RIGIDITY OF MONODROMIES
FOR APPELL’S HYPERGEOMETRIC FUNCTIONS

Yoshishige Haraoka and Tatsuya Kikukawa

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Abstract. For monodromy representations of holonomic systems, the rigidity can be defined. We examine the rigidity of the monodromy representations for Appell’s hypergeometric functions, and get the representations explicitly. The results show how the topology of the singular locus and the spectral types of the local monodromies work for the study of the rigidity.

Keywords: rigidity, monodromy, arrangement of hyperplanes.

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1. INTRODUCTION

A local system on $\mathbb{P}^1 \setminus S$, $S$ being a finite subset, is said to be rigid if it is determined uniquely up to isomorphisms by the local monodromies. Katz [10] defined the index of rigidity, which gives a criterion for the rigidity. The index of rigidity takes a value in even integers up to 2 for irreducible local systems, and it takes the maximal value 2 if and only if the (irreducible) local system is rigid.

We can extend the notion of the rigidity to local systems on $\mathbb{P}^m \setminus S$, where $S$ is a hypersurface. A local system on $\mathbb{P}^m \setminus S$ is identified with an anti-homomorphism

$$\rho : \pi_1(\mathbb{P}^m \setminus S, b) \to \text{GL}(n, \mathbb{C}).$$

Let

$$S = \bigcup_j S_j$$

be the irreducible decomposition of $S$. Fix any irreducible component $S_j$. If $\gamma, \gamma' \in \pi_1(\mathbb{P}^m \setminus S, b)$ both encircle $S_j$ once in the positive direction and do not encircle the other $S_k$’s ($k \neq j$), then $\gamma$ and $\gamma'$ are conjugate to each other in $\pi_1(\mathbb{P}^m \setminus S, b)$. Then we can define the local monodromy of $\rho$ at $S_j$ by the conjugacy class $[\rho(\gamma)]$. The definition of the rigidity is completely similar to the one dimensional case.
The anti-homomorphism $\rho$ is said to be rigid if it is uniquely determined up to isomorphisms by the local monodromies.

On the other hand, it seems difficult to define the index of rigidity for higher the dimensional case, because the rigidity depends on the topology of the hypersurface $S$. For the one dimensional case, the topology of $S$ depends only on the number $\#S$ of the points of $S$. While for higher dimensional cases, there is no such simple topological invariant, so that the fundamental groups may have various presentations. Nevertheless, if we fix one hypersurface, it may be possible to define the index of rigidity for the hypersurface. Bearing this problem in mind, in this paper we study the rigidity of monodromy representations of holonomic systems satisfied by Appell’s hypergeometric functions.

Let $a, a', b, b', c, c'$ be complex numbers satisfying $c, c' \not\in \mathbb{Z}_{\leq 0}$. Appell’s hypergeometric functions $F_1, F_2, F_3$ and $F_4$ are defined by the power series

$$F_1(a, b, b', c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}m!n!} x^m y^n,$$

$$F_2(a, b, b', c, c'; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)(c')_{m+n}m!n!} x^m y^n,$$

$$F_3(a, a', b, b', c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m}(a')_m(b)_m(b')_n}{(c)_{m+n}m!n!} x^m y^n,$$

$$F_4(a, b, c, c'; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(c)_n}{(c)(c')_{m+n}m!n!} x^m y^n,$$

where

$$(a)_m = \frac{\Gamma(a + m)}{\Gamma(a)}.$$

These series converge in a neighborhood of $(x, y) = (0, 0)$. We refer to Appell-Kampé de Fériet [1] and Kimura [11] for basic properties of these functions.

Appell’s hypergeometric functions $F_j$ ($j = 1, 2, 3, 4$) satisfy holonomic systems on $\mathbb{P}^2$ with singular loci $S^{(j)}$, which are given by

$$S^{(1)} = \{ xy(x-1)(y-1)(x-y) = 0 \} \cup L_\infty,$$

$$S^{(2)} = \{ xy(x-1)(y-1)(x+y-1) = 0 \} \cup L_\infty,$$

$$S^{(3)} = \{ xy(x-1)(y-1)(xy-x-y) = 0 \} \cup L_\infty,$$

$$S^{(4)} = \{ xy((x-y)^2 - 2(x+y) + 1) = 0 \} \cup L_\infty,$$

where $L_\infty$ denotes the line at infinity. The singular loci $S^{(2)}, S^{(3)}$ and $S^{(4)}$ can be transformed to $S^{(1)}$ by the following variable changes $(x, y) \rightarrow (x', y')$. For $S^{(2)}$, we take

$$x' = x, \quad y' = 1 - y.$$
for $S^{(3)}$, we take
\[ x' = \frac{x - 1}{x}, \quad y' = \frac{1}{y}; \]
and for $S^{(4)}$, we take
\[ x' = xy, \quad y' = (1 - x)(1 - y). \]

The last one is given by Kato [8, 9]; we note that we find a similar transformation in Bailey [2] (the formula (1) on page 81) in expressing $F_4$ for some reducible case. This formula is originally due to Watson [14, §11.6]. We also find the same transformation as Bailey in Kimura [11, §10], where an integral representation of Euler type for $F_4$ is given.

Moreover, we see that the holonomic system for $F_3$ is transformed to the holonomic system for $F_2$ by these variable changes.

Thus, in this paper, we consider the holonomic systems for $F_1, F_2$ and $F_4$ which have the same singular locus $S^{(1)}$. Then, for $j = 1, 2, 4$, the monodromy representation of the holonomic system for $F_j$ is an anti-homomorphism
\[ \rho_j : \pi_1(\mathbb{P}^2 \setminus S^{(1)}) \to \text{GL}(n_j, \mathbb{C}), \]
where $n_1 = 3$ and $n_2 = n_4 = 4$. Our purpose is to show that these monodromy representations $\rho_j$ are rigid.

In [6] we have shown that the monodromy representation of the holonomic system for $F_4$ with the original singular locus $S^{(4)}$ is rigid. We have also shown in [5] the rigidity of the monodromy representations of the uniformization systems obtained by Kato-Sekiguchi.

2. THE FUNDAMENTAL GROUP

We study the fundamental group $\pi_1(\mathbb{P}^2 \setminus S^{(1)})$. We set
\[ S_1 = \{x = 0\}, \quad S_2 = \{y = 1\}, \quad S_3 = \{x = y\}, \quad S_4 = \{x = 1\}, \quad S_5 = \{y = 0\}. \]

Then we have the irreducible decomposition
\[ S^{(1)} = \bigcup_{j=1}^{5} S_j \cup L_\infty. \]

In order to give generators of the fundamental group, we take a base point and a reference plane. As the base point we take $b = (2, -1/2)$, and as the reference plane we take
\[ F : x + y = \frac{3}{2}. \]
We regard $F$ as a complex line with the coordinate $x$. Then the intersections of $F$ and the irreducible components of $S^{(1)}$ have the following coordinates.

\[
\begin{align*}
F \cap S_1 &: x = 0, \\
F \cap S_2 &: x = \frac{1}{2}, \\
F \cap S_3 &: x = \frac{3}{4}, \\
F \cap S_4 &: x = 1, \\
F \cap S_5 &: x = \frac{3}{2}.
\end{align*}
\]

Let $\gamma_1, \gamma_2, \ldots, \gamma_5$ be loops in $F \setminus S^{(1)}$ with base point $b$ which encircle $x = 0, 1/2, 3/4, 1, 3/2$, respectively, once in the positive direction such that $\gamma_0 = (\gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5)^{-1}$ becomes a loop which encircles $\infty$ once in the positive direction. Thus the loops $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ and $\gamma_5$ encircle the irreducible components $S_1, S_2, S_3, S_4$ and $S_5$, respectively, once in the positive direction. Applying the Zariski-van Kampen theorem [7], we get the following.

**Proposition 2.1.** The fundamental group $\pi_1(\mathbb{P}^2 \setminus S^{(1)}, b)$ has the following presentation:

\[
\pi_1(\mathbb{P}^2 \setminus S^{(1)}, b) = \left\langle \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 \mid \gamma_1 \gamma_2 = \gamma_2 \gamma_1, \gamma_4 \gamma_5 = \gamma_5 \gamma_4, \gamma_1 \gamma_3 \gamma_5 = \gamma_3 \gamma_5 \gamma_1 = \gamma_5 \gamma_1 \gamma_3, \gamma_2 \gamma_3 \gamma_4 = \gamma_3 \gamma_4 \gamma_2 = \gamma_4 \gamma_2 \gamma_3 \right\rangle.
\]

Since $S^{(1)}$ can be regarded as a real arrangement of lines, we can also get the above proposition by applying Randell’s result [13] (see also Theorem 5.57 in [12]).

We set

\[S_0^{(1)} = S^{(1)}|_{\mathbb{C}^2}\]

and

\[S^{(1)'} = S_0^{(1)} \cup L_x^\infty \cup L_y^\infty \subset \mathbb{P}^1 \times \mathbb{P}^1,\]

where $L_x^\infty = \{x = \infty\}$ and $L_y^\infty = \{y = \infty\}$. Then we have the identity of the sets

\[\mathbb{P}^2 \setminus S^{(1)} = \mathbb{P}^1 \times \mathbb{P}^1 \setminus S^{(1)'} = \mathbb{C}^2 \setminus S_0^{(1)},\]

which induces the isomorphisms

\[\pi_1(\mathbb{P}^2 \setminus S^{(1)}, b) \cong \pi_1(\mathbb{P}^1 \times \mathbb{P}^1 \setminus S^{(1)'}, b) \cong \pi_1(\mathbb{C}^2 \setminus S_0^{(1)}, b).\]

We see that, as elements in $\pi_1(\mathbb{P}^1 \times \mathbb{P}^1 \setminus S^{(1)'}, b)$, the loop

\[\gamma_1 \gamma_3 \gamma_4\]
encircles $L_x^\infty$ once in the negative direction, and the loop
\[ \gamma_5 \]
encircles $L_y^\infty$ once in the negative direction.

