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Dedicated to the memory of Professor Kenjiro Okubo

KATZ'S MIDDLE CONVOLUTION AND YOKOYAMA'S EXTENDING OPERATION

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Abstract. We give a concrete relation between Katz's middle convolution and Yokoyama's extension and show the equivalence of both algorithms using these operations for the reduction of Fuchsian systems on the Riemann sphere.

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1. INTRODUCTION

Katz [8] introduces the operations *addition* and *middle convolution* of Fuchsian system

$$\frac{du}{dx} = \sum_{j=1}^{p} \frac{A_j}{x - t_j} u \tag{1.1}$$

of Schlesinger canonical form (SCF) on the Riemann sphere and studies the rigid local systems. It has regular singularities at $x = t_1, \ldots, t_p$ and ∞. Here $A_j \in M(n, \mathbb{C})$ and $M(n, \mathbb{C})$ denotes the space of $n \times n$ matrices with entries in $\mathbb{C}$ and the number n is called the *rank* of the system. Katz shows that any irreducible rigid system of SCF is reduced to a rank 1 system, namely a system with $n = 1$, by a finite iteration of these operations, which implies that any irreducible rigid system of SCF is obtained by applying a finite iteration of these operations to a rank 1 system since these operations are invertible.

The fact that the system is *rigid* is equal to saying that it is free from accessory parameters, namely, the set of conjugacy classes of $A_1, \ldots, A_p$ and $-(A_1 + \ldots + A_p)$ determines the simultaneous conjugacy class of $(A_1, \ldots, A_p)$. But these operations are also useful for the study of non-rigid systems. In fact the Deligne-Simpson problem, the monodromies and integral representations of their solutions, their monodromy preserving deformations and their classification are studied by using these operations (cf. [3, 5, 6, 12, 14, 15] etc.).

Dettweiler and Reiter [2] interpret these operations as those of tuples of matrices $\mathbf{A} = (A_1, \dots, A_p)$ as follows.

The *addition* $M_\mu(\mathbf{A})$ of $\mathbf{A}$ with respect to $\mu = (\mu_1, \dots, \mu_p) \in \mathbb{C}^p$ is given by

$$M_\mu(\mathbf{A}) = M_\mu^p(\mathbf{A}) := (A_1 + \mu_1, \dots, A_p + \mu_p). \tag{1.2}$$

The *convolution* $(G_1, \dots, G_p) \in M(pn, \mathbb{C})^p$ of $\mathbf{A}$ with respect to $\lambda \in \mathbb{C}$ is defined by

$$\begin{aligned} G_j &:= \Big(\delta_{\mu,j}(A_\nu + \delta_{\mu,\nu}\lambda)\Big)_{\substack{1\le\mu\le p\\ 1\le\nu\le p}} \qquad (j = 1, \dots, p) \\ &= \begin{array}{c} \\ j\,\rangle \end{array} \!\!\begin{array}{c} \underset{\smile}{j} \phantom{\quad A_{j+1} \quad \dots \quad A_p} \\ \begin{pmatrix} & & & & & & \\ A_1 & A_2 & \dots & A_j + \lambda & A_{j+1} & \dots & A_p \\ & & & & & & \end{pmatrix} \end{array} \in M(pn, \mathbb{C}). \end{aligned} \tag{1.3}$$

Here G_j are square matrices of size p whose elements are square matrices of size n. In the above $j\,\rangle$ and $\underset{\smile}{j}$ indicate the j-th row and the j-th column of the matrix G_j, respectively, and the (μ, ν) components of G_j are zero matrices when $\mu \neq j$ and they are not written in the above.

$$\mathcal{K} := \left\{ \begin{pmatrix} u_1 \\ \vdots \\ u_p \end{pmatrix} \in \mathbb{C}^{pn}\,;\, A_j u_j = 0,\ j = 1, \dots, p \right\}, \tag{1.4}$$

$$\mathcal{L}_\lambda := \ker(G_1 + \dots + G_p) \tag{1.5}$$

are G_j-invariant, we put $V := \mathbb{C}^{pn}/(\mathcal{K} + \mathcal{L}_\lambda)$ and define linear maps $\bar{G}_j \in \operatorname{End}(V) \simeq M(\dim V, \mathbb{C})$, which are induced by G_j, respectively. Then the *middle convolution* $mc_\lambda(\mathbf{A})$ of $\mathbf{A}$ equals $(\bar{G}_1, \dots, \bar{G}_p)$.

For $\mathbf{A} = (A_1, \dots, A_p)$, $\mathbf{B} = (B_1, \dots, B_p) \in M(n, \mathbb{C})^p$ we define that $\mathbf{A}$ is *conjugate* to $\mathbf{B}$, which is denoted by $\mathbf{A} \sim \mathbf{B}$, if there exists $g \in GL(n, \mathbb{C})$ such that $B_j = gA_jg^{-1}$ and we will sometimes identify $\mathbf{A}$ with $\mathbf{B}$ if $\mathbf{A} \sim \mathbf{B}$. The corresponding systems (1.1) will be also identified.

The Fuchsian system of Okubo normal form (ONF) is

$$(xI_n - T)\frac{du}{dx} = Au \tag{1.6}$$

with a diagonal matrix $T \in M(n, \mathbb{C})$ and a matrix $A \in M(n, \mathbb{C})$. Yokoyama [16] introduces an extension and a restriction of this system when A satisfies a certain genericity condition.

Suppose

$$T = \begin{pmatrix} t_1 I_{n_1} & & \\ & \ddots & \\ & & t_p I_{n_p} \end{pmatrix}, \tag{1.7}$$

where $n = n_1 + \cdots + n_p$ is a partition of n and $t_i \neq t_j$ if $i \neq j$. Put

$$A = \begin{pmatrix} A_{11} & \dots & A_{1p} \\ \vdots & \ddots & \vdots \\ A_{p1} & \dots & A_{pp} \end{pmatrix} \tag{1.8}$$

according to the partition, namely $A_{ij} \in M(n_i, n_j, \mathbb{C})$. Here $M(n_i, n_j, \mathbb{C})$ is the space of $n_i \times n_j$ matrices with entries in $\mathbb{C}$. We note that the system (1.6) of ONF is equal to the system (1.1) of SCF by putting

$$A_j := \Big(\delta_{\mu,j} A_{j,\nu}\Big)_{\substack{1 \le \mu \le p \\ 1 \le \nu \le p}} = \begin{array}{cc} j & \left(\begin{matrix} \\ A_{j1} & A_{j2} & \dots & A_{jp} \\ \\ \end{matrix}\right) \end{array} \in M(n, \mathbb{C}). \tag{1.9}$$

Here A_j are block matrices whose rows except for the j-th row are zero.

Conversely, we have the following lemma.

Lemma 1.1. *Suppose* $(A_1, \dots, A_p) \in M(n, \mathbb{C})^p$ *satisfies*

$$\operatorname{rank} A_1 + \ldots + \operatorname{rank} A_p = n \tag{1.10}$$

and

$$\operatorname{Im} A_1 + \ldots + \operatorname{Im} A_p = \mathbb{C}^n. \tag{1.11}$$

Then there exists $g \in GL(n, \mathbb{C})$ *such that the* ν*-th row of* $g^{-1} A_j g$ *is identically zero if* $\nu \le \operatorname{rank} A_1 + \ldots + \operatorname{rank} A_{j-1}$ *or* $\nu > \operatorname{rank} A_1 + \ldots + \operatorname{rank} A_j$*. Hence, the system of SCF is equivalent to a system of ONF if* (1.10) *and* (1.11) *hold.*

Proof. The assumption implies the existence of a basis $\{v_1, \dots, v_n\}$ of $\mathbb{C}^n$ such that

$$\operatorname{Im} A_j = \sum_{\operatorname{rank} A_1 + \ldots + \operatorname{rank} A_{j-1} < \nu \le \operatorname{rank} A_1 + \ldots + \operatorname{rank} A_j} \mathbb{C} v_j.$$

Then the expression of A_j under this basis has the required property, namely, we may put $g = (v_1, \dots, v_n) \in GL(n, \mathbb{C})$. □

Remark 1.2. If $\operatorname{Im} A_1 + \ldots + \operatorname{Im} A_p \neq \mathbb{C}^n$, the rank of the matrix $(A_1, \dots, A_p)$ of size $n \times np$ is smaller than n and there exists a non-trivial proper subspace V of $\mathbb{C}^n$ such that $A_j V \subset V$ for $1 \le j \le p$. Hence, if a system (1.1) of SCF is linearly irreducible (cf. Definition 3.2), (1.11) is always satisfied. Moreover, if a system (1.6) of ONF is linearly irreducible, it satisfies (1.10) and (1.11) with (1.8) and (1.9).

Yokoyama [16] defines *extensions* $(\hat{T}, \hat{A}) = E_\epsilon(T, A)$ for $\epsilon = 0, 1$ and 2 with respect to two distinct complex numbers ρ_1 and ρ_2 when T, A and A_{ii} $(i = 1, \dots, p)$ are diagonalizable. Here ϵ is the number of the elements of $\{\rho_1, \rho_2\}$ which are not the eigenvalues of A.

Let

$$A_{ii} \sim \begin{pmatrix} \lambda_{i,1} I_{\ell_{i,1}} & & \\ & \ddots & \\ & & \lambda_{i,r_i} I_{\ell_{i,r_i}} \end{pmatrix} \tag{1.12}$$

with $\lambda_{i,j} \neq \lambda_{i,k}$ $(j \neq k)$ and $n_i = \ell_{i,1} + \ldots + \ell_{i,r_i}$ and fix a matrix $P \in GL(n, \mathbb{C})$ so that

$$A' := \begin{pmatrix} \mu_1 I_{m_1} & & \\ & \ddots & \\ & & \mu_q I_{m_q} \end{pmatrix} = P^{-1} A P \sim A, \tag{1.13}$$

where $n = m_1 + \ldots + m_q$ and $\mu_i \neq \mu_j$ $(i \neq j)$. Then $E_2(T, A) = (\hat{T}, \hat{A})$ with

$$\hat{T} := \begin{pmatrix} T & \\ & t_{p+1} I_n \end{pmatrix}, \tag{1.14}$$

$$\hat{A} := \begin{pmatrix} A & P \\ -(A' - \rho_1 I_n)(A' - \rho_2 I_n) P^{-1} & (\rho_1 + \rho_2) I_n - A' \end{pmatrix}. \tag{1.15}$$

When ρ_1 or ρ_2 is an eigenvalue of A, there exists a subspace invariant by $\hat{T}$ and $\hat{A}$ and the *extending operations* E_1 and E_0 of (T, A) are defined as follows. Putting

$$V_k := \left\{ \begin{pmatrix} u \\ v_1 \\ v_2 \end{pmatrix} ;\ u \in \mathbb{C}^n,\ v_1 = 0 \in \mathbb{C}^k \text{ and } v_2 \in \mathbb{C}^{n-k} \right\}, \tag{1.16}$$

we have

$$E_1(T, A) := (\hat{T}|_{V_{m_1}}, \hat{A}|_{V_{m_1}}) \qquad \text{when } \ \rho_1 = \mu_1, \tag{1.17}$$

$$E_0(T, A) := (\hat{T}|_{V_{m_1+m_2}}, \hat{A}|_{V_{m_1+m_2}}) \qquad \text{when } \rho_1 = \mu_1 \text{ and } \rho_2 = \mu_2. \tag{1.18}$$

Restrictions are defined as the inverse operations of these extensions. Yokoyama [16] proves that any irreducible rigid system of ONF with generic spectral parameters $\lambda_{i,j}$ and μ_k is reduced to a rank 1 system by a finite iteration of the extensions and restrictions and obtained the monodromy of the system.

