$ (this is a finite set), $\Omega_2 = \left\{(\alpha,p,q,r)\,;\, \dfrac{d_0}{V(\alpha,p,q,r)-K} < d'\right\}$ and we set

$$\mathcal{L} = \mathcal{L}(\alpha,p,q,r) = \begin{cases} V(\alpha,p,q,r) - K & \text{if } (\alpha,p,q,r) \in \Omega_1, \\ \left[\dfrac{d_0}{d'}\right] + 1 & \text{if } (\alpha,p,q,r) \in \Omega_2. \end{cases}$$

Remark that

$$\frac{d_0}{d'} < \mathcal{L} \le \frac{d_0}{d'} + 1 < V(\alpha,p,q,r) - K + 1$$

holds for $(\alpha,p,q,r) \in \Omega_2$. By this inequality, $\mathcal{L} \le V(\alpha,p,q,r) - K$ holds for all $(\alpha,p,q,r)$, because $\mathcal{L}$ and $V(\alpha,p,q,r) - K + 1$ are natural numbers. By the choice of $\mathcal{L}$, we have $\tilde{C}_2 \le \tilde{C}_3^{p+|q|+|r|}$ by $\tilde{C}_3 = (K + \max\mathcal{L})!^{d'}$. Therefore, $\tilde{C}_2$ can be estimated by the same form as the case $\hat{\alpha}_{ijkl}(X) \not\equiv 0$.

By these observations, (6.13) is estimated by

$$Y_{L,M} \le \sum{}'\sum{}'' G_{\alpha pqrM'} \prod_{k=1}^{p} \tilde{C}_3 Y_{L_k,M_k} \prod_{i=1}^{I}\prod_{j=1}^{d_i}\prod_{l=1}^{q_{ij}} \tilde{C}_3 Y_{L_{ijl},M_{ijl}} \prod_{k=1}^{n}\prod_{l=1}^{r_k} \tilde{C}_3 Y_{L_{kl},M_{kl}+1}, \tag{6.14}$$

Let us consider the following recurrence formula for $\{Z_{L,M}\}_{L\geq K,M\geq 0}$: $Z_{K,M} = \hat{\zeta}_M$ and

$$Z_{L,M} = \sum{}' \sum{}'' G_{\alpha pqrM'} \prod_{k=1}^{p} \tilde{C}_3 Z_{L_k,M_k} \prod_{i=1}^{I} \prod_{j=1}^{d_i} \prod_{l=1}^{q_{ij}} \tilde{C}_3 Z_{L_{ijl},M_{ijl}} \prod_{k=1}^{n} \prod_{l=1}^{r_k} \tilde{C}_3 Z_{L_{kl},M_{kl}+1}. \tag{6.15}$$

This recurrence formula is obtained by the following equation:

$$Z(T,X) = \hat{\zeta}(X)T^K + G_{K+1}\left(T, X, \tilde{C}_3 Z, \left\{\frac{\tilde{C}_3 Z}{T}\right\}, \left\{\tilde{C}_3 S(Z)\right\}\right) \tag{6.16}$$

with $Z(T,X) = O(T^K)$. By the construction of equation (6.16), we have

$$Z(T,X) \gg Y(T,X) = \mathcal{B}_{T,X}^{(d'+1,1)}(W)(T,X),$$

and (6.16) is the same form as (6.9). Therefore, the convergence of a formal solution $Z(T,X)$ follows from the same argument in the case when $\hat{\alpha}_{ijkl}(X) \not\equiv 0$ for some $i,j,k,l$. This completes the proof of Lemma 5.3.

## 7. REFINEMENT OF THEOREM 1.7

In this section, we shall prove Theorem 1.7. To do so, we reduce (2.1) to the more exact form.

