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RIGIDITY OF MONODROMIES FOR APPELL'S HYPERGEOMETRIC FUNCTIONS

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Abstract. For monodromy representations of holonomic systems, the rigidity can be defined. We examine the rigidity of the monodromy representations for Appell's hypergeometric functions, and get the representations explicitly. The results show how the topology of the singular locus and the spectral types of the local monodromies work for the study of the rigidity.

Keywords: rigidity, monodromy, arrangement of hyperplanes.

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1. INTRODUCTION

A local system on $\mathbb{P}^1 \setminus S$, S being a finite subset, is said to be *rigid* if it is determined uniquely up to isomorphisms by the local monodromies. Katz [10] defined the *index of rigidity*, which gives a criterion for the rigidity. The index of rigidity takes a value in even integers up to 2 for irreducible local systems, and it takes the maximal value 2 if and only if the (irreducible) local system is rigid.

We can extend the notion of the rigidity to local systems on $\mathbb{P}^m \setminus S$, where S is a hypersurface. A local system on $\mathbb{P}^m \setminus S$ is identified with an anti-homomorphism

$$\rho : \pi_1(\mathbb{P}^m \setminus S, b) \to \mathrm{GL}(n, \mathbb{C}).$$

Let

$$S = \bigcup_j S_j$$

be the irreducible decomposition of S. Fix any irreducible component S_j. If $\gamma, \gamma' \in \pi_1(\mathbb{P}^m \setminus S, b)$ both encircle S_j once in the positive direction and do not encircle the other S_k's ($k \neq j$), then γ and γ' are conjugate to each other in $\pi_1(\mathbb{P}^m \setminus S, b)$. Then we can define the *local monodromy* of ρ at S_j by the conjugacy class $[\rho(\gamma)]$. The definition of the rigidity is completely similar to the one dimensional case.

The anti-homomorphism ρ is said to be *rigid* if it is uniquely determined up to isomorphisms by the local monodromies.

On the other hand, it seems difficult to define the index of rigidity for higher the dimensional case, because the rigidity depends on the topology of the hypersurface S. For the one dimensional case, the topology of S depends only on the number $\#S$ of the points of S. While for higher dimensional cases, there is no such simple topological invariant, so that the fundamental groups may have various presentations. Nevertheless, if we fix one hypersurface, it may be possible to define the index of rigidity for the hypersurface. Bearing this problem in mind, in this paper we study the rigidity of monodromy representations of holonomic systems satisfied by Appell's hypergeometric functions.

Let a, a', b, b', c, c' be complex numbers satisfying $c, c' \notin \mathbb{Z}_{\leq 0}$. Appell's hypergeometric functions F_1, F_2, F_3 and F_4 are defined by the power series

$$\begin{aligned} F_1(a,b,b',c;x,y) &= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}m!n!}x^m y^n, \\ F_2(a,b,b',c,c';x,y) &= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n m!n!}x^m y^n, \\ F_3(a,a',b,b',c;x,y) &= \sum_{m,n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}m!n!}x^m y^n, \\ F_4(a,b,c,c';x,y) &= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c')_n m!n!}x^m y^n, \end{aligned} \tag{1.1}$$

where

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)}.$$

These series converge in a neighborhood of $(x, y) = (0, 0)$. We refer to Appell-Kampé de Fériet [1] and Kimura [11] for basic properties of these functions.

Appell's hypergeometric functions F_j $(j = 1, 2, 3, 4)$ satisfy holonomic systems on $\mathbb{P}^2$ with singular loci $S^{(j)}$, which are given by

$$\begin{aligned} S^{(1)} &= \{xy(x-1)(y-1)(x-y) = 0\} \cup L_\infty, \\ S^{(2)} &= \{xy(x-1)(y-1)(x+y-1) = 0\} \cup L_\infty, \\ S^{(3)} &= \{xy(x-1)(y-1)(xy-x-y) = 0\} \cup L_\infty, \\ S^{(4)} &= \{xy((x-y)^2 - 2(x+y)+1) = 0\} \cup L_\infty, \end{aligned}$$

where L_∞ denotes the line at infinity. The singular loci $S^{(2)}, S^{(3)}$ and $S^{(4)}$ can be transformed to $S^{(1)}$ by the following variable changes $(x, y) \to (x', y')$. For $S^{(2)}$, we take

$$x' = x, \quad y' = 1 - y;$$

for $S^{(3)}$, we take

$$x' = \frac{x-1}{x}, \quad y' = \frac{1}{y};$$

and for $S^{(4)}$, we take

$$x' = xy, \quad y' = (1-x)(1-y).$$

The last one is given by Kato [8,9]; we note that we find a similar transformation in Bailey [2] (the formula (1) on page 81) in expressing F_4 for some reducible case. This formula is originally due to Watson [14, §11.6]. We also find the same transformation as Bailey in Kimura [11, §10], where an integral representation of Euler type for F_4 is given.

Moreover, we see that the holonomic system for F_3 is transformed to the holonomic system for F_2 by these variable changes.

Thus, in this paper, we consider the holonomic systems for F_1, F_2 and F_4 which have the same singular locus $S^{(1)}$. Then, for $j = 1, 2, 4$, the monodromy representation of the holonomic system for F_j is an anti-homomorphism

$$\rho_j : \pi_1(\mathbb{P}^2 \setminus S^{(1)}) \to \mathrm{GL}(n_j, \mathbb{C}),$$

where $n_1 = 3$ and $n_2 = n_4 = 4$. Our purpose is to show that these monodromy representations ρ_j are rigid.

In [6] we have shown that the monodromy representation of the holonomic system for F_4 with the original singular locus $S^{(4)}$ is rigid. We have also shown in [5] the rigidity of the monodromy representations of the uniformization systems obtained by Kato-Sekiguchi.

2. THE FUNDAMENTAL GROUP

We study the fundamental group $\pi_1(\mathbb{P}^2 \setminus S^{(1)})$. We set

$$S_1 = \{x = 0\}, \quad S_2 = \{y = 1\}, \quad S_3 = \{x = y\}, \quad S_4 = \{x = 1\}, \quad S_5 = \{y = 0\}.$$

Then we have the irreducible decomposition

$$S^{(1)} = \bigcup_{j=1}^{5} S_j \cup L_\infty.$$

In order to give generators of the fundamental group, we take a base point and a reference plane. As the base point we take $b = (2, -1/2)$, and as the reference plane we take

$$F : x + y = \frac{3}{2}.$$

We regard F as a complex line with the coordinate x. Then the intersections of F and the irreducible components of $S^{(1)}$ have the following coordinates.

$$\begin{aligned} F \cap S_1 : &\quad x = 0, \\ F \cap S_2 : &\quad x = \tfrac{1}{2}, \\ F \cap S_3 : &\quad x = \tfrac{3}{4}, \\ F \cap S_4 : &\quad x = 1, \\ F \cap S_5 : &\quad x = \tfrac{3}{2}. \end{aligned}$$

Let $\gamma_1, \gamma_2, \ldots, \gamma_5$ be loops in $F \setminus S^{(1)}$ with base point b which encircle $x = 0, 1/2, 3/4, 1, 3/2$, respectively, once in the positive direction such that

$$\gamma_0 = (\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5)^{-1}$$

becomes a loop which encircles ∞ once in the positive direction. Thus the loops $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ and γ_5 encircle the irreducible components S_1, S_2, S_3, S_4 and S_5, respectively, once in the positive direction. Applying the Zariski-van Kampen theorem [7], we get the following.

Proposition 2.1. *The fundamental group $\pi_1(\mathbb{P}^2 \setminus S^{(1)}, b)$ has the following presentation:*

$$\pi_1(\mathbb{P}^2 \setminus S^{(1)}, b) = \left\langle \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 \;\middle|\; \begin{array}{l} \gamma_1\gamma_2 = \gamma_2\gamma_1, \ \gamma_4\gamma_5 = \gamma_5\gamma_4, \\ \gamma_1\gamma_3\gamma_5 = \gamma_3\gamma_5\gamma_1 = \gamma_5\gamma_1\gamma_3, \\ \gamma_2\gamma_3\gamma_4 = \gamma_3\gamma_4\gamma_2 = \gamma_4\gamma_2\gamma_3 \end{array} \right\rangle.$$

Since $S^{(1)}$ can be regarded as a real arrangement of lines, we can also get the above proposition by applying Randell's result [13] (see also Theorem 5.57 in [12]).

We set

$$S_0^{(1)} = S^{(1)}|_{\mathbb{C}^2}$$

and

$$S^{(1)\prime} = S_0^{(1)} \cup L_\infty^x \cup L_\infty^y \subset \mathbb{P}^1 \times \mathbb{P}^1,$$

where $L_\infty^x = \{x = \infty\}$ and $L_\infty^y = \{y = \infty\}$. Then we have the identity of the sets

$$\mathbb{P}^2 \setminus S^{(1)} = \mathbb{P}^1 \times \mathbb{P}^1 \setminus S^{(1)\prime} = \mathbb{C}^2 \setminus S_0^{(1)},$$

which induces the isomorphisms

$$\pi_1(\mathbb{P}^2 \setminus S^{(1)}, b) \cong \pi_1(\mathbb{P}^1 \times \mathbb{P}^1 \setminus S^{(1)\prime}, b) \cong \pi_1(\mathbb{C}^2 \setminus S_0^{(1)}, b).$$

We see that, as elements in $\pi_1(\mathbb{P}^1 \times \mathbb{P}^1 \setminus S^{(1)\prime}, b)$, the loop

$$\gamma_1\gamma_3\gamma_4$$

encircles L^x_∞ once in the negative direction, and the loop

$$\gamma_2\gamma_3\gamma_5$$

encircles L^y_∞ once in the negative direction.