In this note we clarify the direct relation between Yokoyama's operations and Katz's operations and then relax the assumption to define Yokoyama's operations (cf. Theorem 3.8 and Theorem 4.1). In particular we do not assume that the local monodromies of the system are semisimple (cf. Theorem 6.1). Moreover, we show in Theorem 5.5 that both operations on Fuchsian systems are equivalent in a natural sense. Hence, the property of Katz's operation can be transferred to Yokoyama's operations and vice versa. For example, it is proved by [5] that the middle convolution preserves the deformation equation and therefore so do Yokoyama's operations.

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2. KATZ'S MIDDLE CONVOLUTION

For a partition $\mathbf{m} = (m_1, \ldots, m_N)$ of a positive integer n with $n = m_1 + \ldots + m_N$ and a tuple $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ we define a matrix $L(\mathbf{m}; \lambda) \in M(n, \mathbb{C})$ as a representative of a conjugacy class, which is introduced and effectively used by [13] (cf. [14, §3]):

If $m_1 \geq m_2 \geq \ldots \geq 0$, then

$$L(\mathbf{m}; \lambda) := \Big(A_{ij}\Big)_{\substack{1\le i\le N\\ 1\le j\le N}}, \quad A_{ij} \in M(m_i, m_j, \mathbb{C}),$$
$$A_{ij} = \begin{cases} \lambda_i I_{m_i} & (i = j), \\ I_{m_i, m_j} := \Big(\delta_{\mu\nu}\Big)_{\substack{1\le\mu\le m_i\\ 1\le\nu\le m_j}} = \begin{pmatrix} I_{m_j} \\ 0 \end{pmatrix} & (i = j-1), \\ 0 & (i \neq j,\ j-1). \end{cases} \tag{2.1}$$

For example,

$$L(2,1,1;\lambda_1,\lambda_2,\lambda_3) = \begin{pmatrix} \lambda_1 & 0 & 1 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}.$$

Denoting $Z_{M(n,\mathbb{C})}(A) := \{X \in M(n, \mathbb{C})\,;\, AX = XA\}$, we have

$$\dim\ker \prod_{j=1}^{k} \big(L(\mathbf{m}; \lambda) - \lambda_j\big) = m_1 + \ldots + m_k \quad (k = 1, \ldots, N), \tag{2.2}$$

$$\dim Z_{M(n,\mathbb{C})}\big(L(\mathbf{m}; \lambda)\big) = m_1^2 + \ldots + m_N^2. \tag{2.3}$$

In general we fix a permutation σ of indices $1, \ldots, N$ so that $m_{\sigma(1)} \geq m_{\sigma(2)} \geq \ldots$ and define $L(\mathbf{m}; \lambda) = L(m_{\sigma(1)}, \ldots, m_{\sigma(N)}; \lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(N)})$.

Let $\mathbf{A} = (A_1, \ldots, A_p) \in M(n, \mathbb{C})^p$. Put

$$A_0 = -(A_1 + \ldots + A_p). \tag{2.4}$$

Then Katz [8] defines

$$\operatorname{idx} \mathbf{A} := \sum_{j=0}^{p} \dim Z_{M(n,\mathbb{C})}(A_j) - (p-1)n^2, \tag{2.5}$$

which is called the *index of rigidity*.

If $\mathbf{A}$ is irreducible, $\operatorname{idx} \mathbf{A} \leq 2$. Moreover, an irreducible tuple $\mathbf{A}$ is rigid if and only if $\operatorname{idx} \mathbf{A} = 2$, which is proved by Katz [8, §1.1.1]. Here $\mathbf{A}$ is called *irreducible* if any subspace V of $\mathbb{C}^n$ satisfying $A_j V \subset V$ for $j = 1, \ldots, p$ is $\{0\}$ or $\mathbb{C}^n$ and $\mathbf{A}$ is called *rigid* if $\mathbf{A} \sim \mathbf{B}$ for any $\mathbf{B} \in M(n, \mathbb{C})^p$ satisfying $A_j \sim B_j$ for $j = 1, \ldots, p$ together with $A_1 + \ldots + A_p \sim B_1 + \ldots + B_p$.

Using the representatives $L(\mathbf{m}; \lambda)$ of conjugacy classes of matrices, we define the Riemann scheme of a Fuchsian system of SCF.

Definition 2.1. For $\mathbf{A} \in M(n,\mathbb{C})^p$ we choose a $(p+1)$-tuple of partitions $\mathbf{m} = (\mathbf{m}_0,\dots,\mathbf{m}_p)$ of n and complex numbers $\lambda_{j,\nu}$ so that

$$A_j \sim L\big(\mathbf{m}_j;\lambda_j\big) \text{ with } \mathbf{m}_j = (m_{j,1},\dots,m_{j,n_j}) \text{ and } \lambda_j = (\lambda_{j,1},\dots,\lambda_{j,n_j}) \tag{2.6}$$

for $j = 0,\dots,p$. Here A_0 is determined by (2.4). We define the *Riemann scheme* of the corresponding system (1.1) of SCF by

$$\begin{Bmatrix} x=\infty & x=t_1 & \dots & x=t_p \\ [\lambda_{0,1}]_{(m_{0,1})} & [\lambda_{1,1}]_{(m_{1,1})} & \cdots & [\lambda_{p,1}]_{(m_{p,1})} \\ \vdots & \vdots & & \vdots \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \cdots & [\lambda_{p,n_p}]_{(m_{p,n_p})} \end{Bmatrix}. \tag{2.7}$$

Here A_j is called the *residue matrix* of the system (1.1) at $x = t_j$ $(j = 1,\dots,p)$ and A_0 is the residue matrix of the system at $x = \infty$. We also call (2.7) the Riemann scheme of $\mathbf{A}$. We will allow that some $m_{j,\nu}$ are 0.

The conjugacy classes of components of the middle convolution $mc_\lambda(\mathbf{A})$ of $\mathbf{A}$ are given by Dettweiler and Reiter in terms of the Jordan canonical form. Our normal form (2.1) makes the description of the conjugacy classes easier as follows.

Theorem 2.2 ([2, 3])**.** *Let $\mathbf{A} = (A_1,\dots,A_p) \in M(n,\mathbb{C})^p$. Assume the following conditions:*

$$\bigcap_{\substack{1\le\nu\le p\\ \nu\ne i}} \ker A_\nu \cap \ker(A_i+\tau) = \{0\} \qquad (i=1,\dots,p,\ \textit{for all}\ \tau\in\mathbb{C}), \tag{2.8}$$

$$\sum_{\substack{1\le\nu\le p\\ \nu\ne i}} \operatorname{Im} A_\nu + \operatorname{Im}(A_i+\tau) = \mathbb{C}^n \qquad (i=1,\dots,p,\ \textit{for all}\ \tau\in\mathbb{C}). \tag{2.9}$$

Then $\bar{\mathbf{G}} = (\bar G_1,\dots,\bar G_p) := mc_\lambda(\mathbf{A})$ with $\lambda\in\mathbb{C}$ also satisfies the conditions (2.8) *and* (2.9) *and*

$$\operatorname{idx}\bar{\mathbf{G}} = \operatorname{idx}\mathbf{A}. \tag{2.10}$$

If $\mathbf{A}$ is irreducible, so is $\bar{\mathbf{G}}$. If $\mathbf{A}\sim\mathbf{B}$, then $mc_\lambda(\mathbf{A}) \sim mc_\lambda(\mathbf{B})$. Moreover, we have

$$mc_0(\mathbf{A}) \sim \mathbf{A}, \tag{2.11}$$

$$mc_{\lambda'}\circ mc_\lambda(\mathbf{A}) \sim mc_{\lambda'+\lambda}(\mathbf{A}). \tag{2.12}$$

Let (2.7) *be the Riemann scheme of $\mathbf{A}$. We may assume that*

$$\begin{cases} \lambda_{0,1} = \lambda, \\ \lambda_{i,1} = 0 & (i=1,\dots,p), \\ \lambda_{j,\nu} = \lambda_{j,1} \ \Rightarrow\ m_{j,\nu} \le m_{j,1} & (\nu=1,\dots,n_j,\ j=0,\dots,p). \end{cases} \tag{2.13}$$

Note that $m_{j,1}$ may be 0. Then the Riemann scheme

$$\begin{Bmatrix} x=\infty & x=t_1 & \dots & x=t_p \\ [\lambda]_{(m_{0,1})} & [0]_{(m_{1,1})} & \dots & [0]_{(m_{p,1})} \\ [\lambda_{0,2}]_{(m_{0,2})} & [\lambda_{1,2}]_{(m_{1,2})} & \dots & [\lambda_{p,2}]_{(m_{p,2})} \\ \vdots & \vdots & & \vdots \\ [\lambda_{0,n_0}]_{(m_{0,n_0})} & [\lambda_{1,n_1}]_{(m_{1,n_1})} & \dots & [\lambda_{p,n_p}]_{(m_{p,n_p})} \end{Bmatrix} \tag{2.14}$$

of $\mathbf{A}$ *is transformed into the Riemann scheme*

$$\begin{Bmatrix} x=\infty & x=t_1 & \dots & x=t_p \\ [-\lambda]_{(m_{0,1}-d)} & [0]_{(m_{1,1}-d)} & \dots & [0]_{(m_{p,1}-d)} \\ [\lambda_{0,2}-\lambda]_{(m_{0,2})} & [\lambda_{1,2}+\lambda]_{(m_{1,2})} & \dots & [\lambda_{p,2}+\lambda]_{(m_{p,2})} \\ \vdots & \vdots & & \vdots \\ [\lambda_{0,n_0}-\lambda]_{(m_{0,n_0})} & [\lambda_{1,n_1}+\lambda]_{(m_{1,n_1})} & \dots & [\lambda_{p,n_p}+\lambda]_{(m_{p,n_p})} \end{Bmatrix} \tag{2.15}$$

of $mc_\lambda(\mathbf{A})$ *with*

$$d = m_{0,1} + \ldots + m_{p,1} - (p-1)n. \tag{2.16}$$

Remark 2.3. If $\mathbf{A}$ is irreducible, then (2.8) and (2.9) are valid.