By the same linear change of $t$ variables as in section 3, the vector field is reduced to the following.

$$\sum_{i,j=1}^{d} a_{i,j}(x) t_i \partial_{t_j} \mapsto (\tau^{(1)}, \dots, \tau^{(I)}) \begin{pmatrix} N_1 & & \\ & \ddots & \\ & & N_I \end{pmatrix} \begin{pmatrix} \partial_{\tau^{(1)}} \\ \vdots \\ \partial_{\tau^{(I)}} \end{pmatrix} - \sum_{i,j,k,l} \alpha_{ijkl}(x) \tau_{i,j} \partial_{\tau_{k,l}},$$

where $\alpha_{ijkl}(x) = O(|x|)$ denote holomorphic functions and $N_j$ $(j = 1, \dots, I)$ denotes the nilpotent Jordan block of size $d_j$.

Next, we write the differential operator with respect to $x$ variables by the following form:

$$\sum_{k=1}^{n} b_k(x) \partial_{x_k} = (x_1, \dots, x_n) J(b_1, \dots, b_n)(0) \begin{pmatrix} \partial_{x_1} \\ \vdots \\ \partial_{x_n} \end{pmatrix} - \sum_{k=1}^{n} \hat{b}_k(x) \partial_{x_k},$$

where $\hat{b}_k(x) = O(|x|^2)$ $(k = 1, \dots, n)$. Then we introduce new variables $\xi = (\xi^{(1)}, \dots, \xi^{(J)}) \in \mathbb{C}^n$ $(\xi^{(k)} = (\xi_{k,1}, \dots, \xi_{k,n_k}) \in \mathbb{C}^{n_k})$ by

$$(\xi^{(1)}, \dots, \xi^{(J)}) = (x_1, \dots, x_n) Q, \quad Q^{-1} J(b_1, \dots, b_n)(0) Q = \begin{pmatrix} \hat{N}_1 & & \\ & \ddots & \\ & & \hat{N}_J \end{pmatrix}$$

where $\hat{N}_j$ $(j = 1, \dots, J)$ denotes the nilpotent Jordan block of size $n_j$. By this linear change of variables $x$, the above vector field with respect to $x$ is reduced to

$$(\xi^{(1)}, \dots, \xi^{(J)}) \begin{pmatrix} \hat{N}_1 & & \\ & \ddots & \\ & & \hat{N}_J \end{pmatrix} \begin{pmatrix} \partial_{\xi^{(1)}} \\ \vdots \\ \partial_{\xi^{(J)}} \end{pmatrix} - \sum_{i=1}^{J} \sum_{j=1}^{n_i} \beta_{ij}(\xi) \partial_{\xi_{i,j}},$$

where $\beta_{ij}(\xi) = O(|\xi|^2)$ $(i = 1, \dots, J; j = 1, \dots, n_i)$ denotes a holomorphic function.

Hereafter we rewrite $(\tau, \xi)$ by $(t, x)$ again. Then the equation (2.1) is rewritten as follows.

$$\begin{cases} \Big(\mathcal{N}_1 + \mathcal{N}_2 + c(0)\Big) v = \sum_{i,j,k,l} \alpha_{ijkl}(x) t_{i,j} \partial_{t_{k,l}} v \\ \qquad + \sum_{i=1}^{J} \sum_{j=1}^{n_i} \beta_{ij}(x) \partial_{x_{i,j}} v + \eta(x) v \\ \qquad + \sum_{|\alpha|=K} \zeta_\alpha(x) t^\alpha + g_{K+1}(t, x, v, \partial_t v, \partial_x v), \\ v(t, x) = O(|t|^K), \end{cases} \tag{7.1}$$

where $c(x) = f_u(\boldsymbol{\varphi}(x))$, $\eta(x) = c(0) - c(x) = O(|x|)$ and

$$\mathcal{N}_1 = \sum_{j=1}^{I} \sum_{k=1}^{d_j - 1} \delta t_{j,k+1} \partial_{t_{j,k}}, \quad \mathcal{N}_2 = \sum_{j=1}^{J} \sum_{k=1}^{n_j - 1} \delta x_{j,k+1} \partial_{x_{j,k}}. \tag{7.2}$$