Let

$$\rho : \pi_1(\mathbb{P}^2 \setminus S^{(1)}, b) \to \mathrm{GL}(n, \mathbb{C})$$

be an anti-homomorphism, and set

$$M_j = \rho(\gamma_j)$$

for $0 \leq j \leq 5$. Thanks to Proposition 2.1, we see that the tuple $(M_1, M_2, M_3, M_4, M_5)$ determines ρ, and for the tuple, the relations

$$\begin{aligned} &M_1M_2 = M_2M_1,\ \ M_4M_5 = M_5M_4, \\ &M_5M_3M_1 = M_3M_1M_5 = M_1M_5M_3, \\ &M_4M_3M_2 = M_3M_2M_4 = M_2M_4M_3 \end{aligned} \tag{2.1}$$

hold. By the definition, the conjugacy classes

$$[M_1],\ [M_2],\ [M_3],\ [M_4],\ [M_5],\ [M_0]$$

are the local monodromies at $S_1, S_2, S_3, S_4, S_5, L_\infty$, respectively. Moreover, the conjugacy classes

$$[(M_4M_3M_1)^{-1}],\ [(M_5M_3M_2)^{-1}]$$

are the local monodromies at $L^x_\infty,\ L^y_\infty$, respectively.

3. PFAFFIAN SYSTEMS

3.1. F_1

Let

$$z(x, y) = F_1(a, b, b', c; x, y)$$

be Appell's hypergeometric series F_1, and set

$$u = {}^t(z, xz_x, yz_y).$$

Then, from (1.1), we obtain the Pfaffian system for u of the form

$$du = \left(A_1\frac{dx}{x} + A_2\frac{dy}{y-1} + A_3\frac{d(x-y)}{x-y} + A_4\frac{dx}{x-1} + A_5\frac{dy}{y}\right)u, \tag{3.1}$$

where

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & b'-c+1 & 0 \\ 0 & -b' & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -ab' & -b' & -a-b'+c-1 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -b' & b \\ 0 & b' & -b \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 & 0 \\ -ab & -a-b+c-1 & b \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.2}$$
$$A_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -b \\ 0 & 0 & b-c+1 \end{pmatrix}.$$

Proposition 3.1. *For the solutions of the Pfaffian system* (3.1)*, the following holds:*

(i) *If*

$$b'-c, c-a-b', -b-b', c-a-b, b-c \notin \mathbb{Z},$$

there is no logarithmic solution around the singular locus $S_0^{(1)}$.

(ii) *If* $a-b-b' \notin \mathbb{Z}$*, there is no logarithmic solution around* L_∞.

(iii) *If* $a-b, a-b' \notin \mathbb{Z}$*, there is no logarithmic solution around* L_∞^x *and* L_∞^y.

Proof. Since the eigenvalues of A_1 are $0, 0, b'-c+1$, there is no integral difference among the distinct eigenvalues if $b'-c \notin \mathbb{Z}$. Then in this case, the local monodromy at $S_1 = \{x = 0\}$ is given by $e^{2\pi\sqrt{-1}A_1}$, which is semi-simple. This implies that there is no logarithmic solution around S_1.

The other assertions can be shown similarly. We have only to notice that the residue matrices around $L_\infty, L_\infty^x, L_\infty^y$ are $-(A_1+A_2+A_3+A_4+A_5)$, $-(A_1+A_3+A_4)$, $-(A_2+A_3+A_5)$, respectively, and their eigenvalues are

$$\begin{aligned} -(A_1+A_2+A_3+A_4+A_5) &: \quad a, a, b+b', \\ -(A_1+A_3+A_4) &: \quad a, b, b, \\ -(A_2+A_3+A_5) &: \quad a, b', b'. \end{aligned}$$
□

For later use, we collect the conditions in the proposition:

$$b'-c, c-a-b', -b-b', c-a-b, b-c, a-b-b', a-b, a-b' \notin \mathbb{Z}. \tag{3.3}$$

Then, if (3.3) holds, all the local monodromies are semi-simple. By (3.2), we can see the eigenvalues of the local monodromies:

$$\begin{aligned} S_1 &: 1, 1, e(b'-c), \\ S_2 &: 1, 1, e(-a-b'+c), \\ S_3 &: 1, 1, e(-b-b'), \\ S_4 &: 1, 1, e(-a-b+c), \\ S_5 &: 1, 1, e(b-c), \\ L_\infty &: e(a), e(a), e(b+b'), \\ L_\infty^x &: e(a), e(b), e(b), \\ L_\infty^y &: e(a), e(b'), e(b'), \end{aligned} \tag{3.4}$$

where we use the notation

$$e(\alpha) = e^{2\pi\sqrt{-1}\alpha}.$$

The *spectral type* means the partition which describes the multiplicities of the eigenvalues for a semi-simple matrix. Then the collection of the spectral types of the local monodromies at $S_1, S_2, S_3, S_4, S_5; L_\infty; L_\infty^x, L_\infty^y$ is

$$(21, 21, 21, 21, 21; 21; 21, 21),$$

which we call the *full spectral type* of the Pfaffian system (3.1).

3.2. F_2

Let

$$z(x, y) = F_2(a, b, b', c, c'; x, 1-y)$$

be the transformed F_2, and set

$$u(x, y) = {}^t(z, xz_x, yz_y, xyz_{xy}).$$

Then we obtain from (1.1) the Pfaffian system (3.1) for this u with

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1-c & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1-c \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1-c' & 0 \\ 0 & 0 & 0 & 1-c' \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -abb' & b'(c-a-b-1) & b(c'-a-b'-1) & c+c'-a-b-b'-2 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -ab & c-a-b-1 & -b & -1 \\ 0 & 0 & 0 & 0 \\ abb' & b'(a+b-c+1) & bb' & b' \end{pmatrix},$$

$$A_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -ab' & -b' & c'-a-b'-1 & -1 \\ abb' & bb' & b(a+b'-c'+1) & b \end{pmatrix}.$$

In a similar way as Proposition 3.1, we obtain the following assertion.

Proposition 3.2. *If*

$$\begin{aligned} &c,\ c',\ c+c'-a-b-b',\ c-a-b+b',\ c'-a+b-b', \\ &a-b-b',\ a-b,\ a-b',\ a-b-c',\ a-b'-c \notin \mathbb{Z}, \end{aligned} \tag{3.5}$$

there is no logarithmic solution around the singular locus $S^{(1)}$ and $S^{(1)'}$ of the Pfaffian system for u. In this case, all the local monodromies are semi-simple, and the Pfaffian system has the full spectral type

$$(22, 22, 31, 31, 31; 31; 211, 211).$$

The eigenvalues of the local monodromies are

$$\begin{aligned}
S_1 &: 1, 1, e(-c), e(-c), \\
S_2 &: 1, 1, e(-c'), e(-c'), \\
S_3 &: 1, 1, 1, e(c + c' - a - b - b'), \\
S_4 &: 1, 1, 1, e(c - a - b + b'), \\
S_5 &: 1, 1, 1, e(c' - a + b - b'), \\
L_\infty &: e(a), e(a), e(a), e(b + b'), \\
L_\infty^x &: e(a), e(b), e(b), e(a - c'), \\
L_\infty^y &: e(a), e(b'), e(b'), e(a - c).
\end{aligned}$$

3.3. F_4

Let

$$z(x, y) = F_4(a, b, c, c'; xy, (1 - x)(1 - y))$$

be the transformed F_4, and set

$$u(x, y) = {}^t\Big(z, xz_x, yz_y, xy\Big(z_{xy} + \epsilon\frac{z_x - z_y}{x - y}\Big)\Big),$$

where

$$\epsilon = c + c' - a - b - 1.$$

Then, as is shown in [9], we obtain from (1.1) the Pfaffian system (3.1) for this u with

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 - c & 0 & 0 \\ 0 & \epsilon & 0 & 1 \\ 0 & 0 & 0 & 1 - c \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -ab & \epsilon & -c' & 0 \\ 0 & -(a + \epsilon)(b + \epsilon) & 0 & -c' \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon & -\epsilon & 0 \\ 0 & -\epsilon & \epsilon & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -ab & -c' & \epsilon & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(a + \epsilon)(b + \epsilon) & -c' \end{pmatrix},$$

$$A_5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & \epsilon & 1 \\ 0 & 0 & 1 - c & 0 \\ 0 & 0 & 0 & 1 - c \end{pmatrix}.$$

In a similar way as Proposition 3.1, we obtain the following assertion.

Proposition 3.3. *If*

$$c, c', 2\epsilon, 2a - c - c', 2b - c - c', a - b \notin \mathbb{Z}, \tag{3.6}$$

there is no logarithmic solution around the singular locus $S^{(1)}$ and $S^{(1)'}$ of the Pfaffian system for u. In this case, all the local monodromies are semi-simple, and the Pfaffian system has the full spectral type

$$(22, 22, 31, 22, 22; 211; 22, 22).$$

The eigenvalues of the local monodromies are

$$\begin{aligned}
S_1 &: 1, 1, e(-c), e(-c),\\
S_2 &: 1, 1, e(-c'), e(-c'),\\
S_3 &: 1, 1, 1, e(2(c + c' - a - b)),\\
S_4 &: 1, 1, e(-c'), e(-c'),\\
S_5 &: 1, 1, e(-c), e(-c),\\
L_\infty &: e(c + c'), e(c + c'), e(2a), e(2b),\\
L_\infty^x &: e(a), e(a), e(b), e(b),\\
L_\infty^y &: e(a), e(a), e(b), e(b).
\end{aligned}$$

4. FORMULATION OF THE PROBLEM

As stated in the Introduction, a local system (a representation) is said to be rigid if it is determined by the local monodromies uniquely up to isomorphisms. We are interested in the rigidity of the monodromy representations of the holonomic systems for Appell's hypergeometric functions. If one wants to know the rigidity of a representation ρ, one may look for all representation classes $[\rho']$ with the same local monodromies as ρ. Thus the problem is to determine all representation classes with prescribed local monodromies. In this paper, we slightly extend the problem. In order to explain the motivation, we note the following three facts.

First, the index of rigidity, which gives a criterion for the rigidity of local systems on $\mathbb{P}^1 \setminus S$, is determined by the rank of the local system, the number $\#S$ of the points in S and the spectral types of the local monodromies. Then we expect that, also in higher dimensional cases, the rigidity is determined by the spectral types of the local monodromies.

Second, we see that the eigenvalues of the local monodromies of each ρ_j satisfy some relations which cannot be directly derived from the relations (2.1). For example, if we denote the eigenvalues of the local monodromy at S_j of the monodromy representation ρ_1 for F_1 by $1, 1, e_j$ $(1 \leq j \leq 5)$, we have

$$e_1 e_2 = e_4 e_5.$$

We are interested in how these relations are derived. Then we do not assume these relations a priori, and look for conjugacy classes of monodromy representations with prescribed spectral types.