Let $\rho : \pi_1(\mathbb{P}^2 \setminus S^{(1)}, b) \to \text{GL}(n, \mathbb{C})$ be an anti-homomorphism, and set
\[ M_j = \rho(\gamma_j) \]
for $0 \leq j \leq 5$. Thanks to Proposition 2.1, we see that the tuple $(M_1, M_2, M_3, M_4, M_5)$ determines $\rho$, and for the tuple, the relations
\[ M_1M_2 = M_2M_1, \quad M_3M_4 = M_4M_3, \quad M_5M_6 = M_6M_5, \quad M_1M_2M_3 = M_3M_2M_1 \]
hold. By the definition, the conjugacy classes
\[ [M_1], [M_2], [M_3], [M_4], [M_5], [M_6] \]
are the local monodromies at $S_1, S_2, S_3, S_4, S_5, L_\infty$, respectively. Moreover, the conjugacy classes
\[ [(M_1M_3M_1)^{-1}], [(M_5M_3M_2)^{-1}] \]
are the local monodromies at $L_x^\infty, L_y^\infty$, respectively.

3. PFAFFIAN SYSTEMS

3.1. $F_1$

Let
\[ z(x, y) = F_1(a, b, b', c; x, y) \]
be Appell’s hypergeometric series $F_1$, and set
\[ u = \ell(z, xz, yz). \]
Then, from (1.1), we obtain the Pfaffian system for $u$ of the form
\[ du = \left( A_1 \frac{dx}{x} + A_2 \frac{dy}{y - 1} + A_3 \frac{d(x - y)}{x - y} + A_4 \frac{dx}{x - 1} + A_5 \frac{dy}{y} \right) u, \]
(3.1)
where

\[ A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & b' - c + 1 & 0 \\ 0 & -b' & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -ab' & -b' & -a - b' + c - 1 \end{pmatrix}, \]

\[ A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -b' & b \\ 0 & b' & -b \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 & 0 \\ -ab & -a - b + c - 1 & b \\ 0 & 0 & 0 \end{pmatrix}, \]

\[ A_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -b \\ 0 & b - c + 1 \end{pmatrix}. \]

**Proposition 3.1.** For the solutions of the Pfaffian system (3.1), the following holds:

(i) If

\[ b' - c, c - a - b', -b - b', c - a - b, b - c \notin \mathbb{Z}, \]

there is no logarithmic solution around the singular locus \( S_0^{(1)} \).

(ii) If \( a - b - b' \notin \mathbb{Z} \), there is no logarithmic solution around \( L_{\infty} \).

(iii) If \( a - b, a - b' \notin \mathbb{Z} \), there is no logarithmic solution around \( L_{\infty}^{x} \) and \( L_{\infty}^{y} \).

**Proof.** Since the eigenvalues of \( A_1 \) are 0, 0, \( b' - c + 1 \), there is no integral difference among the distinct eigenvalues if \( b' - c \notin \mathbb{Z} \). Then in this case, the local monodromy at \( S_1 = \{ x = 0 \} \) is given by \( e^{2\pi \sqrt{-1} A_1} \), which is semi-simple. This implies that there is no logarithmic solution around \( S_1 \).

The other assertions can be shown similarly. We have only to notice that the residue matrices around \( L_{\infty}, L_{\infty}^x, L_{\infty}^y \) are \(-(A_1 + A_2 + A_3 + A_4), -(A_1 + A_3 + A_4), -(A_2 + A_3 + A_5)\), respectively, and their eigenvalues are

\[-(A_1 + A_2 + A_3 + A_4 + A_5) : a, a, b + b';\]

\[-(A_1 + A_3 + A_4) : a, b, b;\]

\[-(A_2 + A_3 + A_5) : a, b', b'. \]

For later use, we collect the conditions in the proposition:

\[ b' - c, c - a - b', -b - b', c - a - b, b - c, a - b - b', a - b, a - b' \notin \mathbb{Z}. \] (3.3)

Then, if (3.3) holds, all the local monodromies are semi-simple. By (3.2), we can see the eigenvalues of the local monodromies:

\[ S_1 : 1, 1, e(b' - c), \]
\[ S_2 : 1, 1, e(-a - b' + c), \]
\[ S_3 : 1, 1, e(-b - b'), \]
\[ S_4 : 1, 1, e(-a - b + c), \]
\[ S_5 : 1, 1, e(b - c), \]
\[ L_{\infty} : e(a), e(a), e(b + b'), \]
\[ L_{\infty}^x : e(a), e(b), e(b), \]
\[ L_{\infty}^y : e(a), e(b'), e(b'). \] (3.4)
where we use the notation  
\[ e(\alpha) = e^{2\pi \sqrt{-1} \alpha}. \]

The spectral type means the partition which describes the multiplicities of the eigenvalues for a semi-simple matrix. Then the collection of the spectral types of the local monodromies at \( S_1, S_2, S_3, S_4, S_5; L_\infty; L_\infty^x, L_\infty^y \) is

\[ (21, 21, 21, 21, 21), \]

which we call the full spectral type of the Pfaffian system (3.1).

### 3.2. \( F_2 \)

Let

\[ z(x, y) = F_2(a, b, b'; c, c'; x, 1 - y) \]

be the transformed \( F_2 \), and set

\[ u(x, y) = t(z, xz, yz, xyz). \]

Then we obtain from (1.1) the Pfaffian system (3.1) for this \( u \) with

\[
A_1 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 1 - c & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 - c
\end{pmatrix},
A_2 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 - c' & 0 \\
0 & 0 & 0 & 1 - c'
\end{pmatrix},
A_3 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-abb' & b'(c - a - b - 1) & b(c' - a - b' - 1) & c + c' - a - b - b' - 2 \\
0 & 0 & 0 & 0
\end{pmatrix},
A_4 = \begin{pmatrix}
-\alpha b & c - a - b - 1 & -b & -1 \\
0 & 0 & 0 & 0 \\
abb' & b'(a + b - c + 1) & bb' & b' \\
0 & 0 & 0 & 0
\end{pmatrix},
A_5 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\alpha b' & -b' & c' - a - b' - 1 & -1 \\
abb' & bb' & b(a + b' - c' + 1) & b
\end{pmatrix}.
\]

In a similar way as Proposition 3.1, we obtain the following assertion.

**Proposition 3.2.** If

\[
c, c', c + c' - a - b - b', c - a - b + b', c' - a + b - b',
\]

\[ a - b - b', a - b, a - b', a - b - c', a - b' - c \notin \mathbb{Z}, \quad (3.5)\]
There is no logarithmic solution around the singular locus $S^{(1)}$ and $S^{(1)'}$ of the Pfaffian system for $u$. In this case, all the local monodromies are semi-simple, and the Pfaffian system has the full spectral type

$$(22, 22, 31, 31; 31; 211, 211).$$

The eigenvalues of the local monodromies are

$S_1 : 1, 1, e(-c), e(-c),$

$S_2 : 1, 1, e(-c'), e(-c'),$

$S_3 : 1, 1, 1, e(c + c' - a - b - b'),$

$S_4 : 1, 1, 1, e(c - a - b + b'),$

$S_5 : 1, 1, 1, e(c' - a + b - b'),$

$L_\infty : e(a), e(a), e(b + b'),$

$L'_\infty : e(a), e(b), e(a - c'),$

$L'_\infty : e(a), e(b'), e(a - c).$

3.3. $F_4$

Let

$$z(x, y) = F_4(a, b, c, c'; xy, (1 - x)(1 - y))$$

be the transformed $F_4$, and set

$$u(x, y) = t \left( z, xz_x, yz_y, xy \left( z_{xy} + \epsilon \frac{z_x - z_y}{x - y} \right) \right),$$

where

$$\epsilon = c + c' - a - b - 1.$$  

Then, as is shown in [9], we obtain from (1.1) the Pfaffian system (3.1) for this $u$ with

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 - c & 0 & 0 \\ 0 & \epsilon & 0 & 1 \\ 0 & 0 & 1 - c \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -ab & \epsilon & -c' & 0 \\ 0 & -(a + \epsilon)(b + \epsilon) & 0 & -c' \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon & -\epsilon & 0 \\ 0 & -\epsilon & \epsilon & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -ab & -c' & \epsilon & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(a + \epsilon)(b + \epsilon) & -c' \end{pmatrix},$$

$$A_5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \epsilon & 1 \\ 0 & 0 & 1 - c & 0 \\ 0 & 0 & 0 & 1 - c \end{pmatrix}.$$
In a similar way as Proposition 3.1, we obtain the following assertion.

**Proposition 3.3.** If
\[ c, c', 2a - c - c', 2b - c - c', a - b \notin \mathbb{Z}, \]  
there is no logarithmic solution around the singular locus \( S^{(1)} \) and \( S^{(1)'} \) of the Pfaffian system for \( u \). In this case, all the local monodromies are semi-simple, and the Pfaffian system has the full spectral type \((22, 22, 31, 22, 22, 22, 22)\).

