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MORE ON THE BEHAVIORS OF FIXED POINTS SETS OF MULTIFUNCTIONS AND APPLICATIONS

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Abstract. In this paper, we study the behaviors of fixed points sets of non necessarily pseudo-contractive multifunctions. Rather than comparing the images of the involved multifunctions, we make use of some conditions on the fixed points sets to establish general results on their stability and continuous dependence. We illustrate our results by applications to differential inclusions and give stability results of fixed points sets of non necessarily pseudo-contractive multifunctions with respect to the bounded proximal convergence.

Keywords: multifunction, fixed point, Pompeiu-Hausdorff metric, bounded proximal convergence, differential inclusion.

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1. INTRODUCTION

Results about the behaviors of fixed points sets have brought in recent years the attention of several authors since they not only can be used to describe the dependence of solutions to differential inclusions or partial differential equations but, without claim of completeness, they reinforce the links among several purposes such as stability, optimal control, well-posedness, sensitivity analysis, generalized differentiation, generalized equations, differential inclusions and optimization, see for example, [8, 9, 14, 15, 17, 18, 25, 28, 30, 32] and the references therein.

To our knowledge, the starting works in this direction for families of mappings and multifunctions were done in [27, 29], where the behaviors of fixed points sets have been considered with respect to the Pompeiu-Hausdorff convergence.

Recently, the behaviors of fixed points sets of pseudo-contractive mappings in the settings of complete metric spaces have been investigated and some results have been obtained and illustrated by applications to different topics including applications to differential inclusions, see for instance, [8, 30].

Since the dependence of fixed point sets is a subject which is not limited to the case of Lipschitzian multifunctions (see for instance, [4]), we consider here some conditions and assume that the involved multifunctions have nonempty fixed points sets. We establish some results on the continuous dependence of fixed points sets of non necessarily pseudo-contractive multifunctions and present applications to differential inclusions. Stability results of fixed points sets of multifunction with respect to the bounded proximal convergence which is weaker than both Fisher convergence and Attouch-Wets convergence and stronger than Wijsman convergence are also given.

2. NOTATIONS AND PRELIMINARIES

Throughout this paper, (X, d) stands for a metric space. Given $x \in X$ and $r \in (0, +\infty]$, we denote by $B_r(x)$ (*resp.* $\overline{B}_r(x)$) the open (*resp.* closed) ball around x with radius r, where $B_{+\infty}(x) = \overline{B}_{+\infty}(x) = X$.

The distance from a point $x \in X$ to a set $A \neq \emptyset$ is given by

$$d(x, A) := \inf\{d(x, y) \mid y \in A\}.$$

As usual, $d(x, \emptyset) = +\infty$. For two subsets A and B of X, *the excess* of A over B with respect to d is denoted by $e(A, B)$ and defined by

$$e(A, B) := \sup_{x \in A} d(x, B)$$

and we adopt the convention that $e(\emptyset, B) = 0$ when $B \neq \emptyset$.

The (extended real-valued) distance between A and B with respect to d defined by

$$h(A, B) := \max\{e(A, B), e(B, A)\}$$

is called *the Pompeiu-Hausdorff metric*, see [16].

Let (X, d) be metric spaces. In the sequel, a multifunction T from X to X will be denoted by $T : X \rightrightarrows X$. For a subset M of X, we denote by $\mathcal{F}_M(X)$, the family of all multifunctions from X to X with nonempty closed values on M. That is, $T \in \mathcal{F}_M(X)$ if $T(x) \neq \emptyset$ and $T(x) \in CL(X)$, for every $x \in M$. We write $\mathcal{F}(X)$ instead of $\mathcal{F}_X(X)$.

Let M be a subset of X and let $T \in \mathcal{F}_M(X)$ be a multifunction. Recall that T is said to be λ-*Lipschitzian* on M if

$$h(T(x_1), T(x_2)) \leq \lambda d(x_1, x_2) \quad \text{for all} \quad x_1, x_2 \in M.$$

It is is clear that T is λ-Lipschitzian on M if and only if

$$e(T(x_1), T(x_2)) \leq \lambda d(x_1, x_2) \quad \text{for all} \quad x_1, x_2 \in M.$$

A λ-Lipschitzian multifunction is said to be λ-*contractive* if $\lambda \in [0, 1)$.

Following [8], a multifunction $T : X \rightrightarrows X$ is said to be pseudo-L-Lipschitzian with respect to a subset M of X if

$$e(T(x_1) \cap M, T(x_2)) \leq L d(x_1, x_2) \quad \text{for all} \quad x_1, x_2 \in M.$$

If $L \in [0,1)$, then the multifunction T is called pseudo-L-contractive with respect to M.

In the sequel, for a multifunction $T : X \rightrightarrows X$, we denote by $\mathrm{Fix}(T)$, the fixed points set of T, that is,

$$\mathrm{Fix}(T) = \{x \in X \mid x \in T(x)\}.$$

As well-known, there are several results in the literature about the existence of fixed points of mappings and multifunctions. In the sequel, we will be mainly interested in stability of fixed point sets. For the existence, we refer to [1, 8, 20, 23] and the references therein.

3. THE BEHAVIORS OF FIXED POINTS SETS OF MULTIFUNCTIONS

The following result on the behavior of fixed point sets of multifunctions should be compared to [8, Proposition 2.4] and [9, Proposition 2.4].