Third, for any representation ρ of $\pi_1(\mathbb{P}^2 \backslash S^{(1)})$ and any tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_5) \in (\mathbb{C}^\times)^5$, we get a new representation $\alpha\rho$ defined by

$$(\alpha\rho)(\gamma_j) = \alpha_j \rho(\gamma_j) \quad (1 \le j \le 5),$$

which we call the *multiplication* of ρ by α. It seems natural to consider the representations modulo multiplications.

Thus we formulate our problem as follows.

Problem 4.1. Determine all irreducible representation classes modulo multiplications with prescribed spectral types of local monodromies.

5. RIGIDITY OF THE MONODROMY OF F_1

As we have seen in Section 3.1, the monodromy representation of the Pfaffian system for F_1 is an anti-homomorphism

$$\rho_1 : \pi_1(\mathbb{P}^2 \setminus S^{(1)}, b) \to \mathrm{GL}(3, \mathbb{C})$$

whose spectral type is

$$(21, 21, 21, 21, 21; 21) \tag{5.1}$$

if (3.3) holds. Here we call the collection of the spectral types of the local monodromies at $S_1, S_2, S_3, S_4, S_5; L_\infty$ the *spectral type* of ρ_1. We shall determine the representation classes $[\rho]$ modulo multiplications with the spectral type (5.1).

Let $\rho = (M_1, M_2, M_3, M_4, M_5)$ be an irreducible representation of $\pi_1(\mathbb{P}^2 \setminus S^{(1)}, b)$ with the spectral type (5.1). By a multiplication, we can send the multiple eigenvalue of each M_j to 1, and hence we may assume

$$M_j \sim \begin{pmatrix} 1 & & \\ & 1 & \\ & & e_j \end{pmatrix} \quad (1 \le j \le 5),$$

where $e_j \neq 0, 1$. Since M_1 and M_2 commute, we can send them into diagonal matrices simultaneously by a similar transformation. If we set

$$M_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & e_1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & e_2 \end{pmatrix},$$

we get only reducible representations, which can be shown in a similar way as in the proof of Theorem 1.1 in [6]. Then we set

$$M_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & e_1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & & \\ & e_2 & \\ & & 1 \end{pmatrix}. \tag{5.2}$$

For $j = 3, 4, 5$, we can set

$$M_j = I_3 + \begin{pmatrix} x_j \\ y_j \\ z_j \end{pmatrix} \begin{pmatrix} 1 & p_j & q_j \end{pmatrix},$$

where

$$x_j = e_j - 1 - p_j y_j - q_j z_j.$$

By a similar transformation which keeps M_1, M_2 invariant, we can send $p_3 = q_3 = 1$.

We put these expressions into the relations (2.1). Set

$$\begin{aligned} R_1 &= M_4M_5 - M_5M_4, \\ R_2 &= M_5M_3M_1 - M_3M_1M_5, \\ R_3 &= M_3M_1M_5 - M_1M_5M_3, \\ R_4 &= M_1M_5M_3 - M_5M_3M_1, \\ R_5 &= M_4M_3M_2 - M_3M_2M_4, \\ R_6 &= M_3M_2M_4 - M_2M_4M_3, \\ R_7 &= M_2M_4M_3 - M_4M_3M_2. \end{aligned} \tag{5.3}$$

We denote the (i, j) entry of a matrix A by $A[i, j]$. We have

$$R_2[2, 1] - R_2[2, 2] = (p_5 - 1)y_3(-1 + e_5 + y_5 - p_5y_5 + e_1z_5 - q_5z_5).$$

If $p_5 \neq 1$, $y_3 = 0$ or $-1+e_5+y_5-p_5y_5+e_1z_5-q_5z_5 = 0$ holds, each of which gives only a reducible representation. Then we have $p_5 = 1$. In a similar way, we get $q_4 = 1$. We can show that $z_3z_5 = 0$ yields no irreducible representation. Then we have $z_3z_5 \neq 0$, and in this case q_5, y_3 are determined by $R_4[3, 1], R_4[2, 3]$, respectively in this order. Now we have

$$\begin{aligned} &R_4[1, 3] \cdot y_5(z_3 + {z_3}^2 + e_3z_5) + (R_2[2, 1] - R_2[2, 3])(e_1 - 1)z_3z_5 \\ &\quad = \frac{(e_1 - 1)y_5(z_3 + {z_3}^2 + e_3z_5)(e_5{z_3}^2 - e_1e_3{z_5}^2)}{z_3z_5}. \end{aligned}$$

We can show that $y_5(z_3 + {z_3}^2 + e_3z_5) = 0$ yields no irreducible representation. Then we come to the relation

$$e_5{z_3}^2 - e_1e_3{z_5}^2 = 0.$$

Now we set

$$e_1 = {f_1}^2, \quad e_3 = {f_3}^2, \quad e_5 = {f_5}^2,$$

which makes the above relation

$$(f_5z_3 + f_1f_3z_5)(f_5z_3 - f_1f_3z_5) = 0.$$

Then we have

$$z_5 = -\frac{f_5z_3}{f_1f_3}, \quad \frac{f_5z_3}{f_1f_3}.$$

Here we choose the first value. Then by using the relations (5.3), we can determine remaining parameters, and come to the relation

$$-e_2e_4 + {e_2}^2{f_3}^2 + 2e_2e_4p_4 - 2e_2{f_3}^2p_4 - e_2e_4{p_4}^2 + {f_3}^2{p_4}^2 = 0.$$

By setting

$$e_2 = {f_2}^2, \quad e_4 = {f_4}^2,$$

we reduce this relation to

$$({f_2}^2f_3 + f_2f_4 - f_3p_4 - f_2f_4p_4)({f_2}^2f_3 - f_2f_4 - f_3p_4 + f_2f_4p_4) = 0.$$

Then we have

$$p_4 = \frac{f_2(f_2f_3 - f_4)}{f_3 - f_2f_4}, \quad \frac{f_2(f_2f_3 + f_4)}{f_3 + f_2f_4}.$$

We choose the first value. Then all parameters are written in terms of f_j $(1 \le j \le 5)$. Put them into R_1. Then, for example, we have

$$R_1[3,3] = -\frac{(f_3 - f_2f_4)(f_3 - f_1f_5)(f_4 - f_1f_2f_5)(f_1 - f_3f_5)(f_1f_2 - f_4f_5)}{f_1f_2f_3f_5(-1 + f_1)(1 + f_1)(f_2f_3 - f_4)}.$$

By examining all possibilities for $R_1[3,3] = 0$, we conclude that only the relation

$$f_1f_2 - f_4f_5 = 0$$

gives an irreducible representation. For the other choices of (z_5, p_4), we have the relations

$$f_1f_2 \pm f_4f_5 = 0,$$

and irreducible representations.

In this way, we obtain two representations $\rho^{(1)} = (M_1, M_2, M_3^{(1)}, M_4^{(1)}, M_5^{(1)})$ and $\rho^{(2)} = (M_1, M_2, M_3^{(2)}, M_4^{(2)}, M_5^{(2)})$, where M_1, M_2 are given in (5.2). The explicit forms of $M_j^{(k)}$ $(j = 3, 4, 5)$ are