The eigenvalues of the local monodromies are
\[
S_1 : 1, 1, e(-c), e(-c), \\
S_2 : 1, 1, e(-c'), e(-c'), \\
S_3 : 1, 1, 1, e(2(c + c' - a - b)), \\
S_4 : 1, 1, e(-c'), e(-c'), \\
S_5 : 1, 1, e(-c), e(-c), \\
L_{\infty} : e(c + c'), e(c + c'), e(2a), e(2b), \\
L_{x\infty} : e(a), e(a), e(b), e(b), \\
L_{y\infty} : e(a), e(a), e(b), e(b).
\]

4. FORMULATION OF THE PROBLEM

As stated in the Introduction, a local system (a representation) is said to be rigid if it is determined by the local monodromies uniquely up to isomorphisms. We are interested in the rigidity of the monodromy representations of the holonomic systems for Appell’s hypergeometric functions. If one wants to know the rigidity of a representation \( \rho \), one may look for all representation classes \([\rho']\) with the same local monodromies as \( \rho \).

Thus the problem is to determine all representation classes with prescribed local monodromies. In this paper, we slightly extend the problem. In order to explain the motivation, we note the following three facts.

First, the index of rigidity, which gives a criterion for the rigidity of local systems on \( \mathbb{P}^1 \setminus S \), is determined by the rank of the local system, the number \#\(S\) of the points in \( S \) and the spectral types of the local monodromies. Then we expect that, also in higher dimensional cases, the rigidity is determined by the spectral types of the local monodromies.

Second, we see that the eigenvalues of the local monodromies of each \( \rho_j \) satisfy some relations which cannot be directly derived from the relations (2.1). For example, if we denote the eigenvalues of the local monodromy at \( S_j \) of the monodromy representation \( \rho_1 \) for \( F_1 \) by \( 1, 1, e_j \) \((1 \leq j \leq 5)\), we have
\[ e_1 e_2 = e_4 e_5. \]
We are interested in how these relations are derived. Then we do not assume these relations a priori, and look for conjugacy classes of monodromy representations with prescribed spectral types.

Third, for any representation \( \rho \) of \( \pi_1(\mathbb{P}^2 \setminus S^{(1)}) \) and any tuple \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_5) \in (\mathbb{C}^\times)^5 \), we get a new representation \( \alpha \rho \) defined by
\[
(\alpha \rho)(\gamma_j) = \alpha_j \rho(\gamma_j) \quad (1 \leq j \leq 5),
\]
which we call the multiplication of \( \rho \) by \( \alpha \). It seems natural to consider the representations modulo multiplications.

Thus we formulate our problem as follows.

**Problem 4.1.** Determine all irreducible representation classes modulo multiplications with prescribed spectral types of local monodromies.

5. RIGIDITY OF THE MONODROMY OF \( F_1 \)

As we have seen in Section 3.1, the monodromy representation of the Pfaffian system for \( F_1 \) is an anti-homomorphism
\[
\rho_1 : \pi_1(\mathbb{P}^2 \setminus S^{(1)}, b) \to \text{GL}(3, \mathbb{C})
\]
whose spectral type is
\[
(21, 21, 21, 21, 21) \quad (5.1)
\]
if (3.3) holds. Here we call the collection of the spectral types of the local monodromies at \( S_1, S_2, S_3, S_4, S_5; L_\infty \) the spectral type of \( \rho_1 \). We shall determine the representation classes \([\rho]\) modulo multiplications with the spectral type (5.1).

Let \( \rho = (M_1, M_2, M_3, M_4, M_5) \) be an irreducible representation of \( \pi_1(\mathbb{P}^2 \setminus S^{(1)}, b) \) with the spectral type (5.1). By a multiplication, we can send the multiple eigenvalue of each \( M_j \) to 1, and hence we may assume
\[
M_j \sim \begin{pmatrix} 1 & 1 \\ 1 & e_j \end{pmatrix} \quad (1 \leq j \leq 5),
\]
where \( e_j \neq 0, 1 \). Since \( M_1 \) and \( M_2 \) commute, we can send them into diagonal matrices simultaneously by a similar transformation. If we set
\[
M_1 = \begin{pmatrix} 1 & 1 \\ 1 & e_1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 1 \\ 1 & e_2 \end{pmatrix},
\]
we get only reducible representations, which can be shown in a similar way as in the proof of Theorem 1.1 in [6]. Then we set
\[
M_1 = \begin{pmatrix} 1 & 1 \\ 1 & e_1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 1 \\ e_2 & 1 \end{pmatrix}. \quad (5.2)
\]
For $j = 3, 4, 5$, we can set

$$M_j = I_3 + \begin{pmatrix} x_j \\ y_j \\ z_j \end{pmatrix} \begin{pmatrix} 1 & p_j & q_j \end{pmatrix},$$

where

$$x_j = e_j - 1 - p_j y_j - q_j z_j.$$ 

By a similar transformation which keeps $M_1, M_2$ invariant, we can set $p_3 = q_3 = 1$.

We put these expressions into the relations (2.1). Set

$$R_1 = M_4 M_5 - M_5 M_4,$$
$$R_2 = M_5 M_3 M_1 - M_3 M_1 M_5,$$
$$R_3 = M_3 M_1 M_5 - M_1 M_5 M_3,$$
$$R_4 = M_1 M_5 M_3 - M_5 M_3 M_1,$$
$$R_5 = M_4 M_3 M_2 - M_3 M_2 M_4,$$
$$R_6 = M_3 M_2 M_4 - M_2 M_4 M_3,$$
$$R_7 = M_2 M_4 M_3 - M_4 M_3 M_2. \tag{5.3}$$

We denote the $(i,j)$ entry of a matrix $A$ by $A[i,j]$. We have

$$R_2[2,1] - R_2[2,2] = (p_5 - 1)y_3(-1 + e_5 + y_5 - p_5 y_5 + e_1 z_5 - q_5 z_5).$$

If $p_5 \neq 1$, $y_3 = 0$ or $-1 + e_5 + y_5 - p_5 y_5 + e_1 z_5 - q_5 z_5 = 0$ holds, each of which gives only a reducible representation. Then we have $p_5 = 1$. In a similar way, we get $q_4 = 1$. We can show that $z_3 z_5 = 0$ yields no irreducible representation. Then we have $z_3 z_5 \neq 0$, and in this case $q_5, y_3$ are determined by $R_4[3,1], R_4[2,3]$, respectively in this order. Now we have

$$R_4[1,3] \cdot y_5(z_3 + z_3^2 + e_3 z_5) + (R_2[2,1] - R_2[2,3])(e_1 - 1)z_3 z_5$$
$$= (e_1 - 1)y_6(z_3 + z_3^2 + e_3 z_5)(e_5 z_3^2 - e_1 e_3 z_5^2)_{z_3 z_5}.$$ 

We can show that $y_5(z_3 + z_3^2 + e_3 z_5) = 0$ yields no irreducible representation. Then we come to the relation

$$e_5 z_3^2 - e_1 e_3 z_5^2 = 0.$$ 

Now we set

$$e_1 = f_1^2, \quad e_3 = f_3^2, \quad e_5 = f_5^2,$$

which makes the above relation

$$(f_5 z_3 + f_1 f_3 z_5)(f_5 z_3 - f_1 f_3 z_5) = 0.$$ 

Then we have

$$z_5 = -\frac{f_5 z_3}{f_1 f_3}, \quad \frac{f_5 z_3}{f_1 f_3}.$$
Here we choose the first value. Then by using the relations (5.3), we can determine remaining parameters, and come to the relation

$$-e_2 e_4 + e_2^2 f_3^2 + 2 e_2 e_4 p_4 - 2 e_2^2 f_3^2 p_4 - e_2 e_4 p_4^2 + f_3^2 p_4^2 = 0.$$  

By setting

$$e_2 = f_2^2, \quad e_4 = f_4^2,$$

we reduce this relation to

$$(f_2^2 f_3 + f_2 f_4 - f_3 p_4 - f_2 f_4 p_4)(f_2^2 f_3 - f_2 f_4 - f_3 p_4 + f_2 f_4 p_4) = 0.$$  

Then we have

$$p_4 = \frac{f_2(f_2 f_3 - f_4)}{f_3 - f_2 f_4}, \quad f_2(f_2 f_3 + f_4).$$

We choose the first value. Then all parameters are written in terms of $f_j$ ($1 \leq j \leq 5$).

Put them into $R_1$. Then, for example, we have

$$R_{1[3,3]} = -\frac{(f_3 - f_2 f_4)(f_3 - f_1 f_5)(f_4 - f_1 f_2 f_5)(f_1 - f_3 f_5)(f_1 f_2 - f_4 f_5)}{f_1 f_2 f_3 f_5(-1 + f_3)(1 + f_1)(f_2 f_3 - f_4)}.$$  

By examining all possibilities for $R_{1[3,3]} = 0$, we conclude that only the relation

$$f_1 f_2 - f_4 f_5 = 0$$

gives an irreducible representation. For the other choices of $(z_5, p_4)$, we have the relations

$$f_1 f_2 \pm f_4 f_5 = 0,$$

and irreducible representations.