Theorem 3.1. *Let (X,d) be a metric space. Let $x_0 \in X$ and $r \in (0,+\infty]$ be such that $\overline{B}_r(x_0)$ is a complete subspace. Let $\lambda \in (0,1)$ and $T \in \mathcal{F}_{\overline{B}_r(x_0)}(X)$ with at least one fixed point. Assume that:*

1. *the function $x \to d(x,T(x))$ is lower semicontinuous;*
2. *for any $x \in B_r(x_0)$ and any $y \in T(x) \cap B_r(x_0)$, one has*
$$d(y,T(y)) \le \lambda d(x,y); \tag{3.1}$$
3. *for some $\beta > 0$ such that $\beta < (1-\lambda)r$,*
$$d(x,T(x)) < \lambda\beta \quad \textit{for all} \quad x \in \mathrm{Fix}(S) \cap B_\beta(x_0), \tag{3.2}$$
where $S : X \rightrightarrows X$ is a multifunction.

Then,

$$e(\mathrm{Fix}(S) \cap B_\beta(x_0), \mathrm{Fix}(T)) \le \frac{1}{1-\lambda} \sup_{x \in B_r(x_0)} e(S(x) \cap B_\beta(x_0), T(x)). \tag{3.3}$$

Proof. Assume $\mathrm{Fix}(S) \cap B_\beta(x_0) \neq \emptyset$, otherwise we are finished.

Put $\gamma := \sup_{x \in \mathrm{Fix}(S) \cap B_\beta(x_0)} d(x,T(x))$, which is finite by (3.2), and fix $\varepsilon, \varepsilon' > 0$ so that

$$\sum_{n=0}^{\infty} n\lambda^n \varepsilon' < \frac{\varepsilon}{1-\lambda}.$$

Let $x_1 \in S(x_1)$ be such that $d(x_1,x_0) < \beta$. Thus $d(x_1,T(x_1) \le \gamma$ and $d(x_1,T(x_1)) < \lambda\beta$ from (3.2). Hence there exists $x_2 \in T(x_1)$ such that $d(x_1,x_2) < \min(\gamma+\varepsilon, \lambda\beta)$.

Moreover, as

$$d(x_0,x_2) \le d(x_0,x_1) + d(x_1,x_2) < (1+\lambda)\beta < r,$$

$x_2 \in B_r(x_0)$. Since $x_2 \in T(x_1) \cap B_r(x_0)$ with $x_1 \in B_r(x_0)$ (note that $\beta < r$) and T satisfies (3.1), we get

$$d(x_2, T(x_2)) \leq \lambda d(x_1, x_2),$$

and so we can pick $x_3 \in T(x_2)$ so that

$$d(x_2, x_3) < M_1 := \min(\lambda d(x_1, x_2) + \lambda\varepsilon_1, \lambda^2\beta).$$

Observe that $x_3 \in B_r(x_0)$, since

$$d(x_0, x_3) < (1 + \lambda + \lambda^2)\beta < (1 - \lambda^3)r.$$

Proceeding now by induction and suppose we have constructed a finite sequence $(x_k)_{k=2,..,n}$ such that $x_{k+1} \in T(x_k) \cap B_r(x_0)$ for any $k \in \{2, ..., n-1\}$ and

$$d(x_k, x_{k+1}) < M_{k-1}, \tag{3.4}$$

where

$$M_{k-1} := \min(\lambda d(x_{k-1}, x_k) + \lambda^{k-1}\varepsilon', \lambda^k\beta).$$

Using the contraction assumption for $x_n \in T(x_{n-1}) \cap B_r(x_0)$, it yields

$$d(x_n, T(x_n)) \leq \lambda d(x_{n-1}, x_n) < \min(\lambda d(x_{n-1}, x_n) + \lambda^{n-1}\varepsilon', \lambda^n\beta),$$

since by (3.4), $d(x_{n-1}, x_n) < \lambda^{n-1}\beta$. So that one can pick $x_{n+1} \in T(x_n)$ such that $d(x_n, x_{n+1}) < M_{n-1}$ with

$$M_{n-1} := \min(\lambda d(x_{n-1}, x_n) + \lambda^{n-1}\varepsilon', \lambda^n\beta).$$

Moreover, as $d(x_k, x_{k+1}) < \lambda^k\beta$ for any $k \geq 1$, we get from the following inequalities

$$d(x_0, x_n) \leq d(x_0, x_1) + \sum_{k=1}^{n-1} d(x_k, x_{k+1}) < \sum_{k=0}^{n-1} \lambda^k\beta \tag{3.5}$$

$$< \frac{1-\lambda^n}{1-\lambda}\beta < (1 - \lambda^n)r < r, \tag{3.6}$$

$x_n \in B_r(x_0)$ and the construction is then achieved. Hence the sequence $(x_n)_{n\geq 2}$ is well defined. Let us check now that it is a Cauchy sequence. Indeed, it follows from (3.4) that for any $n \geq 2$ and any $p \in \mathbb{N}$,

$$d(x_n, x_{n+p}) \leq \sum_{k=n}^{n+p-1} d(x_k, x_{k+1}) < \sum_{k=n}^{n+p-1} \lambda^k\beta < \lambda^n\Big(\frac{1-\lambda^p}{1-\lambda}\Big)\beta.$$

We conclude that $(x_n)_{n\geq 2}$ is a Cauchy sequence and as it lies in $\bar{B}_r(x_0)$ which is complete, it converges to some $\bar{x} \in \bar{B}_r(x_0)$. Remark that, by (3.6), the sequence satisfies

$$d(x_0, x_n) < \frac{1-\lambda^n}{1-\lambda}\beta,$$

so that

$$d(x_0, \bar{x}) \leq (1-\lambda)^{-1}\beta < r.$$

Thus $\bar{x} \in B_r(x_0)$. Applying assumption (1) of lower semicontinuity of the function $d(,T(\cdot))$, it results that

$$d(\bar{x}, T(\bar{x})) \leq \liminf_{n\to\infty} d(x_n, T(x_n)) \leq \lambda \liminf_{n\to\infty} d(x_{n-1}, x_n) = 0,$$

since (x_n) converges. Thus $\bar{x} \in T(\bar{x})$ ($T(\bar{x})$ being closed), that is, Fix$T \neq \emptyset$.