Example 2.4. Suppose the Riemann scheme of $\mathbf{A}$ is

$$\begin{Bmatrix} x=\infty & 0 & 1 \\ \mu_{0,1} & \mu_{1,1} & \mu_{2,1} \\ \mu_{0,2} & \mu_{1,2} & \mu_{2,2} \end{Bmatrix}$$

with $\mu_{j,\nu} \in \mathbb{C}$ satisfying $\sum_{j,\nu} \mu_{j,\nu} = 0$ and moreover suppose (2.8) and (2.9).

(i) If $\lambda \neq \mu_{0,\nu}$ and $\mu_{1,\nu}\mu_{2,\nu} \neq 0$ for $\nu = 1$ and 2, the Riemann scheme of $mc_\lambda(\mathbf{A})$ equals

$$\begin{Bmatrix} x=\infty & 0 & 1 \\ [-\lambda]_{(2)} & [0]_{(2)} & [0]_{(2)} \\ \mu_{0,1}-\lambda & \mu_{1,1}+\lambda & \mu_{2,1}+\lambda \\ \mu_{0,2}-\lambda & \mu_{1,2}+\lambda & \mu_{2,2}+\lambda \end{Bmatrix},$$

which follows from Theorem 2.2 with $p = 2$, $m_{j,1} = 0$, $m_{j,\nu+1} = 1$, $\lambda_{j,\nu+1} = \mu_{j,\nu}$ for $0 \le j \le 2$ and $1 \le \nu \le 2$.

(ii) If $\lambda = \mu_{0,1}$ and $\mu_{1,\nu}\mu_{2,\nu} \neq 0$ for $\nu = 1$ and 2, the Riemann scheme of $mc_\lambda(\mathbf{A})$ equals

$$\begin{Bmatrix} x=\infty & 0 & 1 \\ [-\lambda]_{(2)} & 0 & 0 \\ & \mu_{1,1}+\lambda & \mu_{2,1}+\lambda \\ \mu_{0,2}-\lambda & \mu_{1,2}+\lambda & \mu_{2,2}+\lambda \end{Bmatrix},$$

which follows from Theorem 2.2 with $p = 2$, $m_{0,\nu} = 1$, $m_{j,1} = 0$, $m_{j,\nu+1} = 1$, $\lambda_{0,2} = \mu_{0,2}$, $\lambda_{j,\nu+1} = \mu_{j,\nu}$ for $1 \le j \le 2$ and $1 \le \nu \le 2$.

(iii) Similarly, if $\lambda = \mu_{0,1}$ and $\mu_{1,1} = \mu_{2,1} = 0$, the Riemann scheme of $mc_\lambda(\mathbf{A})$ equals $\begin{Bmatrix} x=\infty & 0 & 1 \\ \mu_{0,2}-\lambda & \mu_{1,2}+\lambda & \mu_{2,2}+\lambda \end{Bmatrix}$.

Suppose $\lambda \neq 0$. Since

$$\begin{aligned}\begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_p \end{pmatrix} & \begin{pmatrix} A_1+\lambda & \dots & A_p \\ \vdots & \ddots & \vdots \\ A_1 & \dots & A_p+\lambda \end{pmatrix} \\ & = \begin{pmatrix} A_1+\lambda & \dots & A_1 \\ \vdots & \ddots & \vdots \\ A_p & \dots & A_p+\lambda \end{pmatrix} \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_p \end{pmatrix} \end{aligned} \tag{2.17}$$

and the linear map A_j induces the isomorphism $\mathbb{C}^n/\ker A_j \simeq \operatorname{Im} A_j$, we put

$$\tilde{A} := \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_p \end{pmatrix}, \tag{2.18}$$

$$G'_j := \begin{array}{rc} & \begin{array}{ccccccc} & & & \overset{j}{\smile} & & & \end{array} \\ j) & \begin{pmatrix} A_j & A_j & \dots & A_j+\lambda & A_j & \dots & A_j \end{pmatrix} \end{array} \in M(pn, \mathbb{C}) \tag{2.19}$$

for $j = 1, \dots, p$ and define

$$\begin{aligned} & G'_0 := -(G'_1 + \dots + G'_p), \\ & \bar{G}'_j := G'_j\big|_{\operatorname{Im} \tilde{A}/\ker G'_0} \qquad (j = 0, \dots, p), \end{aligned} \tag{2.20}$$

Lemma 2.5. *Suppose $\lambda \neq 0$ and put $G_0 = -(G_1 + \dots + G_p)$. Then under the above notation*

$$\tilde{A} G_j = G'_j \tilde{A} \qquad (j = 0, \dots, p), \tag{2.21}$$

$$G'_j \operatorname{Im} \tilde{A} \subset \operatorname{Im} \tilde{A}, \quad \ker G'_0 = \bigcap_{\nu=1}^{p} \ker G'_\nu \qquad (j = 0, \dots, p), \tag{2.22}$$

$$\tilde{A}(\mathcal{K} + \mathcal{L}_\lambda) = \ker G'_0 \subset \operatorname{Im} \tilde{A}, \tag{2.23}$$

and therefore $\tilde{A} \in \operatorname{End}(\mathbb{C}^{pn})$ induces the isomorphism

$$\begin{array}{ccc} \Bigl(\operatorname{End}\bigl(\mathbb{C}^{pn}/\mathcal{K} + \mathcal{L}_\lambda\bigr)\Bigr)^p & \simeq & \Bigl(\operatorname{End}\bigl(\operatorname{Im} \tilde{A}/\ker G'_0\bigr)\Bigr)^p \\ \Cup & & \Cup \\ (\bar{G}_1, \dots, \bar{G}_p) = mc_\lambda(A_1, \dots, A_p) & \sim & (\bar{G}'_1, \dots, \bar{G}'_p) \end{array}. \tag{2.24}$$

In particular, if $-\lambda$ is not the eigenvalue of $A_1 + \dots + A_p$, the middle convolution $mc_\lambda(\mathbf{A})$ transforms the system (1.1) *of SCF to the system of ONF*

$$\left(x I_{n'_1 + \dots + n'_p} - \begin{pmatrix} t_1 I_{n'_1} & & \\ & \ddots & \\ & & t_p I_{n'_p} \end{pmatrix} \right) \frac{du}{dx} = \bigl(-G'_0\big|_{\operatorname{Im} A_1 \oplus \dots \oplus \operatorname{Im} A_p}\bigr) u \tag{2.25}$$

with $n'_j = \dim \operatorname{Im} A_j$.

Proof. Note that $\tilde{A}G_0 = G_0'\tilde{A}$, which equals (2.17), and moreover that (2.21) and (2.22) are also clear.

Since $\mathcal{K} = \ker \tilde{A}$ and $\mathcal{L}_\lambda = \ker G_0$, $G_0'\tilde{A}(\mathcal{K} + \mathcal{L}_\lambda) = G_0'\tilde{A} \ker G_0 = \tilde{A}G_0 \ker G_0 = 0$ and therefore $\tilde{A}(\mathcal{K} + \mathcal{L}_\lambda) \subset \ker G_0'$. Let $u \in \ker G_0'$. Putting

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_p \end{pmatrix}, \quad u_j \in \mathbb{C}^n, \quad v := u_1 + \ldots + u_p \ \text{ and } \ \tilde{v} := \begin{pmatrix} v \\ \vdots \\ v \end{pmatrix},$$

we have $A_j v + \lambda u_j = 0$ and therefore $(A_1 + \ldots + A_p)v + \lambda v = 0$. Hence $\tilde{v} \in \ker G_0$ and $u = -\lambda^{-1}\tilde{A}\tilde{v} \in \tilde{A} \ker G_0$. Thus we have (2.23) and the remaining part of the lemma. Here we note that $\ker G_0' = \{0\}$ if $-\lambda$ is not the eigenvalue of $A_1 + \ldots + A_p$. □

3. YOKOYAMA'S EXTENDING OPERATION

In this section we examine the middle convolution of the Fuchsian system (1.6) of ONF with (1.7) and clarify its relation to Yokoyama's extension of the system. Let $\mathbf{A} = (A_1, \ldots, A_p)$ be the tuple given by (1.8) and (1.9). Then it defines the Schlesinger canonical form (1.1) of the system. First we examine the conditions (2.8) and (2.9).

For a partition $n = k_1 + \ldots + k_q$ and matrices $C_j \in M(k_j, \mathbb{C})$ we denote

$$\operatorname{diag}(C_1, \ldots, C_q) := \begin{pmatrix} C_1 & & \\ & \ddots & \\ & & C_q \end{pmatrix} \in M(n, \mathbb{C}),$$

$$O_{k_j} := 0 \in M(k_j, \mathbb{C}).$$

Then A_j equals $\operatorname{diag}(O_{n_1+\ldots+n_{j-1}}, I_{n_j}, O_{n_{j+1}+\ldots+n_p})A$.

Lemma 3.1. *The pair of conditions* (2.8) *and* (2.9) *for a tuple* $\mathbf{A} = (A_1, \ldots, A_p)$ *given by* (1.9) *is equivalent to the pair of conditions*

$$\operatorname{rank} A = n \tag{3.1}$$

and

$$\begin{cases} \operatorname{rank}\bigl(\operatorname{diag}(O_{n_1+\ldots+n_{i-1}}, I_{n_i}, O_{n_{i+1}+\ldots+n_p})(A+\tau)\bigr) = n_i, \\ \operatorname{rank}\bigl((A+\tau)\operatorname{diag}(O_{n_1+\ldots+n_{i-1}}, I_{n_i}, O_{n_{i+1}+\ldots+n_p})\bigr) = n_i \end{cases} \tag{3.2}$$

$$\textit{for any } \tau \in \mathbb{C} \textit{ and } i = 1, \ldots, p.$$

Proof. Note that the condition (2.8) with $\tau = 0$ equals (3.1), which implies (3.2) with $\tau = 0$.

Suppose $\tau \neq 0$ and (3.1). Put $\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_p \end{pmatrix}$ with $u_\nu \in \mathbb{C}^{n_\nu}$. Then

$$\sum_{\nu \neq i} \operatorname{Im} A_\nu = \{\mathbf{u} \in \mathbb{C}^n \,;\, u_i = 0\}$$

and therefore the condition (2.9) is equivalent to the first condition of (3.2). Since

$$\ker(A_i+\tau)=\{\mathbf{u}\in\mathbb{C}^n\,;\,(A_{ii}+\tau)u_i=0 \text{ and } u_\nu=0 \quad (\text{for all } \nu\neq i)\},$$

the condition (2.8) is equivalent to the condition

$$\{u_i\in\mathbb{C}^{n_i}\,;\,(A_{ii}+\tau)u_i=0 \text{ and } A_{\nu,i}u_i=0 \quad (\nu\neq i)\}=\{0\},$$

which is equivalent to the second condition of (3.2). $\square$

Definition 3.2. The system (1.1) of SCF is called *linearly irreducible* if A_j have no non-trivial proper common invariant subspace of $\mathbb{C}^n$, namely, $\mathbf{A}=(A_1,\dots,A_p)$ is irreducible. Then for the system (1.1) we have

$$\text{irreducible} \;\Rightarrow\; \text{linearly irreducible} \;\Rightarrow\; (3.1) \text{ and } (3.2). \tag{3.3}$$

Remark 3.3. The conditions (3.1) and (3.2) are valid if the system (1.6) of ONF is irreducible as a differential equation or linearly irreducible.