In this way, we obtain two representations $\rho^{(1)} = (M_1, M_2, M_3^{(1)}, M_4^{(1)}, M_5^{(1)})$ and $\rho^{(2)} = (M_1, M_2, M_3^{(2)}, M_4^{(2)}, M_5^{(2)})$, where $M_1, M_2$ are given in (5.2). The explicit forms of $M_j^{(k)}$ ($j = 3, 4, 5$) are

$$M_3^{(1)} = I_3 + \begin{pmatrix}
(e_1-e_2)(f_2 f_3 - f_4) & (e_1-e_2)(f_3 f_4 - f_2) & (e_1-e_2)(f_4 f_3 - f_2) \\
(e_1-e_2) & (e_1-e_2) & (e_1-e_2) \\
(f_2 f_3 - f_4) & (f_3 f_4 - f_2) & (f_4 f_3 - f_2)
\end{pmatrix},$$

$$M_4^{(1)} = I_5 + \begin{pmatrix}
(e_1-e_2)(f_2 f_3 - f_4) & (e_1-e_2)(f_3 f_4 - f_2) & (e_1-e_2)(f_4 f_3 - f_2) \\
(e_1-e_2) & (e_1-e_2) & (e_1-e_2) \\
(f_2 f_3 - f_4) & (f_3 f_4 - f_2) & (f_4 f_3 - f_2)
\end{pmatrix},$$

$$M_5^{(1)} = I_5 + \begin{pmatrix}
(e_1-e_2)(f_2 f_3 - f_4) & (e_1-e_2)(f_3 f_4 - f_2) & (e_1-e_2)(f_4 f_3 - f_2) \\
(e_1-e_2) & (e_1-e_2) & (e_1-e_2) \\
(f_2 f_3 - f_4) & (f_3 f_4 - f_2) & (f_4 f_3 - f_2)
\end{pmatrix},$$

(5.4)
and

$$M_{3}^{(2)} = I_{3} + \begin{pmatrix}
(1-e_{1}e_{2})(f_{2}f_{3}+f_{4})(f_{2}+f_{3}f_{4})
& (1-e_{1}e_{2})(f_{2}f_{3}+f_{4})(f_{2}+f_{3}f_{4})
& (1-e_{1}e_{2})(f_{2}f_{3}+f_{4})(f_{2}+f_{3}f_{4})

(1-e_{1})(e_{2}+1)\frac{f_{1}}{f_{2}f_{3}+f_{4}}
& (e_{1}-1)(e_{2}+1)\frac{f_{1}}{f_{2}f_{3}+f_{4}}
& (e_{1}-1)(e_{2}+1)\frac{f_{1}}{f_{2}f_{3}+f_{4}}

(1-e_{1})(e_{2}+1)\frac{f_{1}}{f_{2}f_{3}+f_{4}}
& (e_{1}-1)(e_{2}+1)\frac{f_{1}}{f_{2}f_{3}+f_{4}}
& (e_{1}-1)(e_{2}+1)\frac{f_{1}}{f_{2}f_{3}+f_{4}}
\end{pmatrix},$$

$$M_{4}^{(2)} = I_{3} + \begin{pmatrix}
(1-e_{1}e_{2})(f_{2}f_{4}+f_{3})(f_{2}+f_{3}f_{4})
& (1-e_{1}e_{2})(f_{2}f_{4}+f_{3})(f_{2}+f_{3}f_{4})
& (1-e_{1}e_{2})(f_{2}f_{4}+f_{3})(f_{2}+f_{3}f_{4})

(1-e_{1})(e_{2}+1)\frac{f_{1}}{f_{2}f_{4}+f_{3}}
& (e_{1}-1)(e_{2}+1)\frac{f_{1}}{f_{2}f_{4}+f_{3}}
& (e_{1}-1)(e_{2}+1)\frac{f_{1}}{f_{2}f_{4}+f_{3}}

(1-e_{1})(e_{2}+1)\frac{f_{1}}{f_{2}f_{4}+f_{3}}
& (e_{1}-1)(e_{2}+1)\frac{f_{1}}{f_{2}f_{4}+f_{3}}
& (e_{1}-1)(e_{2}+1)\frac{f_{1}}{f_{2}f_{4}+f_{3}}
\end{pmatrix},$$

$$M_{5}^{(2)} = I_{3} + \begin{pmatrix}
(1-e_{1}e_{2})(f_{2}f_{4}+f_{3})(f_{2}+f_{3}f_{4})
& (1-e_{1}e_{2})(f_{2}f_{4}+f_{3})(f_{2}+f_{3}f_{4})
& (1-e_{1}e_{2})(f_{2}f_{4}+f_{3})(f_{2}+f_{3}f_{4})

(1-e_{1})(e_{2}+1)\frac{f_{1}}{f_{2}f_{4}+f_{3}}
& (e_{1}-1)(e_{2}+1)\frac{f_{1}}{f_{2}f_{4}+f_{3}}
& (e_{1}-1)(e_{2}+1)\frac{f_{1}}{f_{2}f_{4}+f_{3}}

(1-e_{1})(e_{2}+1)\frac{f_{1}}{f_{2}f_{4}+f_{3}}
& (e_{1}-1)(e_{2}+1)\frac{f_{1}}{f_{2}f_{4}+f_{3}}
& (e_{1}-1)(e_{2}+1)\frac{f_{1}}{f_{2}f_{4}+f_{3}}
\end{pmatrix}.$$
Thus we get the following assertion.

**Theorem 5.1.** Let
\[ \rho : \pi_1(\mathbb{P}^2 \setminus S^{(1)}, b) \to \text{GL}(3, \mathbb{C}) \]
be an irreducible anti-homomorphism with spectral type \((2, 1)\) at each irreducible component \(S_j\) \((1 \leq j \leq 5)\). We assume that the local monodromy at each \(S_j\) is given by \(\text{diag}[1, 1, e_j] \) with \(e_j \in \mathbb{C} \setminus \{0, 1\} \) \((1 \leq j \leq 5)\). Then we have
\[ e_1 e_2 = e_4 e_5, \]
and \(\rho\) is isomorphic to one of \(\rho^{(1)} = (M_1, M_2, M_3^{(1)}, M_4^{(1)}, M_5^{(1)})\) and \(\rho^{(2)} = (M_1, M_2, M_3^{(2)}, M_4^{(2)}, M_5^{(2)})\) given by \((5.2), (5.4), (5.5)\), where \(f_2, f_3, f_4\) are square roots of \(e_2, e_3, e_4\), respectively. The local monodromy of \(\rho\) at \(L_\infty\) is given by
\[ \text{diag} \left[ \frac{1}{e_1 e_2}, \frac{1}{e_1 e_2}, \frac{1}{e_3} \right]. \]
The Galois group \(G_1\) given by \((5.6)\) acts on the representations \(\{\rho^{(1)}, \rho^{(2)}\}\) as \((5.7)\).

Since \(\rho^{(1)}\) and \(\rho^{(2)}\) have the same local monodromy at \(S_1, S_2, \ldots, S_5, L_\infty\), we cannot distinguish them by the local monodromies. However, if we regard them as local systems on \(\mathbb{P}^1 \times \mathbb{P}^1 \setminus S^{(1)'},\) we can distinguish them by looking at the local monodromy at \(L_\infty^x\) or at \(L_\infty^y\).

### 6. RIGIDITY OF THE MONODROMY OF \(F_2\)

By Proposition 3.2, the monodromy representation of the Pfaffian system for \(F_2\) is an anti-homomorphism
\[ \rho_2 : \pi_1(\mathbb{P}^2 \setminus S^{(1)}, b) \to \text{GL}(4, \mathbb{C}) \]
whose spectral type is
\[ (22, 21, 31, 31, 31) \] (6.1)
if \((3.5)\) holds. We shall determine the representation classes \([\rho]\) modulo multiplication with the spectral type \((6.1)\).

Let \(\rho = (M_1, M_2, M_3, M_4, M_5)\) be an irreducible representation of \(\pi_1(\mathbb{P}^2 \setminus S^{(1)}, b)\) with the spectral type \((6.1)\). By a multiplication, we can send (one of) the multiple eigenvalues of each \(M_j\) to 1, and hence we may assume
\[ M_j \sim \begin{pmatrix} 1 & 1 \\ e_j & e_j \end{pmatrix} \quad (j = 1, 2), \]
\[ M_j \sim \begin{pmatrix} 1 & 1 \\ 1 & e_j \end{pmatrix} \quad (j = 3, 4, 5), \]
where $e_j \neq 0, 1$. Since $M_1$ and $M_2$ commute, we can send them into diagonal matrices simultaneously by a similar transformation. In order to get an irreducible representation, we set

$$M_1 = \begin{pmatrix} 1 & e_1 \\ e_1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & e_2 \\ e_2 & 1 \end{pmatrix}.$$ \hspace{1cm} (6.2)

For $j = 3, 4, 5$, we can set

$$M_j = I_4 + \begin{pmatrix} x_j \\ z_j \\ y_j \\ w_j \end{pmatrix} \begin{pmatrix} 1 & p_j & q_j & r_j \end{pmatrix},$$

where

$$x_j = e_j - 1 - p_j y_j - q_j z_j - r_j w_j.$$

By a similar transformation which keeps $M_1, M_2$ invariant, we can send $p_3 = q_3 = r_3 = 1$. We define $R_j$ $(1 \leq j \leq 7)$ by (5.3).

