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RUIN PROBABILITY IN A RISK MODEL WITH VARIABLE PREMIUM INTENSITY AND RISKY INVESTMENTS

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Abstract. We consider a generalization of the classical risk model when the premium intensity depends on the current surplus of an insurance company. All surplus is invested in the risky asset, the price of which follows a geometric Brownian motion. We get an exponential bound for the infinite-horizon ruin probability. To this end, we allow the surplus process to explode and investigate the question concerning the probability of explosion of the surplus process between claim arrivals.

Keywords: risk process, infinite-horizon ruin probability, variable premium intensity, risky investments, exponential bound, stochastic differential equation, explosion time, existence and uniqueness theorem, supermartingale property.

Mathematics Subject Classification: 91B30, 60H10, 60G46.

1. INTRODUCTION

Since Lundberg introduced the collective risk model in 1903, the estimation of the ruin probability has been one of the central directions for investigations in risk theory. It is well known that in the Cramér-Lundberg model, which is also called the classical risk model, the infinite-horizon ruin probability decreases exponentially with the initial surplus if the claim sizes have exponential moments and the net profit condition holds. Results concerning bounds and asymptotics for the ruin probability were also obtained for different generalizations of the classical risk model under various assumptions (see, e.g., [2, 7, 20] and the references given there).

Risk models that allow the insurance company to invest are of great interest. The fact that risky investments can be dangerous was first justified mathematically by Kalashnikov and Norberg [12]. They modelled the basic surplus process due to insurance activity and the price of the risky asset by Lévy processes and obtained upper and lower power bounds for the ruin probability when the initial surplus is large enough. Later, Paulsen [18] and Yuen, Wang, Wu [22] considered some generalizations of these results.

Frolova, Kabanov and Pergamenshchikov [5] used the bounds obtained in [12] to show that the ruin occurs with probability 1 in the classical risk model if all surplus is invested in the risky asset, the price of which is modelled by a geometric Brownian motion, and some additional conditions for parameters of the geometric Brownian motion hold. They also showed that if these conditions are not fulfilled, a power asymptotic is true for the ruin probability when the claim sizes are exponentially distributed. The power asymptotic was got by Cai and Xu [4] in the case where the classical risk process is perturbed by a Brownian motion. Moreover, Pergamenshchikov and Zeitouny [19] considered the risk model where the premium intensity is a bounded nonnegative random function and generalized results of [5].

On the other hand, numerous results indicate that risky investments can be used to improve the solvency of the insurance company. For example, Gaier, Grandits and Schachermayer [6] considered the classical risk model under the additional assumptions that the company is allowed to borrow and invest in the risky asset, the price of which follows a geometric Brownian motion. They obtained an upper exponential bound for the ruin probability when the claim sizes have exponential moments and a fixed quantity, which is independent of the current surplus, is invested in the risky asset. It appears that this bound is better then the classical one. For an exponential bound in a model with risky investments see also, for instance, [16].

Numerous investigations are devoted to solving optimal investment problems from the viewpoint of the infinite-horizon ruin probability minimization. For instance, Hipp and Plum [9], Liu and Yang [15], Azcue and Muler [3] considered the optimal investment problem in the classical risk model when the company is allowed to borrow. Asymptotics for the ruin probability under optimal strategies were obtained by Hipp and Schmidli [10], Grandits [8], Schmidli [21] for different assumptions about claim sizes.

We consider a generalization of the classical risk model when the premium intensity depends on the current surplus of the insurance company, which is invested in the risky asset. Our main aim is to show that if the premium intensity grows rapidly with increasing surplus, then an exponential bound for the ruin probability holds under certain conditions in spite of the fact that all surplus is invested in the risky asset. To this end, we allow the surplus process to explode. To be more precise, we let the premium intensity be a quadratic function. In addition, we investigate the question concerning the probability of explosion of the surplus process between claim arrivals in detail.

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space satisfying the usual conditions and all the objects be defined on it. We assume that the insurance company has a nonnegative initial surplus x and denote by $X_t(x)$ its surplus at time $t \ge 0$. For simplicity of notation, we write X_t instead of $X_t(x)$ when no confusion can arise. Let $c\colon \mathbb{R} \to \mathbb{R}_+ \backslash \{0\}$ be a measurable function such that $c(u) = c(0)$ for all $u < 0$ and $c(X_t)$ be a premium intensity that depends on the surplus at time t.

Next, we suppose that the claim sizes form a sequence $(Y_i)_{i\geq 1}$ of nonnegative i.i.d. random variables with finite expectations μ. We denote by τ_i the time when the ith claim arrives. For convenience we set $\tau_0 = 0$.