Assume (3.1) and (3.2) for the system (1.6) of ONF. Put $\lambda=-\rho_1\neq 0$ and apply Lemma 2.5 to $\mathbf{A}=(A_1,\dots,A_p)$ given by (1.9). Then the condition (3.1) assures $\operatorname{Im}A_j\simeq\mathbb{C}^{n_j}$, which is induced by the projection

$$\iota_j:\mathbb{C}^n\ni\mathbf{u}=\begin{pmatrix}u_1\\ \vdots\\ u_p\end{pmatrix}\mapsto u_j\in\mathbb{C}^{n_j}$$

under the notation in the proof Lemma 3.1. Since $\tilde A=\operatorname{diag}(A_1,\dots,A_p)$, under the notation in Lemma 2.5 we have the isomorphism

$$\iota:=\iota_1\oplus\ldots\oplus\iota_p:\operatorname{Im}\tilde A\simeq\mathbb{C}^{n_1+\ldots+n_p}=\mathbb{C}^n$$

and

$$G'_j|_{\operatorname{Im}\tilde A}\simeq G''_j:=\begin{array}{cc} & \begin{array}{ccccccc} \phantom{A_{j1}} & \phantom{A_{j2}} & & j & \phantom{A_{jj+1}} & & \phantom{A_{jp}} \end{array}\\ j) & \begin{pmatrix} A_{j1} & A_{j2} & \dots & A_{jj}-\rho_1 & A_{jj+1} & \dots & A_{jp}\end{pmatrix}\end{array}\in M(n,\mathbb{C}),$$

$$G''_1+\ldots+G''_p=A-\rho_1,$$

$$G'_j|_{\operatorname{Im}\tilde A/\ker G'_0}\simeq\bar G''_j:=G''_j|_{\mathbb{C}^n/\ker(A-\rho_1)}$$

for $j=1,\dots,p$ and therefore the map $\iota\circ\tilde A$ induces the isomorphism

$$\begin{array}{ccc} \Bigl(\operatorname{End}\bigl(\mathbb{C}^{pn}/\mathcal{K}+\mathcal{L}_\lambda\bigr)\Bigr)^p & \simeq & \Bigl(\operatorname{End}\bigl(\mathbb{C}^n/\ker G'_0\bigr)\Bigr)^p\\ \cup & & \cup\\ mc_{-\rho_1}(A_1,\dots,A_p) & \sim & (\bar G''_1,\dots,\bar G''_p)\end{array}. \tag{3.4}$$

In particular, we have the following.

Corollary 3.4. *Suppose the system* (1.6) *of ONF satisfies* (3.1) *and* (3.2)*. If* $-\lambda$ *is not an eigenvalue of* A*, then the middle convolution* $mc_\lambda(A_1,\dots,A_p)$ *with* (1.9) *and* (1.8) *corresponds to the transformation* $A \mapsto A+\lambda$ *of the system* (1.6)*.*

Definition 3.5. We denote this operation of the system of ONF by E_λ and call it a *generic Euler transformation*, which is defined if $-\lambda$ is not the eigenvalue of A. Note that $E_\lambda \circ E_{\lambda'} = E_{\lambda+\lambda'}$.

The transformation $A \mapsto A+\lambda$ of (1.6) corresponds to the Riemann-Liouville integral

$$I_t^\lambda u(x) := \frac{1}{\Gamma(\lambda)} \int_t^x (x-s)^{\lambda-1} u(s)ds \tag{3.5}$$

of the solution $u(x)$ of the system (cf. [9, Chapter 5]). Here $t \in \{t_1,\dots,t_p,\infty\}$.

Definition 3.6. Define the linear maps

$$\begin{array}{cccc} T_{(j,\infty)}: & M(n,\mathbb{C})^p & \to & M(n,\mathbb{C})^p \\ & \Psi & & \Psi \\ & \big(B_1,\dots,B_p\big) & \mapsto & \big(B_1,\dots,B_{j-1},-(B_1+\dots+B_p),B_{j+1},\dots,B_p\big) \end{array}$$

for $j=1,\dots,p$ and

$$\begin{array}{cccc} T_\sigma: & M(n,\mathbb{C})^p & \to & M(n,\mathbb{C})^p \\ & \Psi & & \Psi \\ & \big(B_1,\dots,B_p\big) & \mapsto & \big(B_{\sigma(1)},\dots,B_{\sigma(p)}\big) \end{array}$$

for a permutation σ of the indices $1,\dots,p$. Under the natural identification

$$M(n,\mathbb{C})^p \simeq \{(B_1,\dots,B_{p+1}) \in M(n,\mathbb{C})^{p+1}\,;\, B_{p+1}=0\} \subset M(n,\mathbb{C})^{p+1} \tag{3.6}$$

we have $T_{(p+1,\infty)}(B_1,\dots,B_p) = (B_1,\dots,B_p,-(B_1+\dots+B_p))$.

Remark 3.7.

(i) Let $\mathbf{B} \in M(n,\mathbb{C})^p$. Then $T_{(p+1,\infty)}\mathbf{B}$ is irreducible if and only if $\mathbf{B}$ is irreducible.

(ii) The map $T_{(p+1,\infty)}$ corresponds to the transformation of the Fuchsian system of SCF induced from the automorphism of the Riemann sphere defined by

$$x \mapsto \frac{t_{p+1}x - c}{x - t_{p+1}}.$$

Here $c \in \mathbb{C}$, $c \neq t_{p+1}^2$ and $t_{p+1} \neq t_j$ for $j=1,\dots,p$.

(iii) The middle convolution mc_λ commutes with T_σ, namely,

$$mc_\lambda \circ T_\sigma = T_\sigma \circ mc_\lambda. \tag{3.7}$$

Suppose $\mathbf{A} = (A_1,\dots,A_p)$ given by (1.9) is irreducible. Fix $(\rho_1,\rho_2) \in \mathbb{C}^2$ satisfying $\rho_1\rho_2 \neq 0$ and examine $mc_{\rho_1} \circ M_{(0,\dots,0,\rho_2-\rho_1)} \circ T_{(p+1,\infty)} \circ mc_{-\rho_1}(A_1,\dots,A_p)$.

Under the notation (3.4) we have $mc_{-\rho_1}(A_1,\dots,A_p) \sim (\bar G''_1,\dots,\bar G''_p)$. Since $\bar G''_j = G''_j|_{\mathbb C^n/\ker(A-\rho_1)}$ and $G''_j \ker(A-\rho_1) = 0$ and $\operatorname{rank} G''_j = n_j$, we have $\operatorname{Im}\bar G''_j = \operatorname{Im} G''_j/G''_j\ker(A-\rho_1) \simeq \mathbb C^{n_j}$ for $j=1,\dots,p$. Note that

$$\begin{aligned} M_{(0,\dots,0,\rho_2-\rho_1)}\circ T_{(p+1,\infty)}(\bar G''_1,\dots,\bar G''_p) &= (\bar G''_1,\dots,\bar G''_p,-\bar G''_1-\dots-\bar G''_p+\rho_2-\rho_1)\\ &= \bigl(\bar G''_1,\dots,\bar G''_p,(-A+\rho_2)\big|_{\mathbb C^n/\ker(A-\rho_1)}\bigr),\\ \bar V := \operatorname{Im}(-\bar G''_1-\dots-\bar G''_p+\rho_2-\rho_1) &= \operatorname{Im}\bigl((A-\rho_2)|_{\mathbb C^n/\ker(A-\rho_1)}\bigr) \end{aligned}$$

and $(\bar G''_1,\dots,\bar G''_p,\bar G''_1+\dots+\bar G''_p+\rho_1-\rho_2)$ satisfies the conditions corresponding to (2.8) and (2.9).

Applying Lemma 2.5 to $mc_{\rho_1}(\bar G''_1,\dots,\bar G''_p,-\bar G''_1-\dots-\bar G''_p+\rho_2-\rho_1)$, we have

$$mc_{\rho_1}(\bar G''_1,\dots,\bar G''_p,-\bar G''_1-\dots-\bar G''_p+\rho_2-\rho_1) \sim (A'_1,\dots,A'_p,A'_{p+1}).$$

Here A'_j for $j=1,\dots,p$ and A'_{p+1} are the endomorphisms of the linear space

$$U := \left\{ \begin{pmatrix} u\\ v\end{pmatrix} \ ;\ u\in\mathbb C^n,\ v\in\bar V\right\} \tag{3.8}$$

and they are induced from

$$\begin{aligned} \tilde A'_j &:= \begin{array}{r@{}l} & \begin{array}{ccccccccc} \phantom{A_{j1}} & & & & & \overset{p+j}{\smile} & & & \end{array}\\ j) & \begin{pmatrix} A_{j1} & \dots & A_{jp} & A_{j1} & \dots & A_{jj}-\rho_1 & \dots & A_{jp} \end{pmatrix} \end{array} \in M(2n,\mathbb C),\\ \tilde A'_{p+1} &:= \begin{pmatrix} O_n & O_n\\ -A+\rho_2 & -A+\rho_1+\rho_2 \end{pmatrix} \in M(2n,\mathbb C), \end{aligned}$$

respectively. Since

$$\tilde A'_1+\dots+\tilde A'_{p+1} = \begin{pmatrix} A & A-\rho_1\\ -A+\rho_2 & -A+\rho_2+\rho_1\end{pmatrix} \tag{3.9}$$

and

$$\begin{aligned} &\begin{pmatrix} I_n & \\ & A-\rho_1\end{pmatrix}\begin{pmatrix} A & A-\rho_1\\ -A+\rho_2 & -A+\rho_1+\rho_2\end{pmatrix}\\ &\qquad = \begin{pmatrix} A & I_n\\ -(A-\rho_1)(A-\rho_2) & -A+\rho_1+\rho_2\end{pmatrix}\begin{pmatrix} I_n & \\ & A-\rho_1\end{pmatrix} \end{aligned} \tag{3.10}$$

and the map $A-\rho_1$ defines the isomorphism $\bar V \xrightarrow{\sim} \operatorname{Im}(A-\rho_1)(A-\rho_2)$, we have

$$mc_{\rho_1}(\bar G''_1,\dots,\bar G''_p,-\bar G''_1-\dots-\bar G''_p+\rho_2-\rho_1) \sim (\hat A_1,\dots,\hat A_p,\hat A_{p+1})$$

with

$$\begin{aligned}
\hat{V} &:= \mathbb{C}^n \oplus \operatorname{Im}(A-\rho_1)(A-\rho_2) && (3.11)\\
&= \left\{ \begin{pmatrix} u \\ v \end{pmatrix} ; u \in \mathbb{C}^n,\ v \in \operatorname{Im}(A-\rho_1)(A-\rho_2) \right\} \subset \mathbb{C}^{2n}, \\
\hat{A}' &:= \begin{pmatrix} A & I_n \\ -(A-\rho_1)(A-\rho_2) & -A+\rho_1+\rho_2 \end{pmatrix}, && (3.12)\\
\hat{A} &:= \hat{A}'|_{\hat{V}} \in \operatorname{End}(\hat{V}), && (3.13)\\
\hat{A}_j &:= \bigl(\operatorname{diag}(O_{n_1+\ldots+n_{j-1}}, I_{n_j}, O_{n_{j+1}+\ldots+n_p+n})\hat{A}'\bigr)|_{\hat{V}} \qquad (j=1,\ldots,p), && (3.14)\\
\hat{A}_{p+1} &:= \bigl(\operatorname{diag}(O_n, I_n)\hat{A}'\bigr)|_{\hat{V}}. && (3.15)
\end{aligned}$$

Thus we have the following theorem.