$$\begin{aligned}
&M_3^{(1)} = I_3 + \\
&\begin{pmatrix}
\frac{(1-e_1e_2)(f_2f_3-f_4)(f_2-f_3f_4)}{(e_1-1)(e_2-1)f_2f_4} & \frac{(1-e_1e_2)(f_2f_3-f_4)(f_2-f_3f_4)}{(e_1-1)(e_2-1)f_2f_4} & \frac{(1-e_1e_2)(f_2f_3-f_4)(f_2-f_3f_4)}{(e_1-1)(e_2-1)f_2f_4} \\
-\frac{(f_2f_4-f_3)(f_2-f_3f_4)}{(e_2-1)f_4} & -\frac{(f_2f_4-f_3)(f_2-f_3f_4)}{(e_2-1)f_4} & -\frac{(f_2f_4-f_3)(f_2-f_3f_4)}{(e_2-1)f_4} \\
\frac{(f_2f_3-f_4)(e_1f_2-f_3f_4)}{(e_1-1)f_2f_4} & \frac{(f_2f_3-f_4)(e_1f_2-f_3f_4)}{(e_1-1)f_2f_4} & \frac{(f_2f_3-f_4)(e_1f_2-f_3f_4)}{(e_1-1)f_2f_4}
\end{pmatrix}, \\
&M_4^{(1)} = I_3 + \\
&\begin{pmatrix}
\frac{(1-e_1e_2)(f_2f_4-f_3)(f_2-f_3f_4)}{(e_1-1)(e_2-1)f_2f_3} & \frac{(e_1e_2-1)(f_2f_3-f_4)(f_2-f_3f_4)}{(e_1-1)(e_2-1)f_3} & \frac{(1-e_1e_2)(f_2f_4-f_3)(f_2-f_3f_4)}{(e_1-1)(e_2-1)f_2f_3} \\
\frac{(f_2f_4-f_3)(f_2-f_3f_4)}{(e_2-1)f_2f_3} & -\frac{(f_2f_3-f_4)(f_2-f_3f_4)}{(e_2-1)f_3} & \frac{(f_2f_4-f_3)(f_2-f_3f_4)}{(e_2-1)f_2f_3} \\
\frac{(f_2f_4-f_3)(e_1f_2-f_3f_4)}{(e_1-1)f_2f_3} & -\frac{(f_2f_3-f_4)(e_1f_2-f_3f_4)}{(e_1-1)f_3} & \frac{(f_2f_4-f_3)(e_1f_2-f_3f_4)}{(e_1-1)f_2f_3}
\end{pmatrix}, \qquad (5.4) \\
&M_5^{(1)} = I_3 + \\
&\begin{pmatrix}
\frac{(e_1e_2-1)(f_2f_3-f_4)(e_1f_2-f_3f_4)}{(e_1-1)(e_2-1)f_3e_4} & \frac{(e_1e_2-1)(f_2f_3-f_4)(e_1f_2-f_3f_4)}{(e_1-1)(e_2-1)f_3e_4} & \frac{e_1(e_1e_2-1)(f_2f_3-f_4)(f_2-f_3f_4)}{(e_1-1)(e_2-1)f_3e_4} \\
\frac{f_2(f_2f_4-f_3)(e_1f_2-f_3f_4)}{(e_2-1)f_3e_4} & \frac{f_2(f_2f_4-f_3)(e_1f_2-f_3f_4)}{(e_2-1)f_3e_4} & \frac{e_1f_2(f_2f_4-f_3)(f_2-f_3f_4)}{(e_2-1)f_3e_4} \\
\frac{-(f_2f_3-f_4)(e_1f_2-f_3f_4)}{(e_1-1)f_3e_4} & -\frac{(f_2f_3-f_4)(e_1f_2-f_3f_4)}{(e_1-1)f_3e_4} & -\frac{e_1(f_2f_3-f_4)(f_2-f_3f_4)}{(e_1-1)f_3e_4}
\end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
&M_3^{(2)} = I_3 + \\
&\begin{pmatrix} \frac{(1-e_1e_2)(f_2f_3+f_4)(f_2+f_3f_4)}{(e_1-1)(e_2-1)f_2f_4} & \frac{(1-e_1e_2)(f_2f_3+f_4)(f_2+f_3f_4)}{(e_1-1)(e_2-1)f_2f_4} & \frac{(1-e_1e_2)(f_2f_3+f_4)(f_2+f_3f_4)}{(e_1-1)(e_2-1)f_2f_4} \\ -\frac{(f_2f_4+f_3)(f_2+f_3f_4)}{(e_2-1)f_4} & -\frac{(f_2f_4+f_3)(f_2+f_3f_4)}{(e_2-1)f_4} & -\frac{(f_2f_4+f_3)(f_2+f_3f_4)}{(e_2-1)f_4} \\ \frac{(f_2f_3+f_4)(e_1f_2+f_3f_4)}{(e_1-1)f_2f_4} & \frac{(f_2f_3+f_4)(e_1f_2+f_3f_4)}{(e_1-1)f_2f_4} & \frac{(f_2f_3+f_4)(e_1f_2+f_3f_4)}{(e_1-1)f_2f_4} \end{pmatrix}, \\
&M_4^{(2)} = I_3 + \\
&\begin{pmatrix} \frac{(1-e_1e_2)(f_2f_4+f_3)(f_2+f_3f_4)}{(e_1-1)(e_2-1)f_2f_3} & \frac{(e_1e_2-1)(f_2f_3+f_4)(f_2+f_3f_4)}{(e_1-1)(e_2-1)f_3} & \frac{(1-e_1e_2)(f_2f_4+f_3)(f_2+f_3f_4)}{(e_1-1)(e_2-1)f_2f_3} \\ \frac{(f_2f_4+f_3)(f_2+f_3f_4)}{(e_2-1)f_2f_3} & -\frac{(f_2f_3+f_4)(f_2+f_3f_4)}{(e_2-1)f_3} & \frac{(f_2f_4+f_3)(f_2+f_3f_4)}{(e_2-1)f_2f_3} \\ \frac{(f_2f_4+f_3)(e_1f_2+f_3f_4)}{(e_1-1)f_2f_3} & -\frac{(f_2f_3+f_4)(e_1f_2+f_3f_4)}{(e_1-1)f_3} & \frac{(f_2f_4+f_3)(e_1f_2+f_3f_4)}{(e_1-1)f_2f_3} \end{pmatrix}, \\
&M_5^{(2)} = I_3 + \\
&\begin{pmatrix} \frac{(e_1e_2-1)(f_2f_3+f_4)(e_1f_2+f_3f_4)}{(e_1-1)(e_2-1)f_3e_4} & \frac{(e_1e_2-1)(f_2f_3+f_4)(e_1f_2+f_3f_4)}{(e_1-1)(e_2-1)f_3e_4} & \frac{e_1(e_1e_2-1)(f_2f_3+f_4)(f_2+f_3f_4)}{(e_1-1)(e_2-1)f_3e_4} \\ \frac{f_2(f_2f_4+f_3)(e_1f_2+f_3f_4)}{(e_2-1)f_3e_4} & \frac{f_2(f_2f_4+f_3)(e_1f_2+f_3f_4)}{(e_2-1)f_3e_4} & \frac{e_1f_2(f_2f_4+f_3)(f_2+f_3f_4)}{(e_2-1)f_3e_4} \\ \frac{-(f_2f_3+f_4)(e_1f_2+f_3f_4)}{(e_1-1)f_3e_4} & -\frac{(f_2f_3+f_4)(e_1f_2+f_3f_4)}{(e_1-1)f_3e_4} & -\frac{e_1(f_2f_3+f_4)(f_2+f_3f_4)}{(e_1-1)f_3e_4} \end{pmatrix}.
\end{aligned} \tag{5.5}$$

By using these explicit expressions, we can calculate the local monodromies at infinity in $\mathbb{P}^2$ and in $\mathbb{P}^1 \times \mathbb{P}^1$. For $\rho^{(1)}$, the local monodromy at $L_\infty, L_\infty^x, L_\infty^y$ are given by

$$\operatorname{diag}\left[\frac{1}{e_1e_2}, \frac{1}{e_1e_2}, \frac{1}{e_3}\right], \quad \operatorname{diag}\left[\frac{1}{e_1e_2}, \frac{f_2}{f_3f_4}, \frac{f_2}{f_3f_4}\right], \quad \operatorname{diag}\left[\frac{1}{e_1e_2}, \frac{f_4}{f_2f_3}, \frac{f_4}{f_2f_3}\right],$$

respectively. For $\rho^{(2)}$, the local monodromy at $L_\infty, L_\infty^x, L_\infty^y$ is given by

$$\operatorname{diag}\left[\frac{1}{e_1e_2}, \frac{1}{e_1e_2}, \frac{1}{e_3}\right], \quad \operatorname{diag}\left[\frac{1}{e_1e_2}, -\frac{f_2}{f_3f_4}, -\frac{f_2}{f_3f_4}\right], \quad \operatorname{diag}\left[\frac{1}{e_1e_2}, -\frac{f_4}{f_2f_3}, -\frac{f_4}{f_2f_3}\right],$$

respectively.

A similar transformation which keeps M_1, M_2 invariant is a transformation by a diagonal matrix. Hence we see that $\rho^{(1)}$ and $\rho^{(2)}$ are not isomorphic.

Since f_j is a square root of e_j, we have a Galois group $\mathbb{Z}_2$ generated by

$$\sigma_j : f_j \mapsto -f_j$$

for $j = 2, 3, 4$. Set

$$G_1 = \langle \sigma_2, \sigma_3, \sigma_4 \rangle = (\mathbb{Z}_2)^3. \tag{5.6}$$

Then we see that G_1 acts on $\{\rho^{(1)}, \rho^{(2)}\}$ as

$$\sigma_j(\rho^{(1)}) = \rho^{(2)}, \quad \sigma_j(\rho^{(2)}) = \rho^{(1)} \quad (j = 2, 3, 4). \tag{5.7}$$

Thus we get the following assertion.

Theorem 5.1. *Let*

$$\rho : \pi_1(\mathbb{P}^2 \setminus S^{(1)}, b) \to \mathrm{GL}(3, \mathbb{C})$$

be an irreducible anti-homomorphism with spectral type $(2, 1)$ *at each irreducible component* S_j $(1 \le j \le 5)$*. We assume that the local monodromy at each* S_j *is given by* $\mathrm{diag}[1, 1, e_j]$ *with* $e_j \in \mathbb{C} \setminus \{0, 1\}$ $(1 \le j \le 5)$*. Then we have*

$$e_1 e_2 = e_4 e_5,$$

and ρ *is isomorphic to one of* $\rho^{(1)} = (M_1, M_2, M_3^{(1)}, M_4^{(1)}, M_5^{(1)})$ *and* $\rho^{(2)} = (M_1, M_2, M_3^{(2)}, M_4^{(2)}, M_5^{(2)})$ *given by* (5.2)*,* (5.4)*,* (5.5)*, where* f_2, f_3, f_4 *are square roots of* e_2, e_3, e_4*, respectively. The local monodromy of* ρ *at* L_∞ *is given by*

$$\mathrm{diag}\left[\frac{1}{e_1 e_2}, \frac{1}{e_1 e_2}, \frac{1}{e_3}\right].$$

The Galois group G_1 *given by* (5.6) *acts on the representations* $\{\rho^{(1)}, \rho^{(2)}\}$ *as* (5.7)*.*

Since $\rho^{(1)}$ and $\rho^{(2)}$ have the same local monodromy at $S_1, S_2, \ldots, S_5, L_\infty$, we cannot distinguish them by the local monodromies. However, if we regard them as local systems on $\mathbb{P}^1 \times \mathbb{P}^1 \setminus S^{(1)\prime}$, we can distinguish them by looking at the local monodromy at L^x_∞ or at L^y_∞.

6. RIGIDITY OF THE MONODROMY OF F_2

By Proposition 3.2, the monodromy representation of the Pfaffian system for F_2 is an anti-homomorphism

$$\rho_2 : \pi_1(\mathbb{P}^2 \setminus S^{(1)}, b) \to \mathrm{GL}(4, \mathbb{C})$$

whose spectral type is

$$(22, 22, 31, 31, 31; 31) \tag{6.1}$$

if (3.5) holds. We shall determine the representation classes $[\rho]$ modulo multiplication with the spectral type (6.1).

Let $\rho = (M_1, M_2, M_3, M_4, M_5)$ be an irreducible representation of $\pi_1(\mathbb{P}^2 \setminus S^{(1)}, b)$ with the spectral type (6.1). By a multiplication, we can send (one of) the multiple eigenvalues of each M_j to 1, and hence we may assume

$$M_j \sim \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & e_j & \\ & & & e_j \end{pmatrix} \quad (j = 1, 2),$$

$$M_j \sim \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & e_j \end{pmatrix} \quad (j = 3, 4, 5),$$

where $e_j \neq 0, 1$. Since M_1 and M_2 commute, we can send them into diagonal matrices simultaneously by a similar transformation. In order to get an irreducible representation, we set

$$M_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & e_1 & \\ & & & e_1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & & & \\ & e_2 & & \\ & & 1 & \\ & & & e_2 \end{pmatrix}. \tag{6.2}$$

For $j = 3, 4, 5$, we can set

$$M_j = I_4 + \begin{pmatrix} x_j \\ y_j \\ z_j \\ w_j \end{pmatrix} \begin{pmatrix} 1 & p_j & q_j & r_j \end{pmatrix},$$

where

$$x_j = e_j - 1 - p_j y_j - q_j z_j - r_j w_j.$$

By a similar transformation which keeps M_1, M_2 invariant, we can send $p_3 = q_3 = r_3 = 1$. We define R_j $(1 \leq j \leq 7)$ by (5.3).