Moreover,

$$g_{K+1}(t, x, v, \tau, \xi) = \sum_{V(\alpha,p,q,r) \geq K+1} g_{\alpha pqr}(x) t^\alpha v^p \tau^q \xi^r,$$
$$V(\alpha, p, q, r) = |\alpha| + Kp + (K-1)|q| + K|r|. \tag{7.3}$$

We put $q = (q_{ij})$ $(1 \leq i \leq I, 1 \leq j \leq d_i)$ and $r = (r_{ij})$ $(1 \leq i \leq J, 1 \leq j \leq n_i)$ which are associated with $t = (t_{ij}) \in \mathbb{C}^d$ and $x = (x_{ij}) \in \mathbb{C}^n$. By using these notations and definitions, we obtain the following result:

**Theorem 7.1.** *Let* $\boldsymbol{s}^j = (1, 2, \dots, d_j) \in \mathbb{N}^{d_j}$ $(j = 1, 2, \dots, I)$, $\boldsymbol{\sigma}^j = (1, 2, \dots, n_j) \in \mathbb{N}^{n_j}$ $(j = 1, 2, \dots, J)$ *and let* $n_0 = \max\{d_1, d_2, \dots, d_I, n_1, n_2, \dots, n_J\}$. *We put* $N(\alpha, p, q, r) = \max\{j\,;\, q_{i,j} \neq 0 \text{ or } r_{i,j} \neq 0\}$ *for each nonzero term* $g_{\alpha,p,q,r}(x) t^\alpha u^p \tau^q \xi^r$. *Here we define a positive constant* $n'$ *by*

$$n' = \max_{\alpha,p,q,r} \left\{ \frac{N(\alpha, p, q, r)}{V(\alpha, p, q, r) - K} \right\}. \tag{7.4}$$

*Then under Assumptions 1.1, 1.2, 1.3 and* $c(0) (= f_u(\boldsymbol{\varphi}(0))) \neq 0$, *the formal solution belongs to the Gevrey class of order at most* $(\boldsymbol{s}', \boldsymbol{\sigma}')$ *by*

$$(\boldsymbol{s}', \boldsymbol{\sigma}') = \begin{cases} (\boldsymbol{s} + \boldsymbol{n}_0^1, \boldsymbol{\sigma} + \boldsymbol{n}_0^2) & \text{if } \alpha_{i,j,k,l}(x) \not\equiv 0 \text{ or } \beta_{i,j}(x) \not\equiv 0 \text{ for some } i, j, k, l, \\ (\boldsymbol{s} + \boldsymbol{n}', \boldsymbol{\sigma}) & \text{if } \alpha_{i,j,k,l}(x) \equiv 0 \text{ and } \beta_{i,j}(x) \equiv 0 \text{ for all } i, j, k, l, \end{cases}$$

*where* $\boldsymbol{s} = (\boldsymbol{s}^1, \ldots, \boldsymbol{s}^I) \in \mathbb{N}^d$, $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^1, \ldots, \boldsymbol{\sigma}^J) \in \mathbb{N}^n$, $\boldsymbol{n}_0^1 = (n_0, \ldots, n_0) \in \mathbb{N}^d$, $\boldsymbol{n}_0^2 = (n_0, \ldots, n_0) \in \mathbb{N}^n$ *and* $\boldsymbol{n}' = (n', \ldots, n') \in \mathbb{R}^d$.

Theorem 1.7 is an immediate consequence of Theorem 7.1. Indeed, all the components of $(\boldsymbol{s}^1, \ldots, \boldsymbol{s}^I)$ and $(\boldsymbol{\sigma}^1, \ldots, \boldsymbol{\sigma}^J)$ are estimated by $n_0$ and $n' \leqq n_0$. Therefore, all the components of $\boldsymbol{s}'$ and $\boldsymbol{\sigma}'$ are estimated by $2n_0$, which gives the conclusion of Theorem 1.7.