Theorem 3.8 (Extending operation)**.** *Suppose that the Fuchsian system* (1.6) *of ONF is linearly irreducible. Then for any complex numbers ρ_1 and ρ_2 with $\rho_1\rho_2 \neq 0$, the tuple $(\hat{A}_1, \ldots, \hat{A}_{p+1})$ satisfies*

$$(\hat{A}_1, \ldots, \hat{A}_{p+1}) \sim mc_{\rho_1} \circ M^{p+1}_{(0,\ldots,0,\rho_2-\rho_1)} \circ T_{(p+1,\infty)} \circ mc_{-\rho_1}(A_1, \ldots, A_p)$$

and defines a linearly irreducible Fuchsian system

$$\bigl(xI_{\hat{n}} - \hat{T}\bigr)\frac{du}{dx} = \hat{A}u \tag{3.16}$$

of ONF. Here $\hat{T} = \operatorname{diag}(t_1 I_{n_1}, \ldots, t_p I_{n_p}, t_{p+1} I_{n_{p+1}}) \in \operatorname{End}(\hat{V})$, $\hat{V} \simeq \mathbb{C}^{\hat{n}}$ and the matrices $\hat{A}_1, \ldots, \hat{A}_{p+1}, \hat{A} \in \operatorname{End}(\hat{V})$ are defined by (3.11)–(3.15) *and*

$$\hat{n} = \dim \hat{V} = n + n_{p+1}, \quad n_{p+1} = \dim \operatorname{Im}(A-\rho_1)(A-\rho_2). \tag{3.17}$$

Let

$$\begin{Bmatrix} x=\infty & x=t_1 & \cdots & x=t_p \\ [-\mu_1]_{(m_1)} & [0]_{(n-n_1)} & \cdots & [0]_{(n-n_p)} \\ [-\mu_2]_{(m_2)} & [\lambda_{1,1}]_{(\ell_{1,1})} & \cdots & [\lambda_{p,1}]_{(\ell_{p,1})} \\ \vdots & \vdots & & \vdots \\ [-\mu_q]_{(m_q)} & [\lambda_{1,r_1}]_{(\ell_{1,r_1})} & \cdots & [\lambda_{p,r_p}]_{(\ell_{p,r_p})} \end{Bmatrix} \tag{3.18}$$

be the Riemann scheme of the system (1.6) *of ONF, which is compatible with the notation in* (1.12) *and* (1.13) *etc. when A_{ii} and A are diagonalizable. We may assume*

$$\begin{cases} \rho_1 = \mu_1 \quad \textit{and} \quad \rho_2 = \mu_2, \\ \mu_\nu = \rho_1 \ \Rightarrow\ m_\nu \leq m_1, \\ \mu_\nu = \rho_2 \ \textit{and}\ \nu > 1 \ \Rightarrow\ m_\nu \leq m_2. \end{cases} \tag{3.19}$$

Here m_1 *and* m_2 *may be* 0. *Then the Riemann scheme of the system* (3.16) *equals*

$$\begin{Bmatrix} x=\infty & x=t_1 & \dots & x=t_p & x=t_{p+1} \\ [-\mu_1]_{(n-m_2)} & [0]_{(\hat n-n_1)} & \cdots & [0]_{(\hat n-n_p)} & [0]_{(n)} \\ [-\mu_2]_{(n-m_1)} & [\lambda_{1,1}]_{(\ell_{1,1})} & \cdots & [\lambda_{p,1}]_{(\ell_{p,1})} & [\mu_1+\mu_2-\mu_3]_{(m_3)} \\ & \vdots & & \vdots & \vdots \\ & [\lambda_{1,r_1}]_{(\ell_{1,r_1})} & \cdots & [\lambda_{p,r_p}]_{(\ell_{p,r_p})} & [\mu_1+\mu_2-\mu_q]_{(m_q)} \end{Bmatrix} \tag{3.20}$$

with $\hat n = 2n - m_1 - m_2$.

Remark 3.9.

(i) Suppose that the system (1.6) satisfies (3.1) and (3.2). Then

$$q \ge 2, \tag{3.21}$$

$$\mu_\nu \ne 0 \qquad (\nu = 1, \dots, q), \tag{3.22}$$

$$\ell_{j,\nu} \le n - n_j \qquad (\nu = 1, \dots, r_j,\ j = 1, \dots, p), \tag{3.23}$$

$$m_\nu \le \min\{n_1, \dots, n_p\} \qquad (\nu = 1, \dots, q) \tag{3.24}$$

under the notation in the Theorem 3.8. For example the condition $\ker(A_j - \lambda_{j,\nu}) \cap \bigcap_{\nu\ne j} \ker A_\nu = \{0\}$ with $\dim \bigcap_{\nu \ne j} \ker A_j = n_j$ assures (3.23).

(ii) Yokoyama [16] defines the extending operation for generic parameters $\lambda_{j,\nu}$, μ_ν, ρ_1 and ρ_2 (cf. §1). It is assumed there that A_{ii}, A, $\hat A_{ii}$ and $\hat A$ are diagonalizable, $\operatorname{rank} A_{ii} = n_i$, $\rho_1 \ne \rho_2$ etc. In this note we do not assume these conditions.

(iii) Applying the extending operation to the equation $(x - t_1)\frac{du}{dx} = \lambda u$ with the Riemann scheme $\begin{Bmatrix} x=\infty & x=t_1 \\ -\lambda & \lambda \end{Bmatrix}$, we have a Gauss hypergeometric system with the Riemann scheme $\begin{Bmatrix} x=\infty & x=t_1 & x=t_2 \\ -\rho_1 & 0 & 0 \\ -\rho_2 & \lambda & \rho_1+\rho_2-\lambda \end{Bmatrix}$, which is linearly irreducible. Here λ, ρ_1 and ρ_2 are any complex numbers satisfying $\rho_1\rho_2\lambda(\rho_1 - \lambda)(\rho_2 - \lambda) \ne 0$. In this case $m_1 = m_2 = 0$, $m_3 = 1$, $-\mu_3 = \lambda$ in Theorem 3.8. Theorem 3.8 follows from Theorem 2.2 and the argument just before Theorem 3.8. We will examine the Riemann scheme of (3.16). In fact Theorem 2.2 proves that the operation $M^{p+1}_{(0,\dots,0,\rho_2-\rho_1)} \circ T_{(p+1,\infty)} \circ mc_{-\rho_1}$ transforms (3.18) to

$$\begin{Bmatrix} x=\infty & x=t_1 & \dots & x=t_p & x=t_{p+1} \\ [\rho_1-\rho_2]_{(n-m_1)} & [0]_{(n-n_1-m_1)} & \cdots & [0]_{(n-n_p-m_1)} & [\rho_2-\mu_2]_{(m_2)} \\ & [\lambda_{1,1}-\rho_1]_{(\ell_{1,1})} & \cdots & [\lambda_{p,1}-\rho_1]_{(\ell_{p,1})} & [\rho_2-\mu_3]_{(m_3)} \\ & \vdots & & \vdots & \vdots \\ & [\lambda_{1,r_1}-\rho_1]_{(\ell_{1,r_1})} & \cdots & [\lambda_{p,r_p}-\rho_1]_{(\ell_{p,r_p})} & [\rho_2-\mu_q]_{(m_q)} \end{Bmatrix}$$

and then the farther operation mc_{ρ_1} to this gives (3.20), because $\rho_2 - \mu_2 = 0$ and $\rho_1 - \rho_2 \ne \rho_1$.

4. YOKOYAMA'S RESTRICTING OPERATION

Yokoyama's restriction is the inverse of his extension and we have the following theorem.

Theorem 4.1 (Restricting operation)**.** *Let* (1.6) *be a linearly irreducible Fuchsian system of ONF. Under the notation in Theorem 3.8 we assume* $q = 2$ *and*

$$\mu_1 + \mu_2 \neq \lambda_{p,\nu} \qquad (\nu = 1, \dots, r_p). \tag{4.1}$$

Then $mc_{\mu_1} \circ T_{(p,\infty)} \circ M^p_{(0,\dots,0,\mu_1-\mu_2)} \circ mc_{-\mu_1}(A_1, \dots, A_p)$ *defines a linearly irreducible Fuchsian system*

$$\big(xI_{\check n} - \check T\big)\frac{du}{dx} = \check A u \tag{4.2}$$

of ONF, whose Riemann scheme is

$$\left\{\begin{matrix} x=\infty & x=t_1 & \dots & x=t_{p-1} \\ [-\mu_1]_{(m_1-n_p)} & [0]_{(\check n-n_1)} & \dots & [0]_{(\check n-n_{p-1})} \\ [-\mu_2]_{(m_2-n_p)} & [\lambda_{1,1}]_{(\ell_{1,1})} & \dots & [\lambda_{p-1,1}]_{(\ell_{p-1,1})} \\ [\lambda_{p,1}-\mu_1-\mu_2]_{(\ell_{p,1})} & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ [\lambda_{p,r_p}-\mu_1-\mu_2]_{(\ell_{p,r_p})} & [\lambda_{1,r_1}]_{(\ell_{1,r_1})} & \dots & [\lambda_{p-1,r_{p-1}}]_{(\ell_{p-1,r_{p-1}})} \end{matrix}\right\}. \tag{4.3}$$

Here the rank of the resulting system equals $\check n = n - n_p = n_1 + \dots + n_{p-1}$ *and*

$$\check T = \begin{pmatrix} t_1 I_{n_1} & & \\ & \ddots & \\ & & t_{p-1} I_{n_{p-1}} \end{pmatrix}, \quad \check A = \begin{pmatrix} A_{11} & \dots & A_{1,p-1} \\ \vdots & \ddots & \vdots \\ A_{p-1,1} & \dots & A_{p-1,p-1} \end{pmatrix}. \tag{4.4}$$