We have

$$\begin{aligned} R_7[2,1] - R_7[2,3] &= (1 - e_2)(q_4 - 1)y_4, \\ R_4[3,1] - R_4[3,2] &= (1 - e_1)(p_5 - 1)z_5, \\ R_7[3,2] - R_7[3,4] &= (1 - e_2)(p_4 - r_4)z_4, \\ R_4[2,3] - R_4[2,4] &= (1 - e_1)(q_5 - r_5)y_5. \end{aligned}$$

Since we can see that $y_4 y_5 z_4 z_5 = 0$ yields only reducible representations, we get

$$q_4 = p_5 = 1, \quad r_4 = p_4, \quad r_5 = q_5.$$

Similarly, by eliminating the cases which yield only reducible representations, we can determine $z_4, w_3, z_5, z_3, y_5, y_3, w_4, y_4$ by using $R_1[2,1] - R_1[2,2], R_4[4,1], R_1[4,1], R_4[2,3], R_4[1,3], R_7[2,1], R_7[4,3], R_6[4,4]$, respectively. Then we get

$$R_6[4,3] = ({e_2}^2 e_3 - e_2 e_4 - 2e_2 e_3 p_4 + 2e_2 e_4 p_4 + e_3 {p_4}^2 - e_2 e_4 {p_4}^2)A,$$

where we can see $A \neq 0$ by the irreducibility. If we set

$$e_2 = {f_2}^2, \quad e_3 = {f_3}^2, \quad e_4 = {f_4}^2,$$

the first factor of $R_6[4,3]$ is factored, and we get

$$p_4 = \frac{f_2(f_2 f_3 - f_4)}{f_3 - f_2 f_4}, \quad p_4 = \frac{f_2(f_2 f_3 + f_4)}{f_3 + f_2 f_4}.$$

We choose the first value. Then $R_2[2,1]$ is factored into two polynomials, one of which yields only reducible representations. From the other factor, we get a value of w_5. Putting the value into R_2, we have

$$R_2[4,4] = (-e_1 e_5 {e_1}^2 {f_3}^2 + 2e_1 e_5 q_5 - 2e_1 {f_3}^2 q_5 - e_1 e_5 {q_5}^2 + {f_3}^2 {q_5}^2)B,$$

where we can see $B \neq 0$ by the irreducibility. If we set

$$e_1 = {f_1}^2, \quad e_5 = {f_5}^2,$$

the first factor of $R_2[4,4]$ is factored, and then we get

$$q_5 = \frac{f_1(f_1 f_3 - f_5)}{f_3 - f_1 f_5}, \quad q_5 = \frac{f_1(f_1 f_3 + f_5)}{f_3 + f_1 f_5}.$$

We choose the first value. Then all parameters are written in terms of $f_1, f_2, \dots, f_5$, and we get an irreducible representation. For the other choices of (p_4, q_5), we also get irreducible representations.

In this way, we obtain four representations

$$\rho^{(k)} = (M_1, M_2, M_3^{(k)}, M_4^{(k)}, M_5^{(k)}) \quad (k = 1, 2, 3, 4),$$

where M_1, M_2 are given in (6.2). The explicit forms of $M_j^{(1)}$ are given as follows. We set

$$M_3^{(1)} = I_4 + (a_{ij}), \quad M_4^{(1)} = I_4 + (b_{ij}), \quad M_5^{(1)} = I_4 + (c_{ij}). \tag{6.3}$$

Then we have

$$a_{11} = a_{12} = a_{13} = a_{14} = \frac{(f_2 f_3 + f_4)(f_1 f_3 - f_5)(1 + f_1 f_2 f_4 f_5)}{(e_1 - 1)(e_2 - 1) f_4 f_5},$$

$$a_{21} = a_{22} = a_{23} = a_{24} = -\frac{(f_3 + f_2 f_4)(f_1 f_3 - f_5)(f_2 + f_1 f_4 f_5)}{(e_1 - 1)(e_2 - 1) f_4 f_5},$$

$$a_{31} = a_{32} = a_{33} = a_{34} = \frac{(f_2 f_3 + f_4)(f_1 f_5 - f_3)(f_1 + f_2 f_4 f_5)}{(e_1 - 1)(e_2 - 1) f_4 f_5},$$

$$a_{41} = a_{42} = a_{43} = a_{44} = -\frac{(f_3 + f_2 f_4)(f_1 f_5 - f_3)(f_1 f_2 + f_4 f_5)}{(e_1 - 1)(e_2 - 1) f_4 f_5},$$

$$b_{11} = b_{13} = \frac{(f_3 + f_2 f_4)(f_1 f_3 - f_5)(1 + f_1 f_2 f_4 f_5)}{(e_1 - 1)(e_2 - 1) f_3 f_5},$$

$$b_{12} = b_{14} = \frac{f_2(f_2 f_3 + f_4)(f_1 f_3 - f_5)(1 + f_1 f_2 f_4 f_5)}{(e_1 - 1)(e_2 - 1) f_3 f_5},$$

$$b_{21} = b_{23} = -\frac{(f_3 + f_2 f_4)(f_1 f_3 - f_5)(f_2 + f_1 f_4 f_5)}{(e_1 - 1)(e_2 - 1) f_2 f_3 f_5},$$

$$b_{22} = b_{24} = -\frac{(f_2 f_3 + f_4)(f_1 f_3 - f_5)(f_2 + f_1 f_4 f_5)}{(e_1 - 1)(e_2 - 1) f_3 f_5},$$

$$b_{31} = b_{33} = \frac{(f_3 + f_2 f_4)(f_1 f_5 - f_3)(f_1 + f_2 f_4 f_5)}{(e_1 - 1)(e_2 - 1) f_3 f_5},$$

$$b_{32} = b_{34} = \frac{f_2(f_2 f_3 + f_4)(f_1 f_5 - f_3)(f_1 + f_2 f_4 f_5)}{(e_1 - 1)(e_2 - 1) f_3 f_5},$$

$$b_{41} = b_{43} = -\frac{(f_3 + f_2 f_4)(f_1 f_5 - f_3)(f_1 f_2 + f_4 f_5)}{(e_1 - 1)(e_2 - 1) f_2 f_3 f_5},$$

$$b_{42} = b_{44} = -\frac{(f_2 f_3 + f_4)(f_1 f_5 - f_3)(f_1 f_2 + f_4 f_5)}{(e_1 - 1)(e_2 - 1) f_3 f_5},$$

$$
\begin{aligned}
c_{11} = c_{12} &= \frac{(f_2f_3+f_4)(f_1f_5-f_3)(1+f_1f_2f_4f_5)}{(e_1-1)(e_2-1)f_3f_4},\\
c_{13} = c_{14} &= -\frac{f_1(f_2f_3+f_4)(f_1f_3-f_5)(1+f_1f_2f_4f_5)}{(e_1-1)(e_2-1)f_3f_4},\\
c_{21} = c_{22} &= -\frac{(f_3+f_2f_4)(f_1f_5-f_3)(f_2+f_1f_4f_5)}{(e_1-1)(e_2-1)f_3f_4},\\
c_{23} = c_{24} &= \frac{f_1(f_3+f_2f_4)(f_1f_3-f_5)(f_2+f_1f_4f_5)}{(e_1-1)(e_2-1)f_3f_4},\\
c_{31} = c_{32} &= -\frac{(f_2f_3+f_4)(f_1f_5-f_3)(f_1+f_2f_4f_5)}{(e_1-1)(e_2-1)f_1f_3f_4},\\
c_{33} = c_{34} &= \frac{(f_2f_3+f_4)(f_1f_3-f_5)(f_1+f_2f_4f_5)}{(e_1-1)(e_2-1)f_3f_4},\\
c_{41} = c_{42} &= \frac{(f_3+f_2f_4)(f_1f_5-f_3)(f_1f_2+f_4f_5)}{(e_1-1)(e_2-1)f_1f_3f_4},\\
c_{43} = c_{44} &= -\frac{(f_3+f_2f_4)(f_1f_3-f_5)(f_1f_2+f_4f_5)}{(e_1-1)(e_2-1)f_3f_4}.
\end{aligned}
$$

As in the previous section, we consider the Galois group

$$G_2 = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5 \rangle = (\mathbb{Z}_2)^5, \tag{6.4}$$

where

$$\sigma_j : f_j \mapsto -f_j \quad (1 \le j \le 5).$$

The matrices $M_j^{(k)}$ with $k = 2, 3, 4$ are defined by

$$
\begin{aligned}
M_j^{(2)} &= \sigma_1(M_j^{(1)}),\\
M_j^{(3)} &= \sigma_2(M_j^{(1)}),\\
M_j^{(4)} &= \sigma_3(M_j^{(1)})
\end{aligned}
$$

for $j = 3, 4, 5$. Thus we have

$$\rho^{(2)} = \sigma_1(\rho^{(1)}),\ \rho^{(3)} = \sigma_2(\rho^{(1)}),\ \rho^{(4)} = \sigma_3(\rho^{(1)}). \tag{6.5}$$

We find that every σ_j permutes $(\rho^{(1)}, \rho^{(2)}, \rho^{(3)}, \rho^{(4)})$. If we denote $\rho^{(k)}$ simply by k, the action is given by

$$
\begin{aligned}
\sigma_1(1,2,3,4) &= (2,1,4,3),\\
\sigma_2(1,2,3,4) &= (3,4,1,2),\\
\sigma_3(1,2,3,4) &= (4,3,2,1),\\
\sigma_4(1,2,3,4) &= (3,4,1,2),\\
\sigma_5(1,2,3,4) &= (2,1,4,3).
\end{aligned} \tag{6.6}
$$

Now we look at the $(1,1)$-entries of $M_3^{(k)}$ $(1 \le k \le 4)$:

$$
\begin{aligned}
M_3^{(1)}[1,1] &= 1 + \frac{(f_2f_3+f_4)(f_1f_3-f_5)(1+f_1f_2f_4f_5)}{(e_1-1)(e_2-1)f_4f_5},\\
M_3^{(2)}[1,1] &= 1 + \frac{(f_2f_3+f_4)(-f_1f_3-f_5)(1-f_1f_2f_4f_5)}{(e_1-1)(e_2-1)f_4f_5},\\
M_3^{(3)}[1,1] &= 1 + \frac{(-f_2f_3+f_4)(f_1f_3-f_5)(1-f_1f_2f_4f_5)}{(e_1-1)(e_2-1)f_4f_5},\\
M_3^{(4)}[1,1] &= 1 + \frac{(f_2f_3-f_4)(f_1f_3+f_5)(1+f_1f_2f_4f_5)}{(e_1-1)(e_2-1)f_4f_5},
\end{aligned}
$$

which implies that no two representations in $\{\rho^{(1)}, \rho^{(2)}, \rho^{(3)}, \rho^{(4)}\}$ are isomorphic.