On the other hand, for any $n \geq 2$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \lambda d(x_{n-1}, x_n) + \lambda^{n-1}\varepsilon' \\ &\leq \lambda^2 d(x_{n-2}, x_{n-1}) + 2\lambda^{n-1}\varepsilon' \\ &\vdots \\ &\leq \lambda^{n-1} d(x_1, x_2) + (n-1)\lambda^{n-1}\varepsilon'. \end{aligned}$$

It follows that

$$\begin{aligned} d(x_1, x_n) &\leq \sum_{k=1}^{n-1} d(x_k, x_{k+1}) \leq \sum_{k=0}^{n-2} \lambda^k d(x_1, x_2) + \sum_{k=0}^{n-2} k\lambda^k \varepsilon' \\ &\leq (1-\lambda)^{-1} d(x_1, x_2) + \sum_{k=0}^{n} k\lambda^k \varepsilon' \end{aligned}$$

and, by letting $n \to \infty$, we obtain

$$\begin{aligned} d(x_1, \bar{x}) &\leq (1-\lambda)^{-1}(d(x_1, x_2) + \varepsilon) \\ &\leq (1-\lambda)^{-1}(\gamma + 2\varepsilon), \end{aligned}$$

which leads to $d(x_1, \bar{x}) \leq (1-\lambda)^{-1}\gamma$, since ε is arbitrary. Thus

$$\begin{aligned} d(x_1, \mathrm{Fix}(T)) \leq d(x_1, \bar{x}) &\leq \frac{1}{1-\lambda} \sup_{x \in S(x)\cap B_\beta(x_0)} d\left(x, T\left(x\right)\right) \\ &\leq \frac{1}{1-\lambda} \sup_{x \in B_r(x_0)} e\left(S\left(x\right) \cap B_\beta\left(x_0\right), T(x)\right) \end{aligned} \tag{3.7}$$

and the conclusion (3.3) follows by taking the supremum over $\mathrm{Fix}(S) \cap B_\beta\left(x_0\right)$. □

Remark 3.2.

(i) It is proved in [14] that any multifunction $T : X \rightrightarrows X$ with nonempty closed values on a complete metric space which satisfies assumptions (1) and (2) and such that $d(x_0, T(x_0) < (1-\lambda)\, r$ admits at least a fixed point and $d\left(x_0, \mathrm{Fix}(T)\right) \leq \frac{1}{1-\lambda} d\left(x_0, T(x_0)\right)$. Thus, it is an immediate result that $\mathrm{Fix}(T)$ is nonempty whenever the condition (3.2) is satisfied for some $x_1 \in \mathrm{Fix}(S) \cap B_\beta(x_0)$.

(ii) Let us point out that the assumption (1) in the above theorem could be weakened in such a way that the restriction of the function $x \to d(x, T(x))$ on $B_r(x_0)$ is lower semicontinuous. Note that the lower semicontinuity of the restriction on $B_r(x_0)$ is also weaker than the lower semicontinuity of the function on the subset $B_r(x_0)$ (with respect to the whole space), see [2, 3].

4. APPLICATION TO DIFFERENTIAL INCLUSIONS

Now, we illustrate our results by applications to the stability of solutions sets for differential inclusions in some functional spaces. In this study, the solutions sets for differential inclusions will be the fixed points sets of some suitable multifunctions.

4.1. OVERVIEW ON THE EXISTENCE OF SOLUTIONS OF DIFFERENTIAL INCLUSIONS

We present here the data of the problem and recall the question of the existence of solutions for the differential inclusion:

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) & \text{a.e. on } I, \\ x(0) = \xi, \end{cases} \tag{D.I.}$$

where $F : I \times E \rightrightarrows E$ is a multifunction, E is a separable Banach space, $\xi \in E$ and $I := [0, T]$ with $T > 0$.

Recall that a solution $x(\cdot)$ of the differential inclusion (D.I.) is an element of the space $X := W^{1,1}(I, E)$ of continuous functions $x : I \to E$ such that there exists $u \in \mathcal{L}^1(I, E)$ (the space of Bochner integrable functions from I into E) satisfying

$$x(t) = x(0) + \int_0^t u(s)\, ds \quad \text{for all} \quad t \in I.$$

In the sequel we shall endow the space $W^{1,1}(I, E)$ with the usual norm given by

$$\|x\|_X = \|x(0)\| + \int_0^T \|\dot{x}(s)\|\, ds \tag{4.1}$$

and we denote by $\mathrm{S}_F(\xi)$ the solutions set of the differential inclusion (D.I.).