We have

$$R_7[2, 1] - R_7[2, 3] = (1 - e_2)(q_4 - 1)y_4,$$

$$R_4[3, 1] - R_4[3, 2] = (1 - e_1)(p_5 - 1)z_5,$$

$$R_7[3, 2] - R_7[3, 4] = (1 - e_2)(p_4 - r_4)z_4,$$

$$R_4[2, 3] - R_4[2, 4] = (1 - e_1)(q_5 - r_5)y_5.$$

Since we can see that $y_4 y_5 z_4 z_5 = 0$ yields only reducible representations, we get

$$q_4 = p_5 = 1, \quad r_4 = p_4, \quad r_5 = q_5.$$  

Similarly, by eliminating the cases which yield only reducible representations, we can determine $z_4, w_3, z_5, y_5, y_4, w_4, y_4$ by using $R_1[2, 1] - R_1[2, 2], R_4[4, 1], R_1[4, 1], R_4[2, 3], R_4[1, 3], R_7[2, 1], R_7[4, 3], R_6[4, 4], R_6[4, 4]$, respectively. Then we get

$$R_6[4, 3] = (e_2 e_3 - e_2 e_4 - 2 e_3 e_5 p_4 + 2 e_2 e_4 p_4 + e_3 p_4^2 - e_2 e_4 p_4^2) A,$$

where we can see $A \neq 0$ by the irreducibility. If we set

$$e_2 = f_2^2, \quad e_3 = f_3^2, \quad e_4 = f_4^2,$$

the first factor of $R_6[4, 3]$ is factored, and we get

$$p_4 = \frac{f_2 (f_2 f_3 - f_4)}{f_3 - f_2 f_4}, \quad p_4 = \frac{f_2 (f_2 f_3 + f_4)}{f_3 + f_2 f_4}.$$  

We choose the first value. Then $R_5[2, 1]$ is factored into two polynomials, one of which yields only reducible representations. From the other factor, we get a value of $w_5$. Putting the value into $R_5$, we have

$$R_5[4, 4] = (- e_1 e_5 e_4 f_3^2 + 2 e_1 e_5 q_5 - 2 e_1 f_3^2 q_5 - e_1 e_5 q_5^2 + f_3^2 q_5^2) B,$$
where we can see $B \neq 0$ by the irreducibility. If we set
\[ e_1 = f_1^2, \quad e_5 = f_5^2, \]
the first factor of $R_2[4, 4]$ is factored, and then we get
\[ q_5 = \frac{f_1(f_1 f_3 - f_5)}{f_3 - f_1 f_5}, \quad q_5 = \frac{f_1(f_1 f_3 + f_5)}{f_3 + f_1 f_5}. \]
We choose the first value. Then all parameters are written in terms of $f_1, f_2, \ldots, f_5$, and we get an irreducible representation. For the other choices of $(p_4, q_5)$, we also get irreducible representations.

In this way, we obtain four representations
\[ \rho^{(k)} = (M_1, M_2, M_3^{(k)}, M_4^{(k)}, M_5^{(k)}) \quad (k = 1, 2, 3, 4), \]
where $M_1, M_2$ are given in (6.2). The explicit forms of $M_j^{(1)}$ are given as follows. We set
\[ M_3^{(1)} = I_4 + (a_{ij}), \quad M_4^{(1)} = I_4 + (b_{ij}), \quad M_5^{(1)} = I_4 + (c_{ij}). \] (6.3)
Then we have
\[
\begin{align*}
a_{11} &= a_{12} = a_{13} = a_{14} = \frac{(f_2 f_3 + f_4)(f_1 f_3 - f_5)(1 + f_1 f_2 f_4 f_5)}{(e_1 - 1)(e_2 - 1)f_4 f_5}, \\
b_{11} &= b_{13} = \frac{(f_3 + f_2 f_4)(f_1 f_3 - f_5)(1 + f_1 f_2 f_3 f_5)}{(e_1 - 1)(e_2 - 1)f_3 f_5}, \\
b_{12} &= b_{14} = \frac{f_2(f_2 f_3 + f_4)(f_1 f_3 - f_5)(1 + f_1 f_2 f_4 f_5)}{(e_1 - 1)(e_2 - 1)f_3 f_5}, \\
b_{21} &= b_{23} = \frac{(f_3 + f_2 f_4)(f_1 f_3 - f_5)(f_2 + f_1 f_4 f_5)}{(e_1 - 1)(e_2 - 1)f_2 f_3 f_5}, \\
b_{22} &= b_{24} = \frac{(f_2 f_3 + f_4)(f_1 f_3 - f_5)(f_4 + f_1 f_4 f_5)}{(e_1 - 1)(e_2 - 1)f_4 f_5}, \\
b_{31} &= b_{33} = \frac{(f_3 + f_2 f_4)(f_1 f_5 - f_5)(f_1 + f_2 f_4 f_5)}{(e_1 - 1)(e_2 - 1)f_3 f_5}, \\
b_{32} &= b_{34} = \frac{f_2(f_2 f_3 + f_4)(f_1 f_5 - f_5)(f_3 + f_2 f_4 f_5)}{(e_1 - 1)(e_2 - 1)f_3 f_5}, \\
b_{41} &= b_{43} = \frac{(f_3 + f_2 f_4)(f_1 f_5 - f_5)(f_1 f_2 + f_4 f_5)}{(e_1 - 1)(e_2 - 1)f_2 f_3 f_5}, \\
b_{42} &= b_{44} = \frac{(f_2 f_3 + f_4)(f_1 f_5 - f_5)(f_1 f_2 + f_4 f_5)}{(e_1 - 1)(e_2 - 1)f_3 f_5},
\end{align*}
\]
Rigidity of monodromies for Appell's hypergeometric functions

As in the previous section, we consider the Galois group

\[ M_j = \begin{cases} 
  j & \text{(for } j = 1, 2, 3, 4, 5), \\
  3 & \text{otherwise}.
\end{cases} \]

We find that every \( \sigma_j \) permutes \( \{\rho(1), \rho(2), \rho(3), \rho(4)\} \). If we denote \( \rho(k) \) simply by \( k \), the action is given by

\[
\begin{align*}
\sigma_1(1, 2, 3, 4) &= (2, 1, 4, 3), \\
\sigma_2(1, 2, 3, 4) &= (3, 4, 1, 2), \\
\sigma_3(1, 2, 3, 4) &= (4, 3, 2, 1), \\
\sigma_4(1, 2, 3, 4) &= (3, 4, 1, 2), \\
\sigma_5(1, 2, 3, 4) &= (2, 1, 4, 3).
\end{align*}
\]
Now we look at the \((1,1)\)-entries of \(M_3^{(k)}\) \((1 \leq k \leq 4)\):

\[
M_3^{(1)}[1,1] = 1 + \frac{(f_2 f_3 + f_4)(f_1 f_3 - f_5)(1 + f_1 f_2 f_4 f_5)}{(e_1 - 1)(e_2 - 1)f_4 f_5},
\]

\[
M_3^{(2)}[1,1] = 1 + \frac{(f_2 f_3 + f_4)(-f_1 f_3 - f_5)(1 - f_1 f_2 f_4 f_5)}{(e_1 - 1)(e_2 - 1)f_4 f_5},
\]

\[
M_3^{(3)}[1,1] = 1 + \frac{(-f_2 f_3 + f_4)(f_1 f_3 - f_5)(1 - f_1 f_2 f_4 f_5)}{(e_1 - 1)(e_2 - 1)f_4 f_5},
\]

\[
M_3^{(4)}[1,1] = 1 + \frac{(f_2 f_3 - f_4)(f_1 f_3 + f_5)(1 + f_1 f_2 f_4 f_5)}{(e_1 - 1)(e_2 - 1)f_4 f_5},
\]

which implies that no two representations in \(\{\rho^{(1)}, \rho^{(2)}, \rho^{(3)}, \rho^{(4)}\}\) are isomorphic.

We can calculate the local monodromies at infinity in \(\mathbb{P}^2\) and in \(\mathbb{P}^1 \times \mathbb{P}^1\). The local monodromies at \(L_\infty, L_x^\infty, L_y^\infty\) for \(\rho^{(k)}\) \((1 \leq k \leq 4)\) are all semi-simple, and have the spectral type \((3,1)\) at \(L_\infty\) and \((2,1,1)\) at \(L_x^\infty\) and \(L_y^\infty\). The list of the eigenvalues are given by the following table.

\[
\rho^{(1)}: L_\infty : -\frac{1}{f_1 f_2 f_3 f_4}, -\frac{1}{f_1 f_2 f_3 f_4}, -\frac{1}{f_1 f_2 f_3 f_4}, -\frac{f_4 f_5}{f_1 f_2 f_3 f_4}
\]

\[
L_x^\infty : \frac{f_5}{f_1 f_2 f_3 f_4}, \frac{f_5}{f_1 f_2 f_3 f_4}, -\frac{1}{f_1 f_2 f_3 f_4}, -\frac{f_2}{f_1 f_2 f_3 f_4}
\]

\[
L_y^\infty : -\frac{f_4}{f_1 f_2 f_3 f_4}, -\frac{f_4}{f_1 f_2 f_3 f_4}, -\frac{1}{f_1 f_2 f_3 f_4}, -\frac{f_3}{f_1 f_2 f_3 f_4}
\]

\[
\rho^{(2)}: L_\infty : -\frac{1}{f_1 f_2 f_3 f_4}, -\frac{1}{f_1 f_2 f_3 f_4}, -\frac{1}{f_1 f_2 f_3 f_4}, -\frac{f_4 f_5}{f_1 f_2 f_3 f_4}
\]

\[
L_x^\infty : \frac{f_5}{f_1 f_2 f_3 f_4}, \frac{f_5}{f_1 f_2 f_3 f_4}, -\frac{1}{f_1 f_2 f_3 f_4}, -\frac{f_2}{f_1 f_2 f_3 f_4}
\]

\[
L_y^\infty : -\frac{f_4}{f_1 f_2 f_3 f_4}, -\frac{f_4}{f_1 f_2 f_3 f_4}, -\frac{1}{f_1 f_2 f_3 f_4}, -\frac{f_3}{f_1 f_2 f_3 f_4}
\]

\[
\rho^{(3)}: L_\infty : -\frac{1}{f_1 f_2 f_3 f_4}, -\frac{1}{f_1 f_2 f_3 f_4}, -\frac{1}{f_1 f_2 f_3 f_4}, -\frac{f_4 f_5}{f_1 f_2 f_3 f_4}
\]

\[
L_x^\infty : \frac{f_5}{f_1 f_2 f_3 f_4}, \frac{f_5}{f_1 f_2 f_3 f_4}, -\frac{1}{f_1 f_2 f_3 f_4}, -\frac{f_2}{f_1 f_2 f_3 f_4}
\]