We can calculate the local monodromies at infinity in $\mathbb{P}^2$ and in $\mathbb{P}^1 \times \mathbb{P}^1$. The local monodromies at $L_\infty, L_\infty^x, L_\infty^y$ for $\rho^{(k)}$ $(1 \le k \le 4)$ are all semi-simple, and have the spectral type $(3,1)$ at L_∞ and $(2,1,1)$ at L_∞^x and L_∞^y. The list of the eigenvalues are given by the following table.

$$
\begin{aligned}
\rho^{(1)} : L_\infty &: -\frac{1}{f_1f_2f_3f_4}, -\frac{1}{f_1f_2f_3f_4}, -\frac{1}{f_1f_2f_3f_4}, -\frac{f_4f_5}{f_1f_2e_3},\\
L_\infty^x &: \frac{f_5}{f_1f_3}, \frac{f_5}{f_1f_3}, -\frac{1}{f_1f_2f_4f_5}, -\frac{f_2}{f_1f_4f_5},\\
L_\infty^y &: -\frac{f_4}{f_2f_3}, -\frac{f_4}{f_2f_3}, -\frac{1}{f_1f_2f_4f_5}, -\frac{f_1}{f_2f_4f_5},\\
\rho^{(2)} : L_\infty &: \frac{1}{f_1f_2f_3f_4}, \frac{1}{f_1f_2f_3f_4}, \frac{1}{f_1f_2f_3f_4}, \frac{f_4f_5}{f_1f_2e_3},\\
L_\infty^x &: -\frac{f_5}{f_1f_3}, -\frac{f_5}{f_1f_3}, \frac{1}{f_1f_2f_4f_5}, \frac{f_2}{f_1f_4f_5},\\
L_\infty^y &: -\frac{f_4}{f_2f_3}, -\frac{f_4}{f_2f_3}, \frac{1}{f_1f_2f_4f_5}, \frac{f_1}{f_2f_4f_5},\\
\rho^{(3)} : L_\infty &: \frac{1}{f_1f_2f_3f_4}, \frac{1}{f_1f_2f_3f_4}, \frac{1}{f_1f_2f_3f_4}, \frac{f_4f_5}{f_1f_2e_3},\\
L_\infty^x &: \frac{f_5}{f_1f_3}, \frac{f_5}{f_1f_3}, \frac{1}{f_1f_2f_4f_5}, \frac{f_2}{f_1f_4f_5},\\
L_\infty^y &: \frac{f_4}{f_2f_3}, \frac{f_4}{f_2f_3}, \frac{1}{f_1f_2f_4f_5}, \frac{f_1}{f_2f_4f_5},\\
\rho^{(4)} : L_\infty &: -\frac{1}{f_1f_2f_3f_4}, -\frac{1}{f_1f_2f_3f_4}, -\frac{1}{f_1f_2f_3f_4}, -\frac{f_4f_5}{f_1f_2e_3},\\
L_\infty^x &: -\frac{f_5}{f_1f_3}, -\frac{f_5}{f_1f_3}, -\frac{1}{f_1f_2f_4f_5}, -\frac{f_2}{f_1f_4f_5},\\
L_\infty^y &: \frac{f_4}{f_2f_3}, \frac{f_4}{f_2f_3}, -\frac{1}{f_1f_2f_4f_5}, -\frac{f_1}{f_2f_4f_5}.
\end{aligned}
$$

We see that the representations $\rho^{(1)}$ and $\rho^{(4)}$ cannot be distinguished by looking at the local monodromies in $S^{(1)} \subset \mathbb{P}^2$, and the same holds for $\rho^{(2)}$ and $\rho^{(3)}$. They can be distinguished by the local monodromies in $S^{(1)'} \subset \mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 6.1. *Let*

$$\rho : \pi_1(\mathbb{P}^2 \setminus S^{(1)}, b) \to \mathrm{GL}(4, \mathbb{C})$$

be an irreducible anti-homomorphism with spectral type $(2,2)$ *at* S_1, S_2 *and* $(3,1)$ *at* S_3, S_4, S_5. *We assume that the eigenvalues of the local monodromies at* S_1, S_2 *are given by* $(1,1,e_1,e_1), (1,1,e_2,e_2)$, *respectively, and at* S_3, S_4, S_5 *by* $(1,1,1,e_3), (1,1,1,e_4), (1,1,1,e_5)$, *respectively, where* $e_j \in \mathbb{C} \setminus \{0,1\}$ $(1 \le j \le 5)$. *Then* ρ *is isomorphic to one of* $\rho^{(k)} = (M_1, M_2, M_3^{(k)}, M_4^{(k)}, M_5^{(k)})$ $(1 \le k \le 4)$ *given by* (6.2), (6.3), (6.5), *where* f_j *is a square root of* e_j $(1 \le j \le 5)$. *The local monodromy at* L_∞ *is given by*

$$\mathrm{diag}\left[-\frac{1}{f_1 f_2 f_3 f_4}, -\frac{1}{f_1 f_2 f_3 f_4}, -\frac{1}{f_1 f_2 f_3 f_4}, -\frac{f_4 f_5}{f_1 f_2 e_3}\right]$$

for $\rho^{(1)}$ *and* $\rho^{(4)}$, *and by*

$$\mathrm{diag}\left[\frac{1}{f_1 f_2 f_3 f_4}, \frac{1}{f_1 f_2 f_3 f_4}, \frac{1}{f_1 f_2 f_3 f_4}, \frac{f_4 f_5}{f_1 f_2 e_3}\right]$$

for $\rho^{(2)}$ *and* $\rho^{(3)}$. *The Galois group* G_2 *given by* (6.4) *acts on the representations* $\{\rho^{(1)}, \rho^{(2)}, \rho^{(3)}, \rho^{(4)}\}$ *as* (6.6), *where* k *denotes* $\rho^{(k)}$.

7. RIGIDITY OF THE MONODROMY OF F_4

By Proposition 3.3, the monodromy representation of the Pfaffian system for F_4 is an anti-homomorphism

$$\rho_3 : \pi_1(\mathbb{P}^2 \setminus S^{(1)}, b) \to \mathrm{GL}(4, \mathbb{C})$$

whose spectral type is

$$(22, 22, 31, 22, 22; 211) \tag{7.1}$$

if (3.6) holds. We shall determine the representation classes $[\rho]$ modulo multiplication with the spectral type (7.1).

Let $\rho = (M_1, M_2, M_3, M_4, M_5)$ be an irreducible representation of $\pi_1(\mathbb{P}^2 \setminus S^{(1)}, b)$ with the spectral type (7.1). By a multiplication, we can send (one of) the multiple eigenvalues of each M_j to 1, and hence we may assume

$$M_j \sim \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & e_j & \\ & & & e_j \end{pmatrix} \quad (j = 1, 2, 4, 5),$$

$$M_3 \sim \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & e_3 \end{pmatrix},$$

where $e_j \neq 0, 1$. In the same reason as in the case of F_2, we may assume that M_1, M_2 are given by (6.2). We can set

$$M_3 = I_4 + \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$$

with

$$w = e_3 - 1 - x - y - z.$$

The matrices M_4, M_5 are parametrized as follows. Since

$$\operatorname{rank}(M_4 - I_4) = \operatorname{rank}(M_5 - I_4) = 2,$$

we may assume that the first and the second columns of $M_4 - I_4$ are linearly independent, and also the first and the third columns of $M_5 - I_4$ are linearly independent. Then, by using 2×2 matrices P, Q, U, V, we have

$$\begin{aligned} M_4 &= I_4 + \begin{pmatrix} C \\ U \end{pmatrix} \begin{pmatrix} I_2 & P \end{pmatrix}, \\ M_5 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \left(I_4 + \begin{pmatrix} D \\ V \end{pmatrix} \begin{pmatrix} I_2 & Q \end{pmatrix} \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} C &= (e_4 - 1)I_2 - PU, \\ D &= (e_5 - 1)I_2 - QV. \end{aligned}$$

We set

$$P = (p_{ij}),\ Q = (q_{ij}),\ U = (u_{ij}),\ V = (v_{ij}).$$

Define R_j $(1 \leq j \leq 7)$ by (5.3). We are going to determine the parameters so that $R_j = O$ for $1 \leq j \leq 7$ and $(M_1, M_2, \ldots, M_5)$ is irreducible.