Proof. Suppose $q = 2$. The operation $M^p_{(0,\dots,0,\mu_1-\mu_2)} \circ mc_{-\mu_1}$ transforms (3.18) to

$$\left\{\begin{matrix} x=\infty & x=t_1 & \dots & x=t_{p-1} & x=t_p \\ [0]_{(m_2)} & [0]_{(n-n_1-m_1)} & \dots & [0]_{(n-n_{p-1}-m_1)} & [\mu_1-\mu_2]_{(n-n_p-m_1)} \\ & [\lambda_{1,1}-\mu_1]_{(\ell_{1,1})} & \dots & [\lambda_{p-1,1}-\mu_1]_{(\ell_{p-1,1})} & [\lambda_{p,1}-\mu_2]_{(\ell_{p,1})} \\ & \vdots & & \vdots & \vdots \\ & [\lambda_{1,r_1}-\mu_1]_{(\ell_{1,r_1})} & \dots & [\lambda_{p-1,r_q}-\mu_1]_{(\ell_{p-1,r_{p-1}})} & [\lambda_{p,r_q}-\mu_2]_{(\ell_{p,r_p})} \end{matrix}\right\}$$

and the farther application $mc_{\mu_1} \circ T_{(p,\infty)}$ to the above gives (4.3) because $\mu_1 \neq \mu_1 - \mu_2$ and $\mu_1 \neq \lambda_{p,\nu} - \mu_2$ for $\nu = 1, \dots, r_p$, which corresponds to a system of ONF as is claimed in Lemma 2.5. Here we note that the rank of the resulting system equals

$$\begin{aligned} & m_2 - \big((n-n_1-m_1) + \dots + (n-n_{p-1}-m_1) + 0 - (p-2)m_2\big) \\ &= n_1 + \dots + n_{p-1} - (p-1)n + (p-1)(m_1+m_2) \\ &= n - n_p. \end{aligned}$$

Since the restricting operation defined in the theorem is the inverse of the extending operation in Theorem 3.8, we have (4.4). $\square$

Remark 4.2. Suppose (4.1) is not valid. If we apply E_τ with generic $\tau \in \mathbb{C}$ to the original system of ONF preceding to the restriction, the resulting restriction satisfies (4.1). Note that mc_τ corresponding to the transformations of A, $\lambda_{j,\nu}$ and μ_k to $A+\tau$, $\lambda_{j,\nu}+\tau$ and $\mu_k+\tau$, respectively (cf. Corollary 3.4).

Remark 4.3.

(i) The extension and restriction give transformations between linearly irreducible systems of ONF. These operations do not change their indices of rigidity.

(ii) The system (1.6) is called *strongly reducible* by [16] if there exists a non-trivial proper subspace of $\mathbb{C}^n$ which is invariant under T and A. It is shown there that if the system is not strongly reducible, this property is kept by these operations.

5. EQUIVALENCE OF ALGORITHMS

In this section the system (1.1) of SCF defined by $\mathbf{A} = (A_1, \ldots, A_p) \in M(n, \mathbb{C})^p$ is identified with the system defined by $\mathbf{B} \in M(n, \mathbb{C})^p$ if $\mathbf{A} \sim \mathbf{B}$ and then the system is ONF if a representative of $\mathbf{A}$ under this identification has the form (1.9).

Proposition 5.1. *Let* $\mathbf{A} = (A_1, \ldots, A_p) \in M(n, \mathbb{C})^p$ *with* (2.8) *and* (2.9). *Then* $mc_\lambda(\mathbf{A})$ *is of ONF if and only if* λ *is not the eigenvalue of* $A_0 := -A_1 - \ldots - A_p$. *In this case the corresponding system of ONF is given by* (2.25).

Proof. Putting $d = \dim\ker A_1 + \ldots + \dim\ker A_p + \dim\ker(A_0-\lambda) - (p-1)n$, the rank of the system defined by $mc_\lambda(\mathbf{A})$ equals $n-d$. Lemma 1.1 implies that $mc_\lambda(\mathbf{A})$ is of ONF if and only if $\sum_{j=1}^{p}\big(n - \dim\ker A_j\big) = n-d$, which means $\dim\ker(A_0-\lambda) = 0$. $\square$

Definition 5.2. We denote by $E^p_{\rho_1,\rho_2}$ the extending operation of the system of ONF given in Theorem 3.8 and by R^p the restricting operation given in Theorem 4.1. Then the restricting operation R^p_j is defined by $R^p \circ T_{(j,p)}$ for $j = 1, \ldots, p$. Here (j,p) is the transposition of indices j and p (cf. Definition 3.6). Note that the restricting operation is defined only when $q = 2$.

We have proved that the extension and the restriction of the linear irreducible system of ONF is realized by suitable combinations of additions, middle convolutions and the automorphisms of $\mathbb{P}^1(\mathbb{C})$ written by $T_{(p+1,\infty)}$ and T_σ (cf. Definition 3.6), namely, linear fractional transformations. In fact, we have the following equalities for operations to linearly irreducible systems (1.6) of ONF.

$$E^p_{\rho_1,\rho_2} = mc_{\rho_1} \circ M^{p+1}_{(0,\ldots,0,\rho_2-\rho_1)} \circ T_{(p+1,\infty)} \circ mc_{-\rho_1}, \tag{5.1}$$

$$R^p = mc_{\mu_1} \circ T_{(p,\infty)} \circ M^p_{(0,\ldots,0,\mu_1-\mu_2)} \circ mc_{-\mu_1}, \tag{5.2}$$

$$R^{p+1} \circ E^p_{\rho_1,\rho_2} = \mathrm{id}\,. \tag{5.3}$$

Here $\rho_1\rho_2 \neq 0$ and μ_1 and $\mu_2 \in \mathbb{C}$ are determined by

$$(A-\mu_1)(A-\mu_2) = 0. \tag{5.4}$$

Lemma 5.3. *We have the following relations for $j = 1, \ldots, p$.*

$$R_j^{p+1} \circ E_\epsilon \circ E^p_{\rho_1,\rho_2} = mc_{\rho_1+\epsilon} \circ M^p_{(0,\ldots,0,\underset{\widehat{j}}{\rho_2-\rho_1},0,\ldots,0)} \circ T_{(j,\infty)} \circ mc_{-\rho_1}, \tag{5.5}$$

$$\operatorname{ord} R_j^{p+1} \circ E_\epsilon \circ E^p_{\rho_1,\rho_2}(\mathbf{A}) = \operatorname{ord} \mathbf{A} + \dim \operatorname{Im}(A-\rho_1)(A-\rho_2) - \dim \operatorname{Im} A_j, \tag{5.6}$$

$$\begin{aligned} &R_j^{p+1} \circ E^p_{\rho_1+\epsilon,\rho_1+\rho_2+\rho_3+\epsilon} \circ R_j^{p+1} \circ E_\epsilon \circ E^p_{\rho_1,\rho_2} \\ &\quad = mc_{\rho_1+\epsilon} \circ M^p_{(0,\ldots,0,\underset{\widehat{j}}{\rho_1+\rho_3},0,\ldots,0)} \circ mc_{-\rho_1}. \end{aligned} \tag{5.7}$$

Here ρ_1 and ρ_2 are any non-zero complex numbers, ϵ is a generic complex number and $\operatorname{ord} \mathbf{A}$ *denotes the rank of the corresponding system* (1.1) *of SCF.*

Proof. We may assume $j = 1$. It follows from (5.1) and (5.2) that

$$\begin{aligned} &R_1^{p+1} \circ E_\epsilon \circ E^p_{\rho_1,\rho_2} \\ &= mc_{\rho_1+\epsilon} \circ T_{(p+1,\infty)} \circ M^{p+1}_{(0,\ldots,0,\rho_1-\rho_2)} \circ mc_{-\rho_1-\epsilon} \circ T_{(1,p+1)} \circ mc_\epsilon \\ &\quad \circ mc_{\rho_1} \circ M^{p+1}_{(0,\ldots,0,\rho_2-\rho_1)} \circ T_{(p+1,\infty)} \circ mc_{-\rho_1} \\ &= mc_{\rho_1+\epsilon} \circ T_{(p+1,\infty)} \circ M^{p+1}_{(0,\ldots,0,\rho_1-\rho_2)} \circ T_{(1,p+1)} \circ M^{p+1}_{(0,\ldots,0,\rho_2-\rho_1)} \circ T_{(p+1,\infty)} \\ &\quad \circ mc_{-\rho_1} \\ &= mc_{\rho_1+\epsilon} \circ T_{(p+1,\infty)} \circ M^{p+1}_{(\rho_2-\rho_1,0,\ldots,0,\rho_1-\rho_2)} \circ T_{(1,p+1)} \circ T_{(p+1,\infty)} \circ mc_{-\rho_1} \\ &= mc_{\rho_1+\epsilon} \circ M^p_{(\rho_2-\rho_1,0,\ldots,0)} \circ T_{(1,\infty)} \circ mc_{-\rho_1} \end{aligned}$$

and therefore

$$\begin{aligned} &R_1^{p+1} \circ E^p_{\rho_1+\epsilon,\rho_1+\rho_2+\rho_3+\epsilon} \circ R_1^{p+1} \circ E_\epsilon \circ E^p_{\rho_1,\rho_2} \\ &= mc_{\rho_1+\epsilon} \circ M^p_{(\rho_2+\rho_3,0,\ldots,0)} \circ T_{(1,\infty)} \circ mc_{-\rho_1-\epsilon} \\ &\quad \circ mc_{\rho_1+\epsilon} \circ M^p_{(\rho_2-\rho_1,0,\ldots,0)} \circ T_{(1,\infty)} \circ mc_{-\rho_1} \\ &= mc_{\rho_1+\epsilon} \circ M^p_{(\rho_1+\rho_3,0,\ldots,0)} \circ mc_{-\rho_1}. \end{aligned}$$

The equality (5.6) follows from Theorem 3.8 and Theorem 4.1. Since ϵ is generic, the assumption of Theorem 4.1 is satisfied (cf. Remark 4.2). $\square$

We show Riemann schemes related to (5.7).