Following the work of Filippov [21] (see also [7, 8, 14, 19, 25, 30, 37]), the following assumptions in which $r > 0$, $\theta(\cdot) \in \mathcal{L}^1(I)$ and $x_0 \in X$ with

$$x_0(t) = x_0(0) + \int_0^t u_0(s)\, ds \quad \text{a.e. on } I$$

are classical for the existence of solutions of the differential inclusion (D.I.) on some interval $[0, \tau]$ with $\tau \in (0, T)$:

(H_1) for each $(t,e) \in \bigcup_{t\in I} \{t\}\times \bar{B}_r(x_0(t))$, the set $F(t,e)$ is nonempty and closed, and $F(\cdot,e)$ is measurable. That is, there exists a sequence $(g_n(\cdot,e))_n$ of measurable mappings from I into E such that $g_n(t,e) \in F(t,e)$ a.e. on I for all $n \in \mathbb{N}$ and $F(t,e) \subset \overline{\bigcup_{n\in\mathbb{N}} g_n(t,e)}$ a.e. on I (see [7] for more details);

(H_2) for a.e. $t \in I$, the multifunction $F(t,\cdot)$ is $\theta(t)$-Lipschitzian on $\bar{B}_r(x_0(t))$;

(H_3) $\rho(\cdot) := d(u_0(\cdot), F(\cdot, x_0(\cdot)))$ is in $\mathcal{L}^1(I)$.

Note that Filippov's result has been recently generalized in [14, Theorem 3.1] (see also [26]), where the assumption (H_2) is weakened to the following:

$(\bar{H}_2)$ There exists $\kappa \geq 0$ such that for any $(t,x_1), (t,x_2) \in V_r$, where $V_r := \bigcup_{t\in I}\{t\}\times B_{2r}(x_0(t))$, one has

$$u_1 \in F(t,x_1) \Longrightarrow d(u_1, F(t,x_2)) \leq (\theta(t) + \kappa\|u_1\|)\|x_1 - x_2\|,$$

and the generalized result is as follows:

$$V := \int_0^T e^{-\bar{\theta}t}\|u_0(t)\|\,dt, \quad \bar{V} := \int_0^T e^{-\bar{\theta}t}\rho(t)dt, \quad \text{where} \quad \bar{\theta} := \int_0^T \theta(t)dt.$$

Theorem 4.1 ([14]). *Suppose the assumptions (H_1), $(\bar{H}_2)$ hold, $e^{\bar{\theta}}\beta(r+V) < 1$ and*

$$e^{2\bar{\theta}}\left(\bar{V} + \delta\left(1 - e^{-\bar{\theta}} + \beta\left(V + \bar{V}\right)\right)\right) < r\left(1 - e^{\bar{\theta}}\beta(r+V)\right),$$

where $\delta \in (0,r)$. Then, for all $s_0 \in [0,T]$ and for all $\xi_0 \in B_\delta(x_0(s_0))$, there exists a solution $x \in X$ of

$$\begin{cases} \dot{x}(t) \in F(t,x(t)) & \text{a.e. on } I, \\ x(s_0) = \xi_0. \end{cases}$$

The following lemma is useful for our study.

Lemma 4.2 ([8]). *Let $G : I \rightrightarrows E$ be a measurable multifunction with values in a Banach space E. Let $v_0 : T \to E$ and $\gamma : I \to (0,+\infty)$ be measurable. Then there exists a measurable mapping $v : I \to E$ such that*

$$v(t) \in G(t) \quad \text{and} \quad \|v(t) - v_0(t)\| \leq d(v_0(t), G(t)) + \gamma(t) \quad \text{a.e. on } I.$$

4.2. THE BEHAVIOR OF SOLUTIONS SETS FOR DIFFERENTIAL INCLUSIONS

Our purpose now is to study the behavior of solutions sets $\mathrm{S}_F(\xi)$ when the right hand side of the differential inclusion (D.I.) and the initial point vary. For this aim, consider a family of differential inclusions:

$$\begin{cases} \dot{x}(t) \in F_\omega(t,x(t)) & \text{a.e. on } I, \\ x(0) = \xi, \end{cases} \tag{D.I.$_\mathrm{w}$}$$

where $F_\omega : I \times E \rightrightarrows E$ are multifunctions with nonempty closed values parameterized by $\lambda \in \Lambda$ (Λ is a topological space) and let us assume that there exist $r > 0$, $\omega_0 \in \Lambda$, $\xi_0 \in E$ and $\theta(\cdot) \in \mathcal{L}^1(I)$ such that:

(i) for each $(\omega, e) \in \Lambda \times B_r(\xi_0)$, $F_\omega(\cdot, e)$ is measurable and for any $(t, \omega) \in I \times \Lambda$, $F_\omega(t, \cdot)$ is Hausdorff upper semicontinuous, that is,

$$e\left(F_\omega(t, e), F_\omega(t, e_0)\right) \underset{e \to e_0}{\rightarrow} 0;$$

(ii) there exists $\kappa \geq 0$ such that for any $(t, e_1), (t, e_2) \in I \times B_r(\xi_0)$ and $u_1 \in F_\omega(t, e_1)$ one has

$$d\left(u_1, F_\omega(t, e_2)\right) \leq \left(\theta(t) + \kappa \|u_1\|\right) \|e_1 - e_2\| \quad \text{for all} \quad \omega \in \Lambda;$$

(iii) $\rho(\cdot) := d\left(0, F_{\omega_0}(\cdot, \xi_0)\right)$ is in $\mathcal{L}^1(I)$.

In the sequel, we denote by $T_y : X \rightrightarrows X$ the multifunction defined as follows:

$$z \in T_y(x) \Longleftrightarrow \begin{cases} \text{there exists } u \in \mathcal{L}^1(I, E) \text{ such that } u(t) \in F_\omega(t, x(t)) \text{ a.e. } t \in I, \\ z(t) = \xi + \int_0^t u(s)\, ds \end{cases} \tag{4.2}$$

for any $y := (\omega, \xi) \in \Lambda \times B_r(\xi_0)$ fixed. Observe that $\mathrm{S}_{F_\omega}(\xi) = \mathrm{Fix}(T_y)$.