Note that R_2, R_3, R_4 are relations on $v_{ij}, q_{ij}, x, y, z, e_1, e_3, e_5$, and R_5, R_6, R_7 on $u_{ij}, p_{ij}, x, y, z, e_2, e_3, e_4$. First we consider the relations R_2 and R_4. We have

$$\begin{aligned} R_4[2,3] - R_4[2,4] &= (1 - e_1)(q_{12}v_{11} - v_{12} + q_{22}v_{12}), \\ R_4[4,1] - R_4[4,2] &= (1 - e_1)(v_{21} - q_{11}v_{21} - q_{21}v_{22}). \end{aligned}$$

If we assume $v_{12}v_{22} \neq 0$, these relations determine q_{22} and q_{21}. We assume $q_{12} \neq 0$. Then q_{12} is determined by $R_4[1,3] - R_4[1,4]$. The relation $R_4[3,1] - R_4[3,2]$ contains a factor $(q_{11} - 1)$, and we choose $q_{11} = 1$. Putting these values into R_4, we can determine z by $R_4[4,1]$. We assume $v_{11} \neq v_{12}$. Then x is determined by $R_4[2,3]$, and also y and v_{22} are determined by $R_4[1,3]$ and $R_2[2,1]$, respectively. Now we have

$$\begin{aligned} &R_2[2,3] = \\ &\quad ({e_1}^2 e_3 {v_{11}}^2 - e_1 e_5 {v_{11}}^2 - 2e_1 e_3 v_{11} v_{12} + 2e_1 e_5 v_{11} v_{12} + e_3 {v_{12}}^2 - e_1 e_5 {v_{12}}^2)A \end{aligned}$$

with a non-zero factor A. If we set

$$e_1 = {f_1}^2,\ e_3 = {f_3}^2,\ e_5 = {f_5}^2,$$

the first factor of $R_2[2,3]$ is factored into two factors, and we get

$$v_{11} = \frac{(f_3 - f_1 f_5)v_{12}}{f_1(f_3 f_1 - f_5)},\ v_{11} = \frac{(f_3 + f_1 f_5)v_{12}}{f_1(f_3 f_1 + f_5)}.$$

We choose the first value. Thus we get

$$\begin{aligned}
v_{11} &= \frac{(f_3 - f_1 f_5)v_{12}}{f_1(f_1 f_3 - f_5)}, \\
v_{22} &= \frac{f_1(f_1 f_3 f_5 - 1)v_{21}}{f_3 f_5 - f_1}, \\
q_{11} &= 1, \\
q_{12} &= \frac{(f_1 f_3 - f_5)(f_3 f_5 - f_1)}{f_3(e_1 - 1)v_{21}}, \\
q_{21} &= 0, \\
q_{22} &= \frac{f_1 f_3 - e_1 f_5 - e_3 f_5 + f_1 f_3 e_5 - f_1 f_3 v_{21} + {f_1}^3 f_3 v_{21}}{f_1 f_3(e_1 - 1)v_{21}}, \\
x &= \frac{-e_1 f_3 + f_1 f_5 + {f_1}^3 e_3 f_5 - e_1 f_3 e_5 - f_3 v_{12} + e_1 f_3 v_{12}}{f_1 f_5(e_1 - 1)}, \\
y &= -\frac{f_3 v_{12}}{f_1 f_5}, \\
z &= \frac{f_1 f_3 - e_1 f_5 - e_3 f_5 + f_1 f_3 e_5 - f_1 f_3 v_{21} + {f_1}^3 f_3 v_{21}}{f_5(e_1 - 1)},
\end{aligned}$$

which makes $R_2 = R_4 = O$, and hence $R_3 = O$. In the above, we made several assumptions. If we assume otherwise, we get other sets of $(v_{ij}, q_{ij}, x, y, z)$, which also make $R_2 = R_3 = R_4 = O$.

In a similar way, we solve the relations R_6 and R_7. We have

$$\begin{aligned}
R_7[3,2] - R_7[3,4] &= (1 - e_2)(p_{12}u_{11} - u_{12} + p_{22}u_{12}), \\
R_7[4,1] - R_7[4,3] &= (1 - e_2)(u_{21} - p_{11}u_{21} - p_{21}u_{22}).
\end{aligned}$$

If we assume $u_{12}u_{22} \neq 0$, these relations determine p_{22} and p_{21}. If we assume $p_{12} \neq 0$, its value is determined by $R_7[1,2] - R_7[1,4]$. We choose a solution $p_{11} = 1$ from $R_7[2,1] - R_7[2,3]$. Then y is determined by $R_7[4,1]$. We assume $u_{11} \neq u_{12}$, and then x is determined by $R_7[3,2]$. The values of z and u_{22} are determined by $R_6[1,1] + R_6[3,1]$ and $R_6[2,1] + R_6[4,1]$, respectively. Now we have

$$\begin{aligned}
&R_7[1,2] = \\
&\quad ({e_2}^2 e_3 {u_{11}}^2 - e_2 e_4 {u_{11}}^2 - 2e_2 e_3 u_{11} u_{12} + 2e_2 e_4 u_{11} u_{12} + e_3 {u_{12}}^2 - e_2 e_4 {u_{12}}^2)B
\end{aligned}$$

with some factor B. If we set

$$e_2 = {f_2}^2,\ e_3 = {f_3}^2,\ e_4 = {f_4}^2,$$

the first factor of $R_7[1,2]$ is factored into two factors, and we get

$$u_{11} = \frac{(f_3 - f_2 f_4)u_{12}}{f_2(f_2 f_3 - f_4)},\ u_{11} = \frac{(f_3 + f_2 f_4)u_{12}}{f_2(f_2 f_3 + f_4)}.$$

We choose the first value. Then we get

$$\begin{aligned}
u_{11} &= \frac{(f_3 - f_2 f_4)u_{12}}{f_2(f_2 f_3 - f_4)},\\
u_{22} &= \frac{f_2(f_2 f_3 f_4 - 1)u_{21}}{f_3 f_4 - f_2},\\
p_{11} &= 1,\\
p_{12} &= \frac{(f_2 f_3 - f_4)(f_3 f_4 - f_2)}{f_3(e_2 - 1)u_{21}},\\
p_{21} &= 0,\\
p_{22} &= \frac{f_2 f_3 - e_2 f_4 - e_3 f_4 + f_2 f_3 e_4 - f_2 f_3 u_{21} + {f_2}^3 f_3 u_{21}}{f_2 f_3(e_2 - 1)u_{21}},\\
x &= \frac{-e_2 f_3 + f_2 f_4 + {f_2}^3 e_3 f_4 - e_2 f_3 e_4 - f_3 u_{12} + e_2 f_3 u_{12}}{f_2 f_4(e_2 - 1)},\\
y &= \frac{f_2 f_3 - e_2 f_4 - e_3 f_4 + f_2 f_3 e_4 - f_2 f_3 u_{21} + {f_2}^3 f_3 u_{21}}{f_4(e_2 - 1)},\\
z &= -\frac{f_3 u_{12}}{f_2 f_4},
\end{aligned}$$

which makes $R_6 = R_7 = O$, and hence $R_5 = O$. If we assume otherwise, we get other sets of $(u_{ij}, p_{ij}, x, y, z)$, which also make $R_5 = R_6 = R_7 = O$.

Now we choose a set $(v_{ij}, q_{ij}, x, y, z)$ and a set $(u_{ij}, p_{ij}, x, y, z)$, examine their compatibility, and then check the remaining relation R_1. If we choose the two sets given above, we come to

$$(f_4, f_5) = (f_2, f_1),\ (f_4, f_5) = (-f_2, -f_1).$$

It turns out that both cases give irreducible representations. We choose the first one. In this case, the eigenvalues of ${M_0}^{-1}$ are calculated as

$$e_1 e_2,\ e_1 e_2,\ \frac{A + \sqrt{B}}{C},\ \frac{A - \sqrt{B}}{C},$$

where A, B, C are polynomials in f_1, f_2, f_3 and u_{12}. In particular, B is quadratic in u_{12}. In order to uniformize these eigenvalues, we set

$$u_{12} = \frac{e_2(f_3 - 1)(e_1 + g)(e_2 f_3 + g)}{(e_1 - 1)(e_2 - 1)f_3 g}$$

with a new parameter g. Then the eigenvalues of ${M_0}^{-1}$ are

$$e_1e_2,\ e_1e_2,\ g^2,\ \frac{{e_1}^2{e_2}^2e_3}{g^2}.$$

In this case, we have

$$M_3 = I_4 + (a_{ij}), \quad M_4 = (b_{ij}), \quad M_5 = (c_{ij}) \tag{7.2}$$

with

$$\begin{aligned}
&a_{11} = a_{12} = a_{13} = a_{14} = \frac{(f_3-1)(g+1)(e_1e_2f_3+g)}{(e_1-1)(e_2-1)g},\\
&a_{21} = a_{22} = a_{23} = a_{24} = -\frac{(f_3-1)(e_1f_3+g)(e_2+g)}{(e_1-1)(e_2-1)g},\\
&a_{31} = a_{32} = a_{33} = a_{34} = -\frac{(f_3-1)(e_1+g)(e_2f_3+g)}{(e_1-1)(e_2-1)g},\\
&a_{41} = a_{42} = a_{43} = a_{44} = \frac{(f_3-1)(f_3+g)(e_1e_2+g)}{(e_1-1)(e_2-1)g},\\
&b_{11} = 1 + \frac{(e_2-f_3)(g+1)(e_1e_2f_3+g)}{(e_1-1)(e_2-1)f_3g},\\
&b_{12} = b_{14} = -\frac{e_2(f_3-1)(g+1)(e_1e_2f_3+g)}{(e_1-1)(e_2-1)f_3g},\\
&b_{13} = b_{11} - 1,\\
&b_{21} = b_{23} = \frac{(f_3-1)(e_1f_3+g)(e_2+g)}{(e_1-1)(e_2-1)f_3g},\\
&b_{22} = e_2 + \frac{(e_2f_3-1)(e_1f_3+g)(e_2+g)}{(e_1-1)(e_2-1)f_3g},\\
&b_{24} = b_{22} - e_2,\\
&b_{31} = \frac{(f_3-e_2)(e_1+g)(e_2f_3+g)}{(e_1-1)(e_2-1)f_3g},\\
&b_{32} = b_{34} = \frac{e_2(f_3-1)(e_1+g)(e_2f_3+g)}{(e_1-1)(e_2-1)f_3g},\\
&b_{33} = 1 + b_{31},\\
&b_{41} = b_{43} = -\frac{(f_3-1)(f_3+g)(e_1e_2+g)}{(e_1-1)(e_2-1)f_3g},\\
&b_{42} = -\frac{(e_2f_3-1)(f_3+g)(e_1e_2+g)}{(e_1-1)(e_2-1)f_3g},\\
&b_{44} = e_2 + b_{42},
\end{aligned}$$

$$
\begin{aligned}
c_{11} &= 1 + \frac{(e_1 - f_3)(g+1)(e_1 e_2 f_3 + g)}{(e_1 - 1)(e_2 - 1) f_3 g}, \\
c_{12} &= c_{11} - 1, \\
c_{13} &= c_{14} = -\frac{e_1 (f_3 - 1)(g+1)(e_1 e_2 f_3 + g)}{(e_1 - 1)(e_2 - 1) f_3 g}, \\
c_{21} &= \frac{(f_3 - e_1)(e_1 f_3 + g)(e_2 + g)}{(e_1 - 1)(e_2 - 1) f_3 g}, \\
c_{22} &= 1 + c_{21}, \\
c_{23} &= c_{24} = \frac{e_1 (f_3 - 1)(e_1 f_3 + g)(e_2 + g)}{(e_1 - 1)(e_2 - 1) f_3 g}, \\
c_{31} &= c_{32} = \frac{(f_3 - 1)(e_1 + g)(e_2 f_3 + g)}{(e_1 - 1)(e_2 - 1) f_3 g}, \\
c_{33} &= e_1 + \frac{(e_1 f_3 - 1)(e_1 + g)(e_2 f_3 + g)}{(e_1 - 1)(e_2 - 1) f_3 g}, \\
c_{34} &= c_{33} - e_1, \\
c_{41} &= c_{42} = -\frac{(f_3 - 1)(f_3 + g)(e_1 e_2 + g)}{(e_1 - 1)(e_2 - 1) f_3 g}, \\
c_{43} &= -\frac{(e_1 f_3 - 1)(f_3 + g)(e_1 e_2 + g)}{(e_1 - 1)(e_2 - 1) f_3 g}, \\
c_{44} &= e_1 + c_{43}.
\end{aligned}
$$

We denote by $\rho^{(1)}$ the representation determined by the above $(M_1, M_2, M_3, M_4, M_5)$. Note that, in deriving the entries of M_3, M_4, M_5, we used square roots of $e_1, e_2, \ldots, e_5$, while the result is written in terms of e_1, e_2, a square root of e_3 and a square root of an eigenvalue of ${M_0}^{-1}$.