## 8. EXAMPLES FOR THEOREM 7.1

**Example 8.1.** Let $t = (t_1, t_2) \in \mathbb{C}^2$ and $x = (x_1, x_2, x_3) \in \mathbb{C}^3$, and consider

$$\begin{cases} (t_2\partial_{t_1} + x_2\partial_{x_1} + 1)u(t,x) = (t_1 + t_2)^2 + x_1 t_1 \partial_{t_2} u + t_1 t_2 u \times \partial_{x_3} u, \\ u = O(|t|^2). \end{cases}$$

Since $\alpha_{i,j,k,l}(x) = x_1 \not\equiv 0$, the Gevrey order is estimated by

$$(\boldsymbol{s}', \boldsymbol{\sigma}') = (1,2,1,2,1) + (2,2,2,2,2) = (3,4,3,4,3).$$

**Example 8.2.** Let $t = (t_1, t_2) \in \mathbb{C}^2$ and $x = (x_1, x_2, x_3) \in \mathbb{C}^3$, and consider

$$\begin{cases} (t_2\partial_{t_1} + x_2\partial_{x_1} + 1)u(t,x) = (t_1 + t_2)^2 + t_1 t_2 u \times \partial_{x_3} u, \\ u = O(|t|^2). \end{cases}$$

Since $\alpha_{i,j,k,l}(x) \equiv 0$ and $\beta_{i,j}(x) \equiv 0$, the Gevrey order is estimated by

$$(\boldsymbol{s}', \boldsymbol{\sigma}') = (1,2,1,2,1) + \left(\frac{1}{4}, \frac{1}{4}, 0, 0, 0\right) = \left(\frac{5}{4}, \frac{9}{4}, 1, 2, 1\right).$$

## 9. PROOF OF THEOREM 7.1

In order to prove Theorem 7.1, we prepare lemmas.

**Lemma 9.1.**

(i) *The operator* $P := \mathcal{N}_1 + \mathcal{N}_2 + c(0)$ $(c(0) \neq 0)$ *is invertible on* $\mathbb{C}[t]_L[x]_M$ *for all* $L \geq K$ *and* $M \geq 0$.

(ii) *Let* $\boldsymbol{s} = (\boldsymbol{s}^1, \ldots, \boldsymbol{s}^I) \in \mathbb{N}^d$ *and* $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^1, \ldots, \boldsymbol{\sigma}^J) \in \mathbb{N}^d$ *be as before, and* $T = t_1 + \ldots + t_d \in \mathbb{C}$, $X = x_1 + \ldots + x_n \in \mathbb{C}$. *For* $u(t,x) \in \mathbb{C}[t]_L[x]_M$, *if a majorant relation* $\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}(u)(t,x) \ll W_{L,M} T^L X^M$ $(W_{L,M} \geq 0)$ *holds, then the following majorant relation holds by a positive constant* $C_0$ *independent of* $L$ *and* $M$.

$$\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}(P^{-1}u)(t,x) \quad \ll \quad C_0 W_{L,M} T^L X^M \tag{9.1}$$

**Lemma 9.2.** *Let $\boldsymbol{s} = (\boldsymbol{s}^1, \ldots, \boldsymbol{s}^I) \in \mathbb{N}^d$ and $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^1, \ldots, \boldsymbol{\sigma}^J) \in \mathbb{N}^n$. We put $T = t_1 + \ldots + t_d$ and $X = x_1 + \ldots + x_n$. For a formal power series $W(T, X)$ in $T$ and $X$, if $\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}(u)(t, x) \ll W(T, X)$, then the following majorant relations hold by a positive constant $C_2$:*

$$\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}\Big(\partial_{t_{i,j}} P^{-1} u\Big)(t, x) \ll C_2 \partial_T (T\partial_T)^{j-1} W(T, X), \tag{9.2}$$

$$\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}\Big(\partial_{x_{i,j}} P^{-1} u\Big)(t, x) \ll C_2 \partial_X (X\partial_X)^{j-1} W(T, X). \tag{9.3}$$

The proofs of Lemmas 9.1 and 9.2 are similar to those of Lemmas 5.1 and 5.2, so we omit them.