Lemma 4.3. *Let Λ be a topological space and $\left(F_\omega : I \times E \rightrightarrows E\right)_\omega$ a family of multifunctions with nonempty closed values and satisfying the assumptions* (i)*,* (ii) *and* (iii)*. Assume that the function $\epsilon(\cdot) := e\left(F_{\omega_0}(\cdot, e), F_\omega(\cdot, e)\right)$ is integrable on I for any $(\omega, e) \in \Lambda \times B_r(\xi_0)$. Then the values of T_y are nonempty and closed on $B_r(\xi_0)$.*

Proof. Let us prove first that T_y is nonempty valued on X . Since $\mathrm{S}_{F_{\omega_0}}(\xi_0) \neq \emptyset$ by Theorem 4.1, there exists $x_0 \in X$ such that $x_0(t) = \xi_0 + \int_0^t u_0(s)\, ds$ a.e. on I with $u_0 \in \mathcal{L}^1(I, E)$ and $u_0(t) \in F_{\omega_0}(t, x_0(t))$. Consider now $x \in B_r(\xi_0)$ with $x(t) = x(0) + \int_0^t u(s)\, ds$ a.e. on I and $u \in \mathcal{L}^1(I, E)$. Hence from assumptions (ii) and (4.4), we get

$$d\left(u_0(t), F_\omega(t, x(t))\right) \leq d\left(u_0(t), F_{\omega_0}(t, x(t))\right) + e\left(F_{\omega_0}(t, x(t)), F_\omega(t, x(t))\right) \leq \rho(t),$$

where $\rho(\cdot) : t \to \left(\theta(t) + \kappa \|u_0(t)\|\right) \|x_0(t) - x(t)\| + \epsilon(t)$ is in $\mathcal{L}^1(I)$. And as the multifunction $t \to F_\omega(t, x(t))$ is measurable (see [7, 35]), there exists from Lemma 4.2 (see also [37, Lemma 3.1]), $v \in \mathcal{L}^1(I, E)$ such that

$$v(t) \in F_\omega(t, x(t)) \quad \text{and} \quad \|u_0(t) - v(t)\| \leq \rho(t) \text{ a.e. on } I.$$

Thus $z \in T_y(x)$ by setting $z(t) := \xi + \int_0^t v(s)\, ds$ a.e. on I so that $T_y(x) \neq \emptyset$.

Consider now for $x \in B_r(\xi_0)$, a sequence $(z_n)_n$ such that for any n, $z_n \in T_y(x)$ and (z_n) converges to z in X.

Since $z_n(t) = \xi + \int_0^t u_n(s)\,ds$ and $z(t) = z(0) + \int_0^t u(s)\,ds$ with $u_n, u \in \mathcal{L}^1(I, E)$ and $u_n(t) \in F_\omega(t, x(t))$ a.e. $t \in I$ (for any n), we get

$$z(0) = \xi \quad \text{and} \quad u_n \to u \text{ in } \mathcal{L}^1(I, E).$$

Hence we can pick a subsequence $\left(u_{s(n)}\right)$ such that $u_{s(n)}(t) \to u(t)$ a.e. on I and by the closeness of $F_\omega(t, x(t))$, $u(t) \in F_\omega(t, x(t))$ a.e. $t \in I$. Thus $z \in T_y(x)$, that is, the values of T_y are closed. □

Theorem 4.4. *Let Λ be a topological space and $\left(F_\omega : I \times E \rightrightarrows E\right)_\omega$ a family of multifunctions with nonempty closed values and satisfying* (i), (ii) *and* (iii) *with the condition*

$$\lambda := \kappa r + \bar{\theta} < 1, \tag{4.3}$$

where $\bar{\theta} := \int_0^T \theta(t)dt$. Assume that for any $\beta > 0$ and any $\omega \in \Lambda$, there exists $\epsilon_\beta(\omega, \cdot) \in \mathcal{L}^1(I)$ such that

$$e\left(F_{\omega_0}(t, e), F_\omega(t, e)\right) \leq \epsilon_\beta(\omega, t), \text{ for all } e \in B_\beta(\xi_0) \text{ and } t \in I \tag{4.4}$$

with $\epsilon_\beta(\omega, \cdot) \underset{\omega \to \omega_0}{\longrightarrow} 0$ in $\mathcal{L}^1(I)$, for all $\beta > 0$. Then for all $\beta \in (0, (1-\lambda)r)$, there exist two neighborhoods $\mathcal{U}$ of ω_0 and $\mathcal{V}$ of ξ_0 such that for any $(\omega, \xi) \in \mathcal{U} \times \mathcal{V}$

$$e\left(\mathrm{S}_{F_{\omega_0}}(\xi_0) \cap B_\beta(\xi_0), \mathrm{S}_{F_\omega}(\xi)\right) \leq \frac{1}{1-\lambda}\left(\|\xi - \xi_0\| + \|\epsilon_\beta(\omega, \cdot)\|_{\mathcal{L}^1(I)}\right).$$