\[
L_y^\infty : -\frac{f_4}{f_1 f_2 f_3 f_4}, -\frac{f_4}{f_1 f_2 f_3 f_4}, -\frac{1}{f_1 f_2 f_3 f_4}, -\frac{f_3}{f_1 f_2 f_3 f_4}
\]

\[
\rho^{(4)}: L_\infty : -\frac{1}{f_1 f_2 f_3 f_4}, -\frac{1}{f_1 f_2 f_3 f_4}, -\frac{1}{f_1 f_2 f_3 f_4}, -\frac{f_4 f_5}{f_1 f_2 f_3 f_4}
\]

\[
L_x^\infty : \frac{f_5}{f_1 f_2 f_3 f_4}, \frac{f_5}{f_1 f_2 f_3 f_4}, -\frac{1}{f_1 f_2 f_3 f_4}, -\frac{f_2}{f_1 f_2 f_3 f_4}
\]

\[
L_y^\infty : -\frac{f_4}{f_1 f_2 f_3 f_4}, -\frac{f_4}{f_1 f_2 f_3 f_4}, -\frac{1}{f_1 f_2 f_3 f_4}, -\frac{f_3}{f_1 f_2 f_3 f_4}
\]
We see that the representations \( \rho^{(1)} \) and \( \rho^{(4)} \) cannot be distinguished by looking at the local monodromies in \( S^{(1)} \subset \mathbb{P}^2 \), and the same holds for \( \rho^{(2)} \) and \( \rho^{(3)} \). They can be distinguished by the local monodromies in \( S^{(1)}' \subset \mathbb{P}^1 \times \mathbb{P}^1 \).

**Theorem 6.1.** Let
\[ \rho : \pi_1(\mathbb{P}^2 \setminus S^{(1)}, b) \to \text{GL}(4, \mathbb{C}) \]
be an irreducible anti-homomorphism with spectral type \((2, 2)\) at \( S_1, S_2 \) and \((3, 1)\) at \( S_3, S_4, S_5 \). We assume that the eigenvalues of the local monodromies at \( S_1, S_2 \) are given by \((1, 1, e_1, e_1), (1, 1, e_2, e_2)\), respectively, and at \( S_3, S_4, S_5 \) by \((1, 1, 1, e_3), (1, 1, 1, e_4), (1, 1, 1, e_5)\), respectively, where \( e_j \in \mathbb{C} \setminus \{0, 1\} \) \((1 \leq j \leq 5)\).

Then \( \rho \) is isomorphic to one of \( \rho^{(k)} = (M_1, M_2, M_3^{(k)}, M_4^{(k)}, M_5^{(k)}) \) \((1 \leq k \leq 4)\) given by (6.2), (6.3), (6.5), where \( f_j \) is a square root of \( e_j \) \((1 \leq j \leq 5)\).

The local monodromy at \( L_\infty \) is given by
\[
\text{diag} \left[ \frac{1}{f_1 f_2 f_3 f_4}, \frac{1}{f_1 f_2 f_3 f_4}, \frac{1}{f_1 f_2 f_3 f_4}, \frac{1}{f_1 f_2 f_3 f_4}, -1 \right]
\]
for \( \rho^{(1)} \) and \( \rho^{(4)} \), and by
\[
\text{diag} \left[ \frac{1}{f_1 f_2 f_3 f_4}, \frac{1}{f_1 f_2 f_3 f_4}, \frac{1}{f_1 f_2 f_3 f_4}, \frac{1}{f_1 f_2 f_3 f_4}, -f_4 f_5 \right]
\]
for \( \rho^{(2)} \) and \( \rho^{(3)} \). The Galois group \( G_2 \) given by (6.4) acts on the representations \( \{\rho^{(1)}, \rho^{(2)}, \rho^{(3)}, \rho^{(4)}\} \) as (6.6), where \( k \) denotes \( \rho^{(k)} \).

**7. RIGIDITY OF THE MONODROMY OF \( F_4 \)**

By Proposition 3.3, the monodromy representation of the Pfaffian system for \( F_4 \) is an anti-homomorphism
\[ \rho_3 : \pi_1(\mathbb{P}^2 \setminus S^{(1)}, b) \to \text{GL}(4, \mathbb{C}) \]
whose spectral type is
\[(22, 22, 31, 22, 22; 211)\] (7.1)
if (3.6) holds. We shall determine the representation classes \([\rho]\) modulo multiplication with the spectral type (7.1).

Let \( \rho = (M_1, M_2, M_3, M_4, M_5) \) be an irreducible representation of \( \pi_1(\mathbb{P}^2 \setminus S^{(1)}, b) \) with the spectral type (7.1). By a multiplication, we can send (one of) the multiple eigenvalues of each \( M_j \) to 1, and hence we may assume
\[
M_j \sim \begin{pmatrix}
1 & 1 & e_j \\
1 & e_j & e_j
\end{pmatrix} \quad (j = 1, 2, 4, 5),
\]
\[
M_3 \sim \begin{pmatrix}
1 & 1 \\
1 & e_3
\end{pmatrix}.
\]
where $e_j \neq 0, 1$. In the same reason as in the case of $F_2$, we may assume that $M_1, M_2$ are given by (6.2). We can set

$$M_3 = I_4 + \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} (1 \ 1 \ 1 \ 1)$$

with

$$w = e_3 - 1 - x - y - z.$$

The matrices $M_4, M_5$ are parametrized as follows. Since

$$\text{rank}(M_4 - I_4) = \text{rank}(M_5 - I_4) = 2,$$

we may assume that the first and the second columns of $M_4 - I_4$ are linearly independent, and also the first and the third columns of $M_5 - I_4$ are linearly independent. Then, by using $2 \times 2$ matrices $P, Q, U, V$, we have

$$M_4 = I_4 + \begin{pmatrix} C \\ U \end{pmatrix} (I_2 \ P),$$

$$M_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \left( I_4 + \begin{pmatrix} D \\ V \end{pmatrix} (I_2 \ Q) \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$C = (e_4 - 1)I_2 - PU,$$

$$D = (e_5 - 1)I_2 - QV.$$

We set

$$P = (p_{ij}), \ Q = (q_{ij}), \ U = (u_{ij}), \ V = (v_{ij}).$$

Define $R_j$ ($1 \leq j \leq 7$) by (5.3). We are going to determine the parameters so that $R_j = O$ for $1 \leq j \leq 7$ and $(M_1, M_2, \ldots, M_5)$ is irreducible.

Note that $R_2, R_3, R_4$ are relations on $v_{ij}, q_{ij}, x, y, z, e_1, e_3, e_5$, and $R_5, R_6, R_7$ on $u_{ij}, p_{ij}, x, y, z, e_2, e_3, e_4$. First we consider the relations $R_2$ and $R_4$. We have

$$R_4[2, 3] - R_4[2, 4] = (1 - e_1)(q_{12}v_{11} - v_{12} + q_{22}v_{12}),$$

$$R_4[4, 1] - R_4[4, 2] = (1 - e_1)(v_{21} - q_{11}v_{21} - q_{21}v_{22}).$$

If we assume $v_{12}v_{22} \neq 0$, these relations determine $q_{22}$ and $q_{21}$. We assume $q_{12} \neq 0$. Then $q_{12}$ is determined by $R_4[1, 3] - R_4[1, 4]$. The relation $R_4[3, 1] - R_4[3, 2]$ contains a factor $(q_{11} - 1)$, and we choose $q_{11} = 1$. Putting these values into $R_4$, we can determine $z$ by $R_4[4, 1]$. We assume $v_{11} \neq v_{12}$. Then $x$ is determined by $R_4[2, 3]$, and also $y$ and $v_{22}$ are determined by $R_4[1, 3]$ and $R_4[2, 1]$, respectively. Now we have

$$R_2[2, 3] = (e_1^2e_3v_{11}^2 - e_1e_3v_{11} - 2e_1e_3v_{11}v_{12} + 2e_1e_3v_{11}v_{12} + e_3v_{12}^2 - e_1e_3v_{12}^2)A$$
with a non-zero factor $A$. If we set
e_1 = f_1^2, \ e_3 = f_3^2, \ e_5 = f_5^2,$
the first factor of $R_2[2, 3]$ is factored into two factors, and we get
$$v_{11} = \frac{(f_3 - f_1 f_5) v_{12}}{f_1(f_3 f_1 - f_5)}, \ v_{11} = \frac{(f_3 + f_1 f_5) v_{12}}{f_1(f_3 f_1 + f_5)}.$$ We choose the first value. Thus we get

$$v_{11} = \frac{(f_3 - f_1 f_5) v_{12}}{f_1(f_3 f_1 - f_5)}, \ v_{22} = \frac{f_1(f_3 f_5 - 1) v_{21}}{f_3 f_5 - f_1}, \ q_{11} = 1, \ q_{12} = \frac{(f_1 f_3 - f_5)(f_5 f_3 - f_1)}{f_5 (e_1 - 1) v_{21}}, \ q_{21} = 0, \ q_{22} = \frac{f_1 f_3 - e_1 f_5 - e_3 f_5 + f_1 f_3 e_5 - f_1 f_5 v_{21} + f_1^3 f_5 v_{21}}{f_1 f_5 (e_1 - 1) v_{21}},$$
$$x = -\frac{e_1 f_3 + f_1 f_5 + f_1^3 e_3 f_5 - e_4 f_3 e_5 - f_3 v_{12} + e_1 f_3 v_{12}}{f_1 f_5 (e_1 - 1)}, \ y = -\frac{f_3 v_{12}}{f_1 f_5}, \ z = \frac{f_1 f_5 - e_1 f_5 - e_3 f_5 + f_1 f_3 e_5 - f_1 f_5 v_{21} + f_1^3 f_3 v_{21}}{f_5 (e_1 - 1)},$$

which makes $R_2 = R_4 = O$, and hence $R_3 = O$. In the above, we made several assumptions. If we assume otherwise, we get other sets of $(v_{ij}, q_{ij}, x, y, z)$, which also make $R_2 = R_3 = R_4 = O$.