We put $U(t, x) = Pv(t, x)$ as a new unknown function. Then $U(t, x)$ satisfies the following:

$$\begin{cases} U(t,x) = \displaystyle\sum_{i,j,k,l} \alpha_{ijkl}(x) t_{i,j} \partial_{t_{k,l}} P^{-1} U + \sum_{i=1}^{J}\sum_{j=1}^{n_i} \beta_{ij}(x) \partial_{x_{ij}} P^{-1} U + \eta(x) P^{-1} U \\ \qquad + \displaystyle\sum_{|\alpha|=K} \zeta_\alpha(x) t^\alpha + g_{K+1}(t, x, P^{-1}U, \partial_t P^{-1} U, \partial_x P^{-1} U), \\ U(t,x) = O(|t|^K). \end{cases} \tag{9.4}$$

For equation (9.4), we apply the $(\boldsymbol{s}, \boldsymbol{\sigma})$-Borel transform, then (9.4) is reduced to the following:

$$\begin{aligned} \mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}(U)(t,x) = {} & \mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}\Big(\sum_{i,j,k,l} \alpha_{ijkl}(x) t_{i,j} \partial_{t_{k,l}} P^{-1} U\Big) \\ & + \mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}\Big(\sum_{k=i}^{J}\sum_{j=1}^{n_i} \beta_{ij}(x) \partial_{x_{ij}} P^{-1} U\Big) \\ & + \mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}\Big(\eta(x) P^{-1} U\Big) + \sum_{|\alpha|=K} \frac{\zeta_\alpha(x)|\alpha|!|\beta|!}{(\boldsymbol{s}\cdot\alpha)!(\boldsymbol{\sigma}\cdot\beta)!} t^\alpha \\ & + \mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}\Big\{g_{K+1}(t, x, P^{-1}U, \partial_t P^{-1} U, \partial_x P^{-1} U)\Big\}. \end{aligned} \tag{9.5}$$

By Lemma 5.2, (i) and Lemma 9.2, if a majorant relation $\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}(U)(t, x) \ll W(T, X)$ is satisfied, then there exists a positive constant $C_3$ such that the following majorant relations hold:

$$\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}\Big(\alpha_{ijkl}(x) t_{ij} \partial_{t_{kl}} P^{-1} U\Big)(t, x) \ll C_3 |\alpha_{ijkl}|(X)(T\partial_T)^l W(T, X), \tag{9.6}$$

$$\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}\Big(\beta_{ij}(x) \partial_{x_{i,j}} P^{-1} U\Big)(t, x) \ll C_3 |\beta_{ij}|(X) \partial_X (X\partial_X)^{j-1} W(T, X), \tag{9.7}$$

$$\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}\Big(\eta(x) P^{-1} U\Big)(t, x) \ll C_3 |\eta|(X) W(T, X), \tag{9.8}$$

$$\begin{aligned} & \mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}\Big(g_{K+1}(t, x, P^{-1}U, \partial_t P^{-1} U, \partial_x P^{-1} U)\Big) \\ & \ll |g_{K+1}|(T, X, C_3 W, \{C_3 \partial_T (T\partial_T)^{j-1} W\}_{i,j}, \{C_3 \partial_X (X\partial_X)^{j-1} W\}_{i,j}). \end{aligned} \tag{9.9}$$

On the other hand, for the Borel transform of $\sum \zeta_\alpha(x)t^\alpha$, we have

$$\sum_{|\alpha|=K} \frac{\zeta_\alpha(x)|\alpha|!|\beta|!}{(\boldsymbol{s}\cdot\alpha)!(\boldsymbol{\sigma}\cdot\beta)!}t^\alpha \ll \left(\sum_{|\alpha|=K}|\zeta_\alpha|(X)\right)T^K =: \zeta(X)T^K. \tag{9.10}$$

We remark that $1 \le j \le n_0 = \max\{d_1, \dots, d_I, n_1, \dots, n_J\}$ for $j$ in (9.9).