Proof. Let $\delta \in \left(0, \frac{r}{2}\right)$, $\gamma := \lambda(1-\lambda)\frac{\delta}{2}$ and let $y := (\omega, \xi) \in \Lambda \times B_\gamma(\xi_0)$ fixed. Define the multifunction $T_y : X \rightrightarrows X$ as in the relation (4.2). Then, from the previous lemma, T_y has nonempty closed valued. And clearly, the function $x \to d(x, T_y(x))$ is lower semicontinuous. Let us prove now that assumption (2) of Theorem 3.1 is fulfilled. Indeed, let $x \in B_\delta(\xi_0)$ and let $z \in T_y(x) \cap B_\delta(\xi_0)$ so that $z(t) = \xi + \int_0^t u(s)\,ds$ with $u \in \mathcal{L}^1(I, E)$ and $\dot{z}(t) = u(t) \in F_\omega(t, x(t))$ a.e. on I. We have to show that

$$d(z, T_y(z)) \leq \lambda \|x - z\|.$$

Applying [6, Theorem 2, p. 91], we get some $v \in \mathcal{L}^1(I, E)$ satisfying $v(t) \in F_\omega(t, z(t))$ and $\|u(t) - v(t)\| = d(u(t), F_\omega(t, z(t)))$ a.e. on I. Define $z' \in X$ such that $z'(t) := \xi + \int_0^t v(s)\,ds$ so that $z' \in T_y(z)$ and then, $\dot{z}'(t) = v(t)$. Thus we have the following inequalities

$$\begin{aligned} d(z, T_y(z)) \leq \|z - z'\|_X &= \int_0^T \|u(t) - v(t)\|\,dt \\ &\leq \int_0^T d(u(t), F_\omega(t, z(t)))\,dt \\ &\leq \int_0^T (\theta(t) + \kappa\|u(t)\|)\,\|x(t) - z(t)\|\,dt. \end{aligned}$$

And since

$$\|x(t)-y(t)\| \leq \|x(0)-\xi\| + \int_0^t \left\|\dot{x}(s)-\dot{z}(s)\right\| ds,$$

then

$$\begin{aligned} d(z, T_y(z)) &\leq \int_0^T k(t)\left(\|x(0)-\xi\| + \int_0^t \left\|\dot{x}(s)-\dot{z}(s)\right\| ds\right) dt \\ &\leq K_0 \|x(0)-\xi\| + \int_0^T K_s \left\|\dot{x}(s)-\dot{z}(s)\right\| ds, \end{aligned}$$

where $k(t) := \theta(t) + \kappa \|u(t)\|$ and $K_s := \int_s^T k(t)dt$ for $s \in [0,T)$.

On the other hand, since

$$\int_0^T \|u(t)\| \, dt = \|z-\xi\| < \delta + \gamma < r,$$

thus

$$K_s \leq K_0 < \bar{\theta} + \kappa r.$$

Hence, for any $x \in B_\delta(\xi_0)$ and $z \in T_y(x) \cap B_\delta(\xi_0)$, one has

$$d(z, T_y(z)) \leq \lambda \|x-z\|$$

with $\lambda := \bar{\theta} + \kappa r < 1$ by assumption (4.3). This leads to condition (2) of Theorem 3.1.

It remains to prove that for $\beta \in (0, (1-\lambda)\delta)$

$$d(x, T_y(x)) < \lambda\beta \quad \text{for any} \quad x \in \operatorname{Fix}(T_{y_0}) \cap B_\beta(\xi_0).$$

Let us fix now $\xi \in \mathcal{V} := B_{\bar{\gamma}}(\xi_0)$ with $\bar{\gamma} := \lambda(1-\lambda)\beta$ and take $x \in \operatorname{Fix}(T_{y_0}) \cap B_\beta(\xi_0)$ so that $x(t) = \xi_0 + \int_0^t u_0(s)\,ds$ a.e. on I with $u_0 \in \mathcal{L}^1(I,E)$ and $u_0(t) \in F_{\omega_0}(t,x(t))$. As previously, one can pick $u \in \mathcal{L}^1(I,E)$ satisfying $u(t) \in F_\omega(t,x(t))$ and $\|u_0(t)-u(t)\| = d(u_0(t), F_\omega(t,x(t)))$. Thus for $z(t) := \xi + \int_0^t u(s)\,ds$ a.e. on I, we have $z \in T_y(x)$ and by (4.4) and using the fact that $x(t) \in B_\beta(\xi_0)$ for all t, we have

$$\begin{aligned} \|x-z\| &= \|\xi-\xi_0\| + \int_0^T \|u(s)-u_0(s)\| \, ds \leq \|\xi-\xi_0\| + \int_0^T d(u_0(t), F_\omega(t,x(t)))\, dt \\ &\leq \|\xi-\xi_0\| + \int_0^T e(F_{\omega_0}(t,x(t)), F_\omega(t,x(t)))dt \leq \|\xi-\xi_0\| + \int_0^T \epsilon_\beta(\omega,t)\, dt \\ &\leq \|\xi-\xi_0\| + \|\epsilon_\beta(\omega,\cdot)\|_{\mathcal{L}^1(I)} < \bar{\gamma} + \lambda^2\beta < \lambda\beta \end{aligned}$$

for any $\omega \in \mathcal{U}$, where $\mathcal{U}$ is a neighborhood of ω_0 such that

$$\|\epsilon_\beta(\omega,\cdot)\|_{\mathcal{L}^1(I)} < \lambda^2\beta \quad \text{for any} \quad \omega \in \mathcal{U},$$

and this is possible since $\|\epsilon_\beta(\omega,\cdot)\|_{\mathcal{L}^1(I)} \underset{\omega\to\omega_0}{\to} 0$.