In this way, we obtain a lot of representations. We find that all representations thus obtained are reduced to one of four representations $\rho^{(1)}, \rho^{(2)}, \rho^{(3)}, \rho^{(4)}$ given below.

Let σ, τ be generators of $\mathbb{Z}_2$ defined by

$$
\sigma : f_3 \mapsto -f_3, \ \tau : g \mapsto -g,
$$

and set

$$
G_4 = \langle \sigma, \tau \rangle = (\mathbb{Z}_2)^2, \tag{7.3}
$$

which is Klein's four-group. Then we set

$$
\begin{aligned}
M_j^{(2)} &= \sigma(M_j), \\
M_j^{(3)} &= \tau(M_j), \\
M_j^{(4)} &= \sigma\tau(M_j)
\end{aligned}
$$

for $j=3,4,5$. Now we define $\rho^{(2)},\rho^{(3)},\rho^{(4)}$ by

$$\begin{aligned}\rho^{(2)} &= (M_1, M_2, M_3^{(2)}, M_4^{(2)}, M_5^{(2)}),\\ \rho^{(3)} &= (M_1, M_2, M_3^{(3)}, M_4^{(3)}, M_5^{(3)}),\\ \rho^{(4)} &= (M_1, M_2, M_3^{(4)}, M_4^{(4)}, M_5^{(4)}),\end{aligned} \tag{7.4}$$

where M_1, M_2 are the matrices in (6.2). Then it is evident that the Galois group G_4 acts faithfully on the set $\{\rho^{(1)},\rho^{(2)},\rho^{(3)},\rho^{(4)}\}$. Since the eigenvalues of ${M_0}^{-1}$ do not change for the action of G_4, the eigenvalues of the local monodormy at L_∞ for every $\rho^{(k)}$ are

$$\frac{1}{e_1e_2}, \frac{1}{e_1e_2}, \frac{1}{g^2}, \frac{g^2}{{e_1}^2{e_2}^2e_3}.$$

For each representation $\rho^{(k)}$, the eigenvalues of the local monodromies at L_∞^x and at L_∞^y coincide, and are given as follows:

$$\begin{aligned}\rho^{(1)} &: -\frac{1}{g}, -\frac{1}{g}, -\frac{g}{e_1e_2f_3}, -\frac{g}{e_1e_2f_3},\\ \rho^{(2)} &: -\frac{1}{g}, -\frac{1}{g}, \frac{g}{e_1e_2f_3}, \frac{g}{e_1e_2f_3},\\ \rho^{(3)} &: \frac{1}{g}, \frac{1}{g}, \frac{g}{e_1e_2f_3}, \frac{g}{e_1e_2f_3},\\ \rho^{(4)} &: \frac{1}{g}, \frac{1}{g}, -\frac{g}{e_1e_2f_3}, -\frac{g}{e_1e_2f_3}.\end{aligned}$$

Then we can distinguish the four representations by these local monodromies.

Theorem 7.1. *Let*

$$\rho : \pi_1(\mathbb{P}^2 \setminus S^{(1)}, b) \to \mathrm{GL}(4, \mathbb{C})$$

be an irreducible anti-homomorphism with spectral type $(2,2)$ *at* S_1, S_2, S_4, S_5 *and* $(3,1)$ *at* S_3. *We assume that the eigenvalues of the local monodromies at* S_1, S_2, S_4, S_5 *are given by* $(1,1,e_1,e_1), (1,1,e_2,e_2), (1,1,e_4,e_4), (1,1,e_5,e_5)$, *respectively, and at* S_3 *by* $(1,1,1,e_3)$, *where* $e_j \in \mathbb{C}\setminus\{0,1\}$ $(1 \le j \le 5)$. *Then* ρ *is, up to multiplications, isomorphic to one of* $\rho^{(k)}$ $(1 \le k \le 4)$ *given by* (6.2), (7.2), (7.4), *where* f_3 *is a square root of* e_3. *For these representations,*

$$e_1 = e_5,\ \ e_2 = e_4$$

hold. The local monodromy at L_∞ *is given by*

$$\mathrm{diag}\left[\frac{1}{e_1e_2}, \frac{1}{e_1e_2}, \frac{1}{g^2}, \frac{g^2}{{e_1}^2{e_2}^2e_3}\right].$$

The Galois group G_4 *given by* (7.3) *acts faithfully on the set* $\{\rho^{(1)},\rho^{(2)},\rho^{(3)},\rho^{(4)}\}$.

8. CONCLUSION AND REMARKS

Theorems 5.1, 6.1 and 7.1 show that the monodromy representations for Appell's hypergeometric functions with singular locus

$$S^{(1)} = \{xy(x-1)(y-1)(x-y) = 0\} \cup L_\infty$$

are almost rigid. Namely they are determined by the local monodromies up to a finite number of possibilities. It is remarkable that these representations are determined by prescribing only the local monodromies at singular loci in $\mathbb{C}^2$. The local monodromy at the line at infinity in $\mathbb{P}^2$ or in $\mathbb{P}^1 \times \mathbb{P}^1$ is uniquely determined.

It is also remarkable that these representations are strictly rigid if they are considered in $\mathbb{P}^1 \times \mathbb{P}^1$. Thus the rigidity depends on compactifications of $\mathbb{C}^2$. This fact can be explained by the correspondence of the parameters and the eigenvalues of the local monodromies. For the F_1 case, as listed in (3.4), the eigenvalues of the local monodromy at each component are determined by the parameters (a, b, b', c). If we change the parameters to

$$\left(a, b + \frac{1}{2}, b' + \frac{1}{2}, c + \frac{1}{2}\right),$$

the eigenvalues at $S_1, \ldots, S_5, L_\infty$ are unchanged, while, the eigenvalues at L^x_∞, L^y_∞, $e(b)$ and $e(b')$ change to $-e(b)$ and $-e(b')$. Thus the two distinct representations, which can be distinguished by the local monodromies in $\mathbb{P}^1 \times \mathbb{P}^1$, give the same local monodromies in $\mathbb{P}^2$. For F_2 case, we consider the change of the parameters

$$(a, b, b', c, c') \mapsto \left(a, b + \frac{1}{2}, b' + \frac{1}{2}, c, c'\right).$$

For the F_4 case, we consider the changes of the parameters generated by

$$(a, b, c, c') \mapsto \left(a + \frac{1}{2}, b, c, c'\right),$$
$$(a, b, c, c') \mapsto \left(a, b + \frac{1}{2}, c, c'\right),$$

which form the group $(\mathbb{Z}_2)^2$.

We also note that the relations among the eigenvalues of the local monodromies, such as $e_1e_2 = e_4e_5$ for F_1 or $e_1 = e_5, e_2 = e_4$ for F_4, are consequences of the existence of irreducible representations. In other words, these relations are derived from the spectral type of the representation. Hence our formulation of the problem given in section 4 works well for the study of the rigidity in higher dimensional cases.

Here we remark on the irreducibility of the representations given in Theorems 5.1, 6.1 and 7.1. For each $\rho^{(k)}$, two matrices M_1 and M_2 are diagonal. Then, if $\rho^{(k)}$ is reducible, the invariant subspace is spanned by some of the unit vectors ${}^t(0, \ldots, 1, \ldots, 0)$. Hence we can show the irreducibility by checking the non-vanishing of off-diagonal entries of M_3 or M_4 or M_5. For example, if no off-diagonal entry of

M_3 (or M_4 or M_5) vanishes, the representation is irreducible. We can obtain an exact condition for the irreducibility in the same way as in [6].

In this paper, we started from the spectral types of existing representations, and looked for representations with the spectral types. It is not known for what spectral types irreducible representations exist. This problem may be concerned with defining the index of rigidity in higher dimensional cases.

On the other hand, we can obtain infinitely many representations of $\pi_1(\mathbb{P}^2 \setminus S^{(1)})$ from rigid Fuchsian ordinary differential equations with four singular points by using the middle convolution defined by [4]. Also, the middle convolution connects the monodromy representations for F_2 and F_4 with the one for F_1. It would be interesting to study the representations of $\pi_1(\mathbb{P}^2 \setminus S^{(1)})$ by the middle convolution.

In the famous paper [3] Gérard and Levelt showed that, if the singular locus consists of normally crossing hyperplanes, any solution of the Pfaffian system becomes elementary. We can give another proof for this assertion by considering the monodromy. Namely, if the hyperplanes of the singular locus are normally crossing, the fundamental group becomes abelian, so that any representation becomes reducible. The singular locus $S^{(1)}$ is chosen by Gérard-Levelt as the simplest hyperplane arrangement which gives a non-elementary Pfaffian system. Now we recognize that the topology of the singular locus fairly determines the analytic behaviors of the solutions. Then we consider it important to characterize hypersurfaces such that the fundamental groups of the complement space admit irreducible representations. This problem will be fundamental in the theory of hyperplane arrangements or in algebraic geometry.

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