Remark 5.4. By the extension $E^p_{-\lambda_{0,1},-\lambda_{0,2}}$, we have

$$\begin{Bmatrix} x=\infty & x=t_1 & \dots & x=t_p \\ [\lambda_{0,1}]_{(m_{0,1})} & [0]_{(m_{1,1})} & \dots & [0]_{(m_{p,1})} \\ [\lambda_{0,2}]_{(m_{0,2})} & [\lambda_{1,2}]_{(m_{1,2})} & \dots & [\lambda_{p,2}]_{(m_{p,2})} \\ \vdots & \vdots & & \vdots \end{Bmatrix} \mapsto$$

$$\begin{Bmatrix} x=\infty & x=t_1 & \dots & x=t_{p+1} \\ [\lambda_{0,1}]_{(n-m_{0,2})} & [0]_{(m_{1,1}+n-m_{0,1}-m_{0,2})} & \dots & [0]_{(n)} \\ [\lambda_{0,2}]_{(n-m_{0,1})} & [\lambda_{1,2}]_{(m_{1,2})} & \dots & [\lambda_{0,3}-\lambda_{0,1}-\lambda_{0,2}]_{(m_{0,3})} \\ & \vdots & & \vdots \end{Bmatrix}.$$

Here $n = m_{j,1}+m_{j,2}+\dots$ and $m_{1,1}+\dots+m_{p,1} = (p-1)n$. By applying the restriction $R_1^{p+1}\circ E_\epsilon$ to this result we have

$$\begin{Bmatrix} x=\infty & x=t_1 & x=t_2 & \dots \\ [\lambda_{0,1}-\epsilon]_{(m_{1,1}-m_{0,2})} & [0]_{(m_{1,1})} & [0]_{(m_{2,1}-m_{0,1}-m_{0,2}+m_{1,1})} & \dots \\ [\lambda_{0,2}-\epsilon]_{(m_{1,1}-m_{0,1})} & [\lambda_{0,3}-\lambda_{0,1}-\lambda_{0,2}+\epsilon]_{(m_{0,3})} & [\lambda_{2,2}+\epsilon]_{m_{2,2}} & \dots \\ [\lambda_{1,2}+\lambda_{0,1}+\lambda_{0,2}-\epsilon]_{(m_{1,2})} & \vdots & \vdots & \dots \\ [\lambda_{1,3}+\lambda_{0,1}+\lambda_{0,2}-\epsilon]_{(m_{1,3})} & & & \\ \vdots & & & \end{Bmatrix}$$

whose rank equals $n-(m_{0,1}+m_{0,2}-m_{1,1})$. By applying the extending operation $E^p_{-\lambda_{0,1}+\epsilon,-\lambda_{1,2}-\lambda_{0,1}-\lambda_{0,2}+\epsilon}$ to what we obtained we have

$$\begin{Bmatrix} x=\infty & x=t_1 \\ [\lambda_{0,1}-\epsilon]_{(n-m_{0,1}-m_{0,2}+m_{1,1}-m_{1,2})} & [0]_{(n-m_{0,1}+m_{1,1}-m_{1,2})} \\ [\lambda_{1,2}+\lambda_{0,1}+\lambda_{0,2}-\epsilon]_{(n-m_{0,1})} & [\lambda_{0,3}-\lambda_{0,1}-\lambda_{0,2}+\epsilon]_{(m_{0,3})} \\ & [\lambda_{0,4}-\lambda_{0,1}-\lambda_{0,2}+\epsilon]_{(m_{0,4})} \\ & \vdots \end{Bmatrix.}$$

$$\begin{Bmatrix.} x=t_2 & \dots & x=t_{p+1} \\ [0]_{(n-2m_{0,1}-m_{0,2}+m_{1,1}-m_{1,2}+m_{2,1})} & \dots & [0]_{(n-m_{0,1}-m_{0,2}+m_{1,1})} \\ [\lambda_{2,2}+\epsilon]_{(m_{2,2})} & \dots & [-2\lambda_{0,1}-\lambda_{1,2}+\epsilon]_{(m_{1,1}-m_{0,1})} \\ \vdots & \dots & [\lambda_{1,3}-\lambda_{1,2}-\lambda_{0,1}+\epsilon]_{(m_{1,3})} \\ & & \vdots \end{Bmatrix}$$

and by applying the restriction R_1^{p+1} to this result we finally have

$$\begin{Bmatrix} x=\infty & x=t_1 & x=t_2 & \cdots \\ [\lambda_{0,1}-\epsilon]_{(m_{0,1}-d)} & [0]_{(m_{1,1})} & [0]_{(m_{2,1}-d)} & \cdots \\ [\lambda_{0,2}+\lambda_{0,1}+\lambda_{1,2}-\epsilon]_{(m_{0,2})} & [-2\lambda_{0,1}-\lambda_{1,2}+\epsilon]_{(m_{1,1}-m_{0,1})} & [\lambda_{2,2}+\epsilon]_{(m_{2,2})} & \cdots \\ [\lambda_{0,3}+\lambda_{0,1}+\lambda_{1,2}-\epsilon]_{(m_{0,3})} & [\lambda_{1,3}-\lambda_{1,2}-\lambda_{0,1}+\epsilon]_{(m_{1,3})} & \vdots & \cdots \\ [\lambda_{0,4}+\lambda_{0,1}+\lambda_{1,2}-\epsilon]_{(m_{0,4})} & [\lambda_{1,4}-\lambda_{1,2}-\lambda_{0,1}+\epsilon]_{(m_{1,4})} & & \\ \vdots & \vdots & & \end{Bmatrix}$$

with $d = m_{0,1}-m_{1,1}+m_{1,2}$.

Theorem 5.5. *Suppose* $\mathbf{A} = (A_1, \dots, A_p) \in M(n, \mathbb{C})^p$ *is irreducible and suppose* $\mathbf{B} = (B_1, \dots, B_p) \in M(n, \mathbb{C})^p$ *is obtained from* $\mathbf{A}$ *by a finite iteration of additions, middle convolutions and operations* $T_{(p,\infty)}$ *and* T_σ *in Definition 3.6. Let* α *and* β *be generic complex numbers so that* $mc_\alpha(\mathbf{A})$ *and* $mc_\beta(\mathbf{B})$ *are of ONF. Then* $mc_\beta(\mathbf{B})$ *can be obtained from* $mc_\alpha(\mathbf{A})$ *by a finite iteration of the suitable operations* $R_j^{p+1} \circ E_\epsilon \circ E^p_{\rho_1,\rho_2}$ *with* $\rho_1\rho_2 \neq 0$*, namely, extensions, restrictions and generic Euler transformations. Here* $\alpha = 0$ *is generic if* $\mathbf{A}$ *is of ONF.*

Proof. The theorem follows from Lemma 5.3, since $R_j^{p+1} \circ E^p_{\rho_1,\rho_1} = T_{(j,\infty)}$, $T_{(i,j)} = T_{(j,\infty)} \circ T_{(i,\infty)} \circ T_{(j,\infty)}$, $M_\mu \circ M_{\mu'} = M_{\mu+\mu'}$, $mc_\lambda \circ mc_{\lambda'} = mc_{\lambda+\lambda'}$ and $mc_0 = id$. For example, $mc_{\alpha+\epsilon} \circ M^p_{(0,\dots,\rho_3,\dots,0)}(\mathbf{A}) = mc_{\alpha+\epsilon} \circ M^p_{(0,\dots,\rho_3,\dots,0)} \circ mc_{-\alpha} \circ (mc_\alpha(\mathbf{A}))$ etc. □

6. REDUCTION PROCESS

For the system (1.1) of SCF the *spectral type* of $\mathbf{A} = (A_1, \dots, A_p)$ denoted by $\operatorname{spt}\mathbf{A}$ is the tuple of $p+1$ partitions of n

$$\operatorname{spt}\mathbf{A} := \mathbf{m} = (m_{0,1}, \dots, m_{0,n_0}; m_{1,1}, \dots, m_{1,n_1}; \dots; m_{p,1}, \dots, m_{p,n_p}) \tag{6.1}$$

under the notation (2.6). This tuple may be expressed by

$$m_{0,1} \dots m_{0,n_0}, m_{1,1} \dots m_{1,n_1}, \dots, m_{p,1} \dots m_{p,n_p} \tag{6.2}$$

and in this case (2.3) shows

$$\operatorname{idx}\mathbf{A} = \sum_{\substack{1 \le \nu \le n_j \\ 0 \le j \le p}} m_{j,\nu}^2 - (p-1)(\operatorname{ord}\mathbf{A})^2. \tag{6.3}$$

We put $n_j = 1$ and $m_{j,1} = \operatorname{ord}\mathbf{m} := m_{0,1} + \dots + m_{0,n_0}$ if $j > p$. Moreover, we put $m_{j,\nu} = 0$ if $\nu > n_j$.

For $p+1$ non-negative integers $\tau = (\tau_0, \dots, \tau_p)$ we define

$$d_\tau(\mathbf{m}) := m_{0,\tau_0} + \dots + m_{p,\tau p} - (p-1)\operatorname{ord}\mathbf{A} \tag{6.4}$$

and $\tau(\mathbf{m}) = \bigl(\tau(\mathbf{m})_0, \dots, \tau(\mathbf{m})_p\bigr)$ so that

$$m_{j,\tau(\mathbf{m})_j} \ge m_{j,\nu} \qquad (\nu = 1, \dots, n_j,\ j = 0, \dots, p). \tag{6.5}$$

Moreover, we put

$$d_{\max}(\mathbf{m}) := d_{\tau(\mathbf{m})}(\mathbf{m}). \tag{6.6}$$

Suppose $\mathbf{A}$ is irreducible. Put

$$mc_{\max}(\mathbf{A}) := mc_{\lambda_{0,\tau(\mathbf{m})_0} + \dots + \lambda_{p,\tau(\mathbf{m})_p}} \circ M_{(-\lambda_{1,\tau(\mathbf{m})_1}, \dots, -\lambda_{p,\tau(\mathbf{m})_p})}(\mathbf{A}) \tag{6.7}$$

under the notation (2.6). If $n > 1$, then Theorem 2.2 proves

$$\begin{cases} \operatorname{spt} mc_{\max}(\mathbf{A}) = \partial_{\max}(\mathbf{m}) := (\dots; m'_{j,1}, \dots, m'_{j,n_j}; \dots), \\ m'_{j,\nu} = m_{j,\nu} - d_{\max}(\mathbf{m})\delta_{\nu,\tau(\mathbf{m})_j} \qquad (\nu = 1, \dots, n_j,\ j = 0, \dots, p), \end{cases} \tag{6.8}$$

$$\operatorname{ord} \partial_{\max}(\mathbf{m}) = \operatorname{ord} \mathbf{m} - d_{\max}(\mathbf{m}). \tag{6.9}$$

If $\mathbf{A}$ is rigid, namely, $\operatorname{idx} \mathbf{m} = 2$, then we have $d_{\max}(\mathbf{m}) > 0$, because

$$\operatorname{idx} \mathbf{m} + \sum_{j=0}^{p} \sum_{\nu=1}^{n_j} (m_{j,\tau(\mathbf{m})_j} - m_{j,\nu}) \cdot m_{j,\nu} = \left(\sum_{j=0}^{p} m_{j,\tau(\mathbf{m})_j} - (p-1) \operatorname{ord} \mathbf{m} \right) \cdot \operatorname{ord} \mathbf{m}$$

and thus we have $\operatorname{ord} mc_{\max}(\mathbf{A}) < \operatorname{ord} \mathbf{A}$. Hence, if the system of SCF is linearly irreducible and rigid, the system is connected to a rank 1 system by a finite iteration of additions and middle convolutions and conversely any linearly irreducible system of SCF is constructed from a rank 1 system by a finite iteration of additions and middle convolutions (cf. [2, 8, 10, 12, 14]).