Observe finally that for any $x \in \mathrm{Fix}\,(T_{y_0}) \cap B_\beta(\xi_0)$, we have

$$d(x, T_y(x)) \le \|x - z\| < \bar{\gamma} + \lambda^2\beta < \lambda\beta.$$

Hence following Theorem 3.1, we obtain that for any $y = (\omega,\xi) \in \mathcal{U} \times \mathcal{V}$,

$$e\left(\mathrm{Fix}\,(T_{y_0}) \cap B_\beta(\xi_0), \mathrm{Fix}\,(T_y)\right) \le \frac{1}{1-\lambda} \sup_{x\in B_r(\xi_0)} d\left(T_{y_0}(x) \cap B_\beta(\xi_0), T_y(x)\right).$$

Observe now that for any $x \in B_r(\xi_0)$,

$$e\left(T_{y_0}(x) \cap B_\beta(\xi_0), T_y(x)\right) \le \|\xi - \xi_0\| + \|\epsilon_\beta(\omega,\cdot)\|_{\mathcal{L}^1(I)}.$$

Indeed, let $x \in B_r(\xi_0)$ and let $z \in T_{y_0}(x) \cap B_\beta(\xi_0)$ so that $z(t) = \xi + \int_0^t u(s)\,ds$ with $u \in \mathcal{L}^1(I,E)$ and $z'(t) = u(t) \in F_{\omega_0}(t, x(t))$ a.e. on I. Thus, for any $z' \in T_y(x)$, we have $d(z, T_y(x)) \le \|z - z'\|$ so that for all $v \in \mathcal{L}^1(I,E)$ such that $v(t) \in F_\omega(t,x(t))$ a.e. on I, we have

$$d(z, T_y(x)) \le \|\xi - \xi_0\| + \int_0^T \|u(s) - v(s)\|\,ds.$$

Taking the infimum over $F_\omega(t, x(t))$, we get (see [35])

$$d(z, T_y(x)) \le \|\xi - \xi_0\| + \int_0^T d\left(u(t), F_\omega(t, x(t))\right) dt$$

and since $u(t) \in F_{\omega_0}(t, x(t))$,

$$\begin{aligned} d(z, T_y(x)) &\le \|\xi - \xi_0\| + \int_0^T e\left(F_{\omega_0}(t, x(t)), F_\omega(t, x(t))\right) dt \\ &\le \|\xi - \xi_0\| + \|\epsilon_\beta(\omega,\cdot)\|_{\mathcal{L}^1(I)}, \end{aligned}$$

which leads to the conclusion. □

5. STABILITY OF FIXED POINTS SETS WITH RESPECT TO PROXIMAL CONVERGENCE

Now, we will be concerned with the stability of fixed point sets of multifunctions with respect to proximal convergence and especially, the bounded proximal convergence which has been first introduced in [31] for optimization purposes. The bounded proximal convergence fits very well in the collection of all set convergence and has been the subject of interest of many authors. For more details on the bounded proximal convergence and its corresponding hyperspace topology, the bounded proximal topology, as well as for others notions of set convergence which abound in the literature, we refer to [5, 10–13, 24, 28, 33, 34, 36] and the references therein.

Let (X, d) be a metric space. We denote by $B(X)$ the set of nonempty closed and bounded subsets of X.

A sequence $(A_n)_n$ of subsets of X is said to be *upper bounded proximal convergent to* A if

$$\lim_{n\to+\infty} e\left(A_n \cap B, A\right) = 0 \quad \text{for all} \quad B \in B(X).$$

The sequence $(A_n)_n$ is said to be bounded proximal convergent to A if it is lower and upper bounded proximal convergent to A. Note that $(A_n)_n$ lower bounded proximal converges to A if $A \subset \liminf_{n\to+\infty} A_n$, where $\liminf_{n\to+\infty} A_n$ is given in the classical sense of Painelevé-Kuratowski by

$$\begin{aligned}\liminf_{n\to+\infty} A_n &= \left\{x \in X \mid \limsup_{n\to+\infty} d(x, A_n) = 0\right\} \\ &= \left\{x \in X \mid \text{ for each } n \text{ there exists } x_n \in A_n \text{ such that } x_n \to x\right\}.\end{aligned}$$

It is well-known that the bounded proximal convergence is weaker than both Fisher convergence and Attouch-Wets convergence and stronger than Wijsman convergence, see [36, Corollary 9.4]. Note that the Fisher convergence is the sequential proximal convergence.

The following result on the continuous dependence of fixed point sets of multifunctions will be useful in the sequel.

Lemma 5.1. *Let (X, d) be a metric space. Suppose all the conditions of Theorem 3.1 are satisfied. Then, for every subset B of X, we have*

$$e(\operatorname{Fix}(S) \cap B_\beta(x_0) \cap B, \operatorname{Fix}(T)) \le \frac{1}{1-\lambda} \sup_{x\in B_r(x_0)} e\left(S(x) \cap B_\beta(x_0) \cap B, T(x)\right).$$

Proof. The proof comes easily from the fact that the multifunction $S \cap B$ satisfies all the conditions of Theorem 3.1 whenever S satisfies them. □

An adaptation of the upper bounded proximal convergence to the framework of multifunctions yields the following definition. Let M be a subset of X, a sequence of multifunctions $(T_n)_n$ is said to be *upper bounded proximal convergent* to a multifunction T in M if

$$\lim_{n\to+\infty} \sup_{x\in M} e\left(T_n(x) \cap B, T(x)\right) = 0, \quad \text{for all} \quad B \in B(X).$$

Clearly, if $(T_n)_n$ is upper bounded proximal convergent to T in M, then for every subset G of X (not necessarily closed), the sequence $(T_n \cap G)_n$ is upper bounded proximal convergent to T in M.

Now, we derive the following result on the stability of fixed point sets of multifunctions with respect to the bounded proximal convergence.