In a similar way, we solve the relations $R_6$ and $R_7$. We have

$$R_7[3, 2] - R_7[3, 4] = (1 - e_2)(p_{12} u_{11} - u_{12} + p_{22} u_{12}), \ R_7[4, 1] - R_7[4, 3] = (1 - e_2)(u_{21} - p_{11} u_{12} - p_{21} u_{22}).$$

If we assume $u_{12} u_{22} \neq 0$, these relations determine $p_{22}$ and $p_{21}$. If we assume $p_{12} \neq 0$, its value is determined by $R_7[1, 2] - R_7[1, 4]$. We choose a solution $p_{11} = 1$ from $R_7[2, 1] - R_7[2, 3]$. Then $y$ is determined by $R_7[4, 1]$. We assume $u_{11} \neq u_{12}$, and then $x$ is determined by $R_7[3, 2]$. The values of $z$ and $u_{22}$ are determined by $R_6[1, 1] + R_6[3, 1]$ and $R_6[2, 1] + R_6[4, 1]$, respectively. Now we have

$$R_7[1, 2] = (e_2^2 e_3 u_{11}^2 - e_2 e_4 u_{11}^2 - 2 e_2 e_3 u_{11} u_{12} + 2 e_2 e_4 u_{11} u_{12} + e_3 u_{12}^2 - e_2 e_4 u_{12}^2) B$$
with some factor $B$. If we set
\[ e_2 = f_2^2, \quad e_3 = f_3^2, \quad e_4 = f_4^2, \]
the first factor of $R_7[1,2]$ is factored into two factors, and we get
\[ u_{11} = \frac{(f_5 - f_2 f_4)u_{12}}{f_2(f_2 f_3 - f_4)}, \quad u_{11} = \frac{(f_3 + f_2 f_4)u_{12}}{f_2(f_2 f_3 + f_4)}. \]
We choose the first value. Then we get
\[
\begin{align*}
u_{11} &= \frac{(f_5 - f_2 f_4)u_{12}}{f_2(f_2 f_3 - f_4)}, \\
u_{22} &= \frac{f_3(f_2 f_3 f_4 - 1)u_{21}}{f_3 f_4 - f_2}, \\
p_{11} &= 1, \\
p_{12} &= \frac{(f_2 f_3 - f_4)(f_3 f_4 - f_2)}{f_3(e_2 - 1)u_{21}}, \\
p_{21} &= 0, \\
p_{22} &= \frac{f_2 f_3 - e_2 f_4 - e_3 f_4 + f_2 f_3 e_4 - f_2 f_3 u_{21} + f_2^3 f_3 u_{21}}{f_2 f_3(e_2 - 1)u_{21}}, \\
x &= \frac{-e_2 f_3 + f_2 f_4 + f_2^2 e_3 f_4 - e_2 f_3 e_4 - f_3 u_{12} + e_2 f_3 u_{12}}{f_2 f_4(e_2 - 1)}, \\
y &= \frac{f_2 f_3 - e_2 f_4 - e_3 f_4 + f_2 f_3 e_4 - f_2 f_3 u_{21} + f_2^3 f_3 u_{21}}{f_4(e_2 - 1)}, \\
z &= \frac{-f_3 u_{12}}{f_2 f_4},
\end{align*}
\]
which makes $R_6 = R_7 = O$, and hence $R_5 = O$. If we assume otherwise, we get other sets of $(u_{ij}, p_{ij}, x, y, z)$, which also make $R_5 = R_6 = R_7 = O$.

Now we choose a set $(v_{ij}, q_{ij}, x, y, z)$ and a set $(u_{ij}, p_{ij}, x, y, z)$, examine their compatibility, and then check the remaining relation $R_1$. If we choose the two sets given above, we come to
\[(f_4, f_5) = (f_2, f_1), \quad (f_4, f_5) = (-f_2, -f_1).\]
It turns out that both cases give irreducible representations. We choose the first one. In this case, the eigenvalues of $M_0^{-1}$ are calculated as
\[ e_1 e_2, \quad e_1 e_2, \quad \frac{A + \sqrt{B}}{C}, \quad \frac{A - \sqrt{B}}{C}, \]
where $A, B, C$ are polynomials in $f_1, f_2, f_3$ and $u_{12}$. In particular, $B$ is quadratic in $u_{12}$. In order to uniformize these eigenvalues, we set
\[ u_{12} = \frac{e_2(f_3 - 1)(e_1 + g)(e_2 f_3 + g)}{(e_1 - 1)(e_2 - 1)f_3 g}, \]
with a new parameter $g$. Then the eigenvalues of $M_0^{-1}$ are

$$e_1 e_2, \ e_1 e_2, \ g^2, \ \frac{e_1 e_2^2 e_3}{g^2}.$$ 

In this case, we have

$$M_3 = I_4 + (a_{ij}), \ M_4 = (b_{ij}), \ M_5 = (c_{ij}) \quad (7.2)$$

with

$$a_{11} = a_{12} = a_{13} = a_{14} = \frac{(f_3 - 1)(g + 1)(e_1 e_2 f_3 + g)}{(e_1 - 1)(e_2 - 1) f_3},$$

$$a_{21} = a_{22} = a_{23} = a_{24} = -\frac{(f_3 - 1)(e_1 f_3 + g)(e_2 + g)}{(e_1 - 1)(e_2 - 1) g},$$

$$a_{31} = a_{32} = a_{33} = a_{34} = -\frac{(f_3 - 1)(e_1 + g)(e_2 f_3 + g)}{(e_1 - 1)(e_2 - 1) g},$$

$$a_{41} = a_{42} = a_{43} = a_{44} = \frac{(f_3 - 1)(f_3 + g)(e_1 e_2 + g)}{(e_1 - 1)(e_2 - 1) g},$$

$$b_{11} = 1 + \frac{(e_2 - f_3)(g + 1)(e_1 e_2 f_3 + g)}{(e_1 - 1)(e_2 - 1) f_3 g},$$

$$b_{12} = b_{14} = -\frac{e_2(f_3 - 1)(g + 1)(e_1 e_2 f_3 + g)}{(e_1 - 1)(e_2 - 1) f_3 g},$$

$$b_{13} = b_{11} - 1,$$

$$b_{21} = b_{23} = \frac{(f_3 - 1)(e_1 f_3 + g)(e_2 + g)}{(e_1 - 1)(e_2 - 1) f_3 g},$$

$$b_{22} = e_2 + \frac{(e_2 f_3 - 1)(e_1 f_3 + g)(e_2 + g)}{(e_1 - 1)(e_2 - 1) f_3 g},$$

$$b_{24} = b_{22} - e_2,$$

$$b_{31} = \frac{(f_3 - e_2)(e_1 + g)(e_2 f_3 + g)}{(e_1 - 1)(e_2 - 1) f_3 g},$$

$$b_{32} = b_{34} = -\frac{e_2(f_3 - 1)(e_1 + g)(e_2 f_3 + g)}{(e_1 - 1)(e_2 - 1) f_3 g},$$

$$b_{33} = 1 + b_{31},$$

$$b_{41} = b_{43} = -\frac{(f_3 - 1)(f_3 + g)(e_1 e_2 + g)}{(e_1 - 1)(e_2 - 1) f_3 g},$$

$$b_{42} = -\frac{(e_2 f_3 - 1)(f_3 + g)(e_1 e_2 + g)}{(e_1 - 1)(e_2 - 1) f_3 g},$$

$$b_{44} = e_2 + b_{42}.$$
\[c_{11} = 1 + \frac{(e_1 - f_3)(g + 1)(e_1 e_2 f_3 + g)}{(e_1 - 1)(e_2 - 1)f_3 g},\]
\[c_{12} = c_{11} - 1,\]
\[c_{13} = c_{14} = -\frac{e_1 (f_3 - 1)(g + 1)(e_1 e_2 f_3 + g)}{(e_1 - 1)(e_2 - 1)f_3 g},\]
\[c_{21} = \frac{(f_3 - e_1)(e_1 f_3 + g)(e_2 + g)}{(e_1 - 1)(e_2 - 1)f_3 g},\]
\[c_{22} = 1 + c_{21},\]
\[c_{23} = c_{24} = \frac{e_1 (f_3 - 1)(e_1 f_3 + g)(e_2 + g)}{(e_1 - 1)(e_2 - 1)f_3 g},\]
\[c_{31} = c_{32} = \frac{(f_3 - 1)(e_1 + g)(e_2 f_3 + g)}{(e_1 - 1)(e_2 - 1)f_3 g},\]
\[c_{33} = e_1 + \frac{(e_1 f_3 - 1)(e_1 + g)(e_2 f_3 + g)}{(e_1 - 1)(e_2 - 1)f_3 g},\]
\[c_{34} = c_{33} - e_1,\]
\[c_{41} = c_{42} = -\frac{(f_3 - 1)(f_3 + g)(e_1 e_2 + g)}{(e_1 - 1)(e_2 - 1)f_3 g},\]
\[c_{43} = -\frac{(e_1 f_3 - 1)(f_3 + g)(e_1 e_2 + g)}{(e_1 - 1)(e_2 - 1)f_3 g},\]
\[c_{44} = e_1 + c_{43}.\]

We denote by \(\rho^{(1)}\) the representation determined by the above \((M_1, M_2, M_3, M_4, M_5)\). Note that, in deriving the entries of \(M_3, M_4, M_5\), we used square roots of \(e_1, e_2, \ldots, e_5\), while the result is written in terms of \(e_1, e_2\), a square root of \(e_3\) and a square root of an eigenvalue of \(M_0^{-1}\).