Since $|\beta_{ij}|(X) = O(X^2)$, we put a holomorphic function $|\hat{\beta}_{ij}|(X) = |\beta_{ij}|(X)/X = O(X)$. Then the following relation holds.

$$|\beta_{ij}|(X)\partial_X(X\partial_X)^{j-1}W =: |\hat{\beta_{ij}}|(X)(X\partial_X)^jW.$$

We consider the following equation.

$$\begin{aligned} W = \sum_{i,j,k,l}\tilde{\alpha}_{ijkl}(X)(T\partial_T)^lW + \sum_{i=1}^J\sum_{j=1}^{n_i}\tilde{\beta}_{ij}(X)(X\partial_X)^jW + \tilde{\eta}(X)W + \zeta(X)T^K \\ + |g_{K+1}|(T, X, C_3W, \{C_3\partial_T(T\partial_T)^{j-1}W\}_{i,j}, \{C_3\partial_X(X\partial_X)^{j-1}W\}_{i,j}), \end{aligned} \tag{9.11}$$

with $W = O(T^K)$, where $\tilde{\alpha}_{ijkl}(X) = C_3|\alpha_{ijkl}|(X)$, $\tilde{\beta}_{ij}(X) = C_3|\hat{\beta_{ij}}|(X)$ and $\tilde{\eta}(X) = C_3|\eta|(X)$ all vanish at $X = 0$. By the construction of this equation, it is easily seen that

$$\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}(U)(t,x) \ll W(T,X).$$

Now we put $F(X) = 1 - \tilde{\eta}(X)$. Since $F(0) = 1$, by multiplying $1/F(X)$ for both sides, the equation (9.11) is reduced to the following.

$$\begin{aligned} W = \sum_{i,j,k,l}\hat{\alpha}_{ijkl}(X)(T\partial_T)^lW + \sum_{i=1}^J\sum_{j=1}^{n_i}\bar{\beta}_{ij}(X)(X\partial_X)^jW + \hat{\zeta}(X)T^K \\ + G_{K+1}(T, X, C_3W, \{C_3\partial_T(T\partial_T)^{j-1}W\}_{i,j}, \{C_3\partial_X(X\partial_X)^{j-1}(W)\}_{i,j}), \end{aligned} \tag{9.12}$$

where $\hat{\alpha}_{ijkl}(X) = \tilde{\alpha}_{ijkl}(X)/F(X) = O(X)$ and the others are similarly defined. Especially $\bar{\beta}_{ij}(X) = O(X)$.

For the equation (9.12), the following lemma holds.

**Lemma 9.3.**

(i) *If $\hat{\alpha}_{ijkl}(X) \not\equiv 0$ or $\bar{\beta}_{ij}(X) \not\equiv 0$ for some $i, j, k, l$, then the formal solution $W(T,X)$ of* (9.12) *belongs to the Gevrey class $\mathcal{G}_{T,X}^{(n_0+1,n_0+1)}$, where $n_0 = \max\{d_1, \dots, d_I, n_1, \dots, n_J\}$.*

(ii) *If $\hat{\alpha}_{ijkl}(X) \equiv 0$ and $\bar{\beta}_{ij}(X) \equiv 0$ for all $i, j, k, l$, then the formal solution $W(T,X)$ of* (9.12) *belongs to the Gevrey class $\mathcal{G}_{T,X}^{(n'+1,1)}$, where $n'$ is the constant defined by* (7.4)*.*

Since Lemma 9.3, (i) can be proved by similarly to that of Lemma 5.3, and Lemma 9.3, (ii) is a special case of Theorem 1.6 in the previous paper [12], we omit the proof of Lemma 9.3.

By Lemma 9.3, $W(T,X) \in \mathcal{G}_{T,X}^{(n_0+1,n_0+1)}$ or $\mathcal{G}_{T,X}^{(n'+1,1)}$. On the other hand, the majorant relation $\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}(U)(t,x) \ll W(T,X)$. Therefore, we have

$$\mathcal{B}_{t,x}^{(\boldsymbol{s},\boldsymbol{\sigma})}(U)(t,x) \in \mathcal{G}_{T,X}^{(n_0+1,n_0+1)} \text{ or } \mathcal{G}_{T,X}^{(n'+1,1)},$$

which proves Theorem 3.3.

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