Theorem 5.2. *Let (X,d) be a metric space. Let $x_0 \in X$ and $r \in (0,+\infty]$ be such that $\overline{B}_r(x_0)$ is a complete subspace. Let $\lambda \in (0,1)$ and $T \in \mathcal{F}_{\overline{B}_r(x_0)}(X)$ with at least one fixed point. Assume that the conditions* (1) *and* (2) *of Theorem 3.1 are satisfied for T and the condition* (3) *is satisfied for every T_n, where $(T_n)_n$ is a sequence of multifunctions. If the sequence $(T_n \cap B_\beta(x_0))_n$ is upper bounded proximal convergent to T in $B_r(x_0)$, then the sequence $(\mathrm{Fix}\,(T_n \cap B_\beta(x_0)))_n$ is upper bounded proximal converging to* $\mathrm{Fix}\,(T)$.

Proof. By hypothesis, for all $B \in B(X)$ and $\varepsilon > 0$ there exists N such that for all $n \geq N$ we have

$$\sup_{x \in B_r(x_0)} e\left(T_n(x) \cap B_\beta(x_0) \cap B, T(x)\right) < (1-\lambda)\,\varepsilon.$$

By applying Lemma 5.1, we obtain

$$\begin{aligned}\forall B \in B(X)\ \forall \varepsilon > 0\ \exists N\ \forall n \geq N\ \ & e\left(\mathrm{Fix}\,(T_n) \cap B_\beta\,(x_0) \cap B, \mathrm{Fix}(T)\right)\\ & \leq \frac{1}{1-\lambda} \sup_{x \in B_r(x_0)} e\left(T_n(x) \cap B_\beta(x_0) \cap B, T(x)\right) < \varepsilon\end{aligned}$$

which completes the proof. □

Here some results on the stability of fixed point sets of multifunctions. The following result should be compared to [22, Corollary 4.7].

Corollary 5.3. *Suppose the conditions of the above theorem hold and the sequence $(T_n \cap B_\beta(x_0))$ is upper bounded proximal converging to T in $B_r(x_0)$. If $(x_n)_n$ is a converging sequence to x and $x_n \in \mathrm{Fix}\,(T_n) \cap B_\beta(x_0)$, for every n, then $x \in \mathrm{Fix}(T)$.*

Proof. By assumptions, we have $d\,(x_n, x_0) < \beta < (1-\lambda)\,r$, for every n. Since $(x_n)_n$ is a convergent sequence to x, then the set

$$B = \{x_n \mid n\} \cup \{x\}$$

is closed and bounded (compact). Let $\varepsilon \in (0, \lambda r)$. By the above theorem, let N_ε be such that

$$e\left(\mathrm{Fix}\,(T_n) \cap B_\beta\,(x_0) \cap B, \mathrm{Fix}(T)\right) < \varepsilon \quad \text{for all} \quad n \geq N_\varepsilon.$$

For every $n \geq N_\varepsilon$, choose $x'_n \in \mathrm{Fix}(T)$ such that $d\,(x_n, x'_n) < \varepsilon$. Clearly, $x'_n \in B_r(x_0)$.

Now, let $k_0 \in \mathbb{N}$ be such that $\frac{1}{k_0} \in (0, \lambda r)$. By choosing $\varepsilon = \frac{1}{k}$ with $k \geq k_0$, we construct a subsequence $(x_{n_k})_k$ of $(x_n)_n$ and a sequence $\left(x'_{n_k}\right)_k$ such that:

1. $x'_{n_k} \in \mathrm{Fix}(T) \cap B_r(x_0)$ for every $k \geq k_0$,
2. $d\left(x_{n_k}, x'_{n_k}\right) < \frac{1}{k}$.

The sequence $\left(x'_{n_k}\right)_k$ is converging to x and by condition (1) on the semicontinuity and since $x'_{n_k} \in T\left(x'_{n_k}\right)$, for every k, we have

$$d(x, T(x)) \leq \liminf_{k \to +\infty} d\left(x'_{n_k}, T\left(x'_{n_k}\right)\right) = 0.$$

Since $x \in \overline{B}_r(x_0)$, it results that $T(x)$ is closed and then, $x \in T(x)$ which completes the proof. □

The following result should be compared to [27, Theorem 1].

Corollary 5.4. *Suppose the conditions of the above corollary hold with X a real Banach space, $(x_n)_n$ is a weakly converging sequence to x and $x_n \in \mathrm{Fix}(T_n) \cap B_\beta(x_0)$, for every n. If in addition $T(x)$ is weakly closed and the condition (1) of Theorem 3.1 holds for weakly converging sequences, then $x \in \mathrm{Fix}(T)$.*

Proof. By the same proof as in Corollary 5.3, we construct a subsequence $(x_{n_k})_k$ of $(x_n)_n$ and a sequence $\left(x'_{n_k}\right)_k$ such that:

1. $x'_{n_k} \in \mathrm{Fix}(T) \cap B_r(x_0)$ for every $k \geq k_0$,
2. $d\left(x_{n_k}, x'_{n_k}\right) < \frac{1}{k}$.

The sequence $\left(x'_{n_k}\right)_k$ is weakly converging to x and we have

$$d(x, T(x)) \leq \liminf_{k \to +\infty} d\left(x'_{n_k}, T\left(x'_{n_k}\right)\right) = 0.$$

Since $T(x)$ is weakly closed, we have $x \in T(x)$. □

Remark 5.5. Note that Corollary 5.4 is obtained under the upper bounded proximal convergence for multifunctions considered in the paper which is a definition of a kind of "*uniform convergence*" in the sense of the bounded proximal convergence. In view of [27, Theorem 2] and even if the Banach space E satisfies Opial's condition, it is not clear whether the result remains true when we take only the "*pointwise convergence*" in the sense of the bounded proximal convergence.

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