In this way, we obtain a lot of representations. We find that all representations thus obtained are reduced to one of four representations \(\rho^{(1)}, \rho^{(2)}, \rho^{(3)}, \rho^{(4)}\) given below.

Let \(\sigma, \tau\) be generators of \(\mathbb{Z}_2\) defined by
\[\sigma : f_3 \mapsto -f_3, \quad \tau : g \mapsto -g,\]
and set
\[G_4 = (\sigma, \tau) = (\mathbb{Z}_2)^2,\]
(7.3)

which is Klein’s four-group. Then we set
\[M^{(2)}_j = \sigma(M_j),\]
\[M^{(3)}_j = \tau(M_j),\]
\[M^{(4)}_j = \sigma \tau(M_j).\]
for $j = 3, 4, 5$. Now we define $\rho^{(2)}, \rho^{(3)}, \rho^{(4)}$ by

$$
\rho^{(2)} = (M_1, M_2, M_3^{(2)}, M_4^{(2)}, M_5^{(2)}),
\rho^{(3)} = (M_1, M_2, M_3^{(3)}, M_4^{(3)}, M_5^{(3)}),
\rho^{(4)} = (M_1, M_2, M_3^{(4)}, M_4^{(4)}, M_5^{(4)}),
$$

where $M_1, M_2$ are the matrices in (6.2). Then it is evident that the Galois group $G_4$ acts faithfully on the set $\{\rho^{(1)}, \rho^{(2)}, \rho^{(3)}, \rho^{(4)}\}$. Since the eigenvalues of $M_0^{-1}$ do not change for the action of $G_4$, the eigenvalues of the local monodromy at $L_\infty$ for every $\rho^{(k)}$ are

$$
\frac{1}{e_1 e_2}, \frac{1}{e_1 e_2}, \frac{1}{g^2}, \frac{e_1^2 e_2^2 e_3}{e_1 e_2}.
$$

For each representation $\rho^{(k)}$, the eigenvalues of the local monodromies at $L_\infty$ coincide, and are given as follows:

$$
\rho^{(1)} : -\frac{1}{g}, -\frac{1}{g}, -\frac{g}{e_1 e_2 f_3}, -\frac{g}{e_1 e_2 f_3},
\rho^{(2)} : -\frac{1}{g}, -\frac{g}{e_1 e_2 f_3}, \frac{g}{e_1 e_2 f_3},
\rho^{(3)} : \frac{1}{g}, \frac{1}{g}, \frac{g}{e_1 e_2 f_3}, \frac{g}{e_1 e_2 f_3},
\rho^{(4)} : \frac{1}{g}, \frac{1}{g}, \frac{g}{e_1 e_2 f_3}, -\frac{g}{e_1 e_2 f_3}.
$$

Then we can distinguish the four representations by these local monodromies.

**Theorem 7.1.** Let

$$
\rho : \pi_1(\mathbb{P}^2 \setminus S^{(1)}, b) \to \text{GL}(4, \mathbb{C})
$$

be an irreducible anti-homomorphism with spectral type (2, 2) at $S_1, S_2, S_4, S_5$ and (3, 1) at $S_3$. We assume that the eigenvalues of the local monodromies at $S_1, S_2, S_4, S_5$ are given by $(1, 1, e_1, e_1), (1, 1, e_2, e_2), (1, 1, e_4, e_4), (1, 1, e_5, e_5)$, respectively, and at $S_3$ by $(1, 1, e_3)$, where $e_j \in \mathbb{C} \setminus \{0, 1\}$ ($1 \leq j \leq 5$). Then $\rho$ is, up to multiplications, isomorphic to one of $\rho^{(k)}$ ($1 \leq k \leq 4$) given by (6.2), (7.2), (7.4), where $f_3$ is a square root of $e_3$. For these representations,

$$
e_1 = e_5, \ e_2 = e_4
$$

hold. The local monodromy at $L_\infty$ is given by

$$
\text{diag} \begin{bmatrix}
\frac{1}{e_1 e_2}, & \frac{1}{e_1 e_2}, & \frac{1}{g^2}, & \frac{g^2}{e_1^2 e_2^2 e_3}
\end{bmatrix}.
$$

The Galois group $G_4$ given by (7.3) acts faithfully on the set $\{\rho^{(1)}, \rho^{(2)}, \rho^{(3)}, \rho^{(4)}\}$. 


8. CONCLUSION AND REMARKS

Theorems 5.1, 6.1 and 7.1 show that the monodromy representations for Appell’s hypergeometric functions with singular locus

\[ S^{(1)} = \{ xy(x-1)(y-1)(x-y) = 0 \} \cup L_\infty \]

are almost rigid. Namely they are determined by the local monodromies up to a finite number of possibilities. It is remarkable that these representations are determined by prescribing only the local monodromies at singular loci in \( \mathbb{C}^2 \). The local monodromy at the line at infinity in \( \mathbb{P}^2 \) or in \( \mathbb{P}^1 \times \mathbb{P}^1 \) is uniquely determined.

It is also remarkable that these representations are strictly rigid if they are considered in \( \mathbb{P}^1 \times \mathbb{P}^1 \). Thus the rigidity depends on compactifications of \( \mathbb{C}^2 \). This fact can be explained by the correspondence of the parameters and the eigenvalues of the local monodromies. For the \( F_1 \) case, as listed in (3.4), the eigenvalues of the local monodromy at each component are determined by the parameters \((a, b, b', c)\). If we change the parameters to

\[ (a, b + \frac{1}{2}, b' + \frac{1}{2}, c + \frac{1}{2}) \],

the eigenvalues at \( S_1, \ldots, S_5, L_\infty \) are unchanged, while, the eigenvalues at \( L_{\infty}^x, L_{\infty}^y, e(b) \) and \( e(b') \) change to \(-e(b)\) and \(-e(b')\). Thus the two distinct representations, which can be distinguished by the local monodromies in \( \mathbb{P}^1 \times \mathbb{P}^1 \), give the same local monodromies in \( \mathbb{P}^2 \). For \( F_2 \) case, we consider the change of the parameters

\[ (a, b, b', c, c') \mapsto (a, b + \frac{1}{2}, b' + \frac{1}{2}, c, c') \].

For the \( F_4 \) case, we consider the changes of the parameters generated by

\[ (a, b, c, c') \mapsto (a + \frac{1}{2}, b, c, c'), \]

\[ (a, b, c, c') \mapsto (a, b + \frac{1}{2}, c, c') \],

which form the group \((\mathbb{Z}_2)^2\).

We also note that the relations among the eigenvalues of the local monodromies, such as \( e_1 e_2 = e_4 e_5 \) for \( F_1 \) or \( e_1 = e_5, e_2 = e_4 \) for \( F_4 \), are consequences of the existence of irreducible representations. In other words, these relations are derived from the spectral type of the representation. Hence our formulation of the problem given in section 4 works well for the study of the rigidity in higher dimensional cases.

Here we remark on the irreducibility of the representations given in Theorems 5.1, 6.1 and 7.1. For each \( \rho^{(k)} \), two matrices \( M_1 \) and \( M_2 \) are diagonal. Then, if \( \rho^{(k)} \) is reducible, the invariant subspace is spanned by some of the unit vectors \( t_i(0, \ldots, 1, \ldots, 0) \). Hence we can show the irreducibility by checking the non-vanishing of off-diagonal entries of \( M_3 \) or \( M_4 \) or \( M_5 \). For example, if no off-diagonal entry of
$M_3$ (or $M_4$ or $M_5$) vanishes, the representation is irreducible. We can obtain an exact condition for the irreducibility in the same way as in [6].

In this paper, we started from the spectral types of existing representations, and looked for representations with the spectral types. It is not known for what spectral types irreducible representations exist. This problem may be concerned with defining the index of rigidity in higher dimensional cases.

On the other hand, we can obtain infinitely many representations of $\pi_1(P^2 \setminus S^{(1)})$ from rigid Fuchsian ordinary differential equations with four singular points by using the middle convolution defined by [4]. Also, the middle convolution connects the monodromy representations for $F_2$ and $F_4$ with the one for $F_3$. It would be interesting to study the representations of $\pi_1(P^2 \setminus S^{(1)})$ by the middle convolution.

In the famous paper [3] Gérard and Levelt showed that, if the singular locus consists of normally crossing hyperplanes, any solution of the Pfaffian system becomes elementary. We can give another proof for this assertion by considering the monodromy. Namely, if the hyperplanes of the singular locus are normally crossing, the fundamental group becomes abelian, so that any representation becomes reducible. The singular locus $S^{(1)}$ is chosen by Gérard-Levelt as the simplest hyperplane arrangement which gives a non-elementary Pfaffian system. Now we recognize that the topology of the singular locus fairly determines the analytic behaviors of the solutions. Then we consider it important to characterize hypersurfaces such that the fundamental groups of the complement space admit irreducible representations. This problem will be fundamental in the theory of hyperplane arrangements or in algebraic geometry.

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Yoshishige Haraoka
haraoka@kumamoto-u.ac.jp

Kumamoto University
Department of Mathematics
Kumamoto 860-8555, Japan

Tatsuya Kikukawa

Kumamoto High School
Shin-Oe 1-8, Kumamoto 862-0972, Japan

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