Since any rank 1 system is transformed into ONF by a suitable addition, Theorem 5.5 implies the following theorem, which is given in [16, Theorem 4.6] when the parameters $\lambda_{i,\nu}$ and μ_j are generic.

Theorem 6.1. *Any linearly irreducible rigid system of ONF is connected to a rank* 1 *system of ONF by a finite iteration of extensions, restrictions and generic Euler transformations.*

Remark 6.2.

(i) For a given $\mathbf{A} \in M(n, \mathbb{C})^p$, if there exists j with $d_{\max}(\operatorname{spt} \mathbf{A}) > m_{j,\tau(\operatorname{spt} \mathbf{A})_j}$, $\mathbf{A}$ is not irreducible. This is a consequence of Theorem 3.8.

(ii) It follows from Proposition 5.1 that $mc_{\max}(\mathbf{A})$ is not of ONF for any linearly irreducible system (1.1) of SCF.

(iii) In virtue of Lemma 6.3 a more explicit construction of the reduction process within ONF using extensions, restrictions and generic Euler transformations is obtained as follows.

Put $\mathbf{m} = \operatorname{spt}(\mathbf{A})$ for a linearly irreducible system (1.6) of ONF. Assume that $\mathbf{m}$ satisfies the assumption of Lemma 6.3 and $\lambda_{j,1} = 0$ for $j = 1, \dots, p$. Then Lemma 6.3 assures that we can find $j \geq 1$ with

$$m_{0,1} - m_{j,1} + m_{j,2} > 0 \tag{6.10}$$

because $d_{\max}(\mathbf{m}) = m_{0,1}$. Applying the operation (5.7) with $\rho_1 = \lambda_{0,1}$, $\rho_2 = \lambda_{0,2}$ and $\rho_3 = \lambda_{1,2}$, it follows from Remark 5.4 that the resulting $\mathbf{A}'$ satisfies

$$\operatorname{ord} \mathbf{A}' = \operatorname{ord} \mathbf{A} - m_{0,1} + m_{j,1} - m_{j,2} < \operatorname{ord} \mathbf{A}. \tag{6.11}$$

(iv) The existence of $j \geq 1$ satisfying (6.10) is given by [16, Lemma 4.2] when the index of rigidity of the system of ONF equals 2. Note that any linearly irreducible rigid system of SCF with rank > 1 always satisfies the assumption of Lemma 6.3.

Lemma 6.3. *Let* $\mathbf{m}$ *be a spectral type of a linearly irreducible system* (1.1) *of SCF with* $\operatorname{ord}\mathbf{m} > 1$*. Put* $\mathbf{m}' = \partial_{\max}(\mathbf{m})$*. We may assume* $m_{j,1} \geq m_{j,2} \geq \ldots \geq m_{j,n_j}$ *for* $j = 0, \ldots, p$*. If* $d_{\max}(\mathbf{m}) > 0$ *and* $d_{\max}(\mathbf{m}') > 0$*, then*

$$\sum_{j=0}^{p} \max\{0, d_{\max}(\mathbf{m}) - (m_{j,1} - m_{j,2})\} > d_{\max}(\mathbf{m}). \tag{6.12}$$

Proof. Put $d = d_{\max}(\mathbf{m})$. Since $\max\{m'_{j,1}, \ldots, m'_{j,n_j}\} = \max\{m_{j,2}, m_{j,1} - d\}$, the assumption implies

$$\sum_{j=0}^{p} \max\{m_{j,2}, m_{j,1} - d\} > (p-1)\operatorname{ord}\mathbf{m}' = (p-1)(n-d).$$

Hence we have

$$\begin{aligned}\sum_{j=0}^{p} \max\{d - (m_{j,1} - m_{j,2}), 0\} &> (p-1)(n-d) - \sum_{j=1}^{p}(m_{j,1} - d)\\ &= (p-1)(n-d) - (p-1)n + pd = d.\end{aligned}$$

□

A linearly irreducible system (1.1) of SCF satisfying $d_{\max}(\operatorname{spt}\mathbf{A}) \leq 0$ is called *fundamental*, which is not rigid and not of ONF. It is known that the fundamental systems of SCF with different spectral types cannot be connected by any iteration of middle convolutions, additions, $T_{j,\infty}$ and T_σ. It is also known that the spectral type of an irreducible system is indivisible if the index of rigidity is not negative (cf. [14, Theorem 10.2]). Moreover there exist a finite number of spectral types of fundamental systems with a fixed index of rigidity (cf. [1], [14, Proposition 8.1 and Remark 10 (ii)–(iii)], [15, Proposition 7.13]). Here $\mathbf{m} = (\ldots; m_{j,1}, \ldots, m_{j,n_j}; \ldots)$ is indivisible if there exists no non-trivial common divisor of $\{m_{j,\nu}\,;\, j = 0, 1, \ldots,\ \nu = 1, 2, \ldots\}$ and two tuples are identified if permutations of indices ν within the same j and a permutation of indices j transform one of the two into the other.

It is shown by [1] that the fundamental systems correspond to the positive imaginary roots in the closure of a negative Weyl chamber of a Kac-Moody root system with a star-shaped Dynkin diagram (cf. [7], [14, §7], [15, §7.1]). Moreover, the rigidity index of the system equals the norm of the corresponding root. Any linearly irreducible system of SCF which is not rigid is connected to a fundamental system by an iteration of $mc_{\max}$ and therefore we have the following theorem.

Definition 6.4. Let $\mathbf{m}$ in (6.1) be a spectral type of a system of SCF. We may assume $m_{j,1} \geq m_{j,2} \geq \ldots \geq m_{j,n_j}$ for $j = 0, \ldots, p$. We put

$$\operatorname{Oidx}\mathbf{m} := (p-1)\cdot\operatorname{ord}\mathbf{m} - \max_{0\leq k\leq p} \sum_{\substack{0\leq j\leq p\\ j\neq k}} \max\{m_{j,1}, m_{j,2}, \ldots\} \tag{6.13}$$

and define that $\mathbf{m}$ is of *Okubo type* if $\operatorname{Oidx}\mathbf{m} = 0$.

Remark 6.5. Suppose the spectral type $\mathbf{m}$ of a linearly irreducible system (1.1) of SCF is of Okubo type. We may assume

$$(n - m_{1,1}) + \ldots + (n - m_{p,1}) = n \tag{6.14}$$

under the notation (2.7) after applying a linear fractional transformation $T_{(j,\infty)}$ to the system if necessary. Applying the addition $M_{(-\lambda_{1,1},\ldots,-\lambda_{p,1})}$ to the system, the resulting system satisfies $\operatorname{rank} A_1 + \ldots + \operatorname{rank} A_p \le n$. Since the system is linearly irreducible, (1.10) and (1.11) are satisfied and the system is of ONF because of Lemma 1.10.

Theorem 6.6. *Suppose the system* (1.6) *of ONF is linearly irreducible and not rigid. Then the corresponding system of SCF is connected to a fundamental system of SCF with a spectral type* $\mathbf{m}$ *by an iteration of additions and middle convolutions.*
By a finite iteration of extensions, restrictions and generic Euler transformations, the system (1.6) *is connected to a system of ONF with rank* $\operatorname{ord}\mathbf{m} + \operatorname{Oidx}\mathbf{m}$. *The resulting system of ONF has the minimal rank among systems of ONF connected to the system* (1.6) *by suitable iterations of extensions, restrictions and generic Euler transformations and it is obtained by a single middle convolution of a fundamental system of SCF* (*cf. Proposition 5.1*).

Proof. The tuple of matrices $\mathbf{A} = (A_1, \ldots, A_n)$ given in (1.9) is transformed into a tuple $\mathbf{B}$ of a fundamental system of SCF by an iteration of $mc_{\max}$. Applying suitable additions and a linear fractional transformation to this fundamental system, we may assume that the Riemann scheme (2.6) of the fundamental system satisfies

$$\lambda_{1,1} = \lambda_{2,1} = \ldots = \lambda_{p,1} = 0 \quad \text{and} \quad \sum_{j=1}^{p}\sum_{\nu=2}^{n_j} m_{j,\nu} = \operatorname{ord}\mathbf{m} + \operatorname{Oidx}\mathbf{m}.$$

We apply mc_β with a generic complex number β to this system and then Theorem 2.2 and Proposition 5.1 show that the resulting system is of ONF whose rank equals $\operatorname{ord}\mathbf{m} + \operatorname{Oidx}\mathbf{m}$.

Lemma 6.3 assures that a system of ONF with the minimal rank in the theorem is connected to a fundamental system of SCF by an addition and a single middle convolution. Then Definition 6.4 and Theorem 2.2 show that the minimal rank is not smaller than $\operatorname{ord}\mathbf{m} + \operatorname{Oidx}\mathbf{m}$. □

We will give some examples.

Example 6.7. There exist 4 different spectral types of fundamental systems with index of rigidity 0 (cf. [11], [14, Proposition 8.1]):

type	ord	fundamental system	ord	ONF
$\tilde{D}_4$	2	11,11,11,11	3	111,21,21,21
$\tilde{E}_6$	3	111,111,111	4	1111,211,211
$\tilde{E}_7$	4	1111,1111,22	5	11111,2111,32
$\tilde{E}_8$	6	111111,222,33	7	1111111,322,43

The following is the list of spectral types of fundamental systems with index of rigidity -2 (cf. [14, Proposition 8.4]):

type	ord	fundamental system	ord	ONF	
GA_5	2	11,11,11,11,11	4	211,31,31,31,31	
	3	111,111,21,21	4	1111,211,31,31	
	4	1111,22,22,31	5	11111,32,32,41	
	4	1111,1111,211	5	11111,2111,311	
$D_4^{(2)}$	4	211,22,22,22	6	2211,42,42,42	222,411,42,42
	5	11111,221,221	6	111111,321,321	
	5	11111,11111,32	6	111111,21111,42	
	6	111111,2211,33	7	1111111,3211,43	
$D_6^{(2)}$	6	2211,222,222	8	22211,422,422	2222,422,4211
	8	11111111,332,44	9	111111111,432,54	
$E_7^{(2)}$	8	22211,2222,44	10	222211,4222,64	22222,42211,64
	10	22222,3331,55	12	222222,5331,75	
$E_8^{(2)}$	12	2222211,444,66	14	22222211,644,86	

Here we give the spectral types of systems of ONF with the minimal rank corresponding to a fundamental system, which are not necessarily unique but transformed to each other by suitable iterations of extensions, restrictions and generic Euler transformations. More examples can be obtained from [15, §13.1].

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