Let $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ be the shifted moment generating function of Y_i such that $h(0) = 0$, i.e.

$$h(r) = \mathbb{E}\big[\mathrm{e}^{rY_i}\big] - 1.$$

We make the following classical assumption concerning $h(r)$: there exists $r_\infty \in (0, +\infty]$ such that $h(r) < +\infty$ for all $r \in [0, r_\infty)$ and $\lim_{r\uparrow r_\infty} h(r) = +\infty$ (see [7, p. 2]). It is easily seen that $h(r)$ is increasing, convex, and continuous on $[0, r_\infty)$.

The number of claims on the time interval $[0, t]$ is a Poisson process $(N_t)_{t\geq 0}$ with constant intensity $\lambda > 0$. Thus, the total claims on $[0, t]$ equal $\sum_{i=1}^{N_t} Y_i$. We set $\sum_{i=1}^{0} Y_i = 0$ if $N_t = 0$.

In addition, we assume that all surplus is invested in the risky asset, the price of which equals S_t at time t. We model the process $(S_t)_{t\geq 0}$ by a geometric Brownian motion. Thus,

$$dS_t = S_t(a\,dt + b\,dW_t), \tag{1.1}$$

where $a \in \mathbb{R}$, $b > 0$, and $(W_t)_{t\geq 0}$ is a standard Brownian motion. We suppose that the random variables $(Y_i)_{i\geq 1}$ and the processes $(N_t)_{t\geq 0}$ and $(W_t)_{t\geq 0}$ are independent.

Let $(\mathfrak{F}_t)_{t\geq 0}$ be a filtration generated by $(Y_i)_{i\geq 1}$, $(N_t)_{t\geq 0}$, and $(W_t)_{t\geq 0}$, i.e.

$$\mathfrak{F}_t = \sigma\big((N_s)_{0\leq s\leq t}, (W_s)_{0\leq s\leq t}, Y_1, Y_2, \ldots, Y_{N_t}\big).$$

Under the above assumptions, the surplus process $(X_t)_{t\geq 0}$ follows the equation

$$X_t = x + \int_0^t c(X_s)\,ds + \int_0^t \frac{X_s}{S_s}\,dS_s - \sum_{i=1}^{N_t} Y_i, \quad t \geq 0. \tag{1.2}$$

Substituting (1.1) into (1.2) yields

$$X_t = x + \int_0^t c(X_s)\,ds + a\int_0^t X_s\,ds + b\int_0^t X_s\,dW_s - \sum_{i=1}^{N_t} Y_i, \quad t \geq 0. \tag{1.3}$$

The ruin time is defined as $\tau(x) = \inf\{t \geq 0\colon X_t(x) < 0\}$. We suppose that $\tau(x) = \infty$ if $X_t(x) \geq 0$ for all $t \geq 0$. To simplify notation, we let τ stand for $\tau(x)$. The corresponding infinite-horizon ruin probability is given by $\psi(x) = \mathbb{P}\big[\inf_{t\geq 0} X_t(x) < 0\big]$, which is equivalent to $\psi(x) = \mathbb{P}[\tau(x) < \infty]$.

The rest of the paper is organized in the following way. Section 2 deals with the detailed investigation of the question concerning the probability of explosion of the risk process between claim arrivals. In Section 3 we formulate and prove the existence and uniqueness theorem for stochastic differential equations that describe the surplus process. In Section 4 we establish the supermartingale property for an auxiliary exponential process. This property allows us to get an exponential bound for the ruin probability under certain conditions. Finally, in Section 5 we consider the case where the premium intensity is a quadratic function and obtain an exponential bound for the ruin probability. In addition, Appendices 6 and 7 have some lemmas and theorems, which are used in Section 2.

2. AUXILIARY RESULTS

Consider now the following stochastic differential equation

$$X_t = x + \int_0^t p(X_s)\,ds + b\int_0^t X_s\,dW_s, \quad t \ge 0, \tag{2.1}$$

where $x > 0$, $b > 0$, $(W_t)_{t\ge 0}$ is a standard Brownian motion, $p\colon \mathbb{R} \to \mathbb{R}_+$ is a locally Lipschitz continuous function such that $p(u)$ is strictly increasing on $\mathbb{R}_+$ and $p(u) = p(0)$ for all $u < 0$.

Equation (2.1) describes the surplus process between two successive jumps of $(N_t)_{t\ge 0}$ up to the first exit time of $(X_t)_{t\ge 0}$ from $[0, +\infty)$ in the model considered above provided that one puts the corresponding restrictions on $c(u)$, sets $p(u) = c(u) + au$ for $u \ge 0$, and takes the surplus at time of the last jump of $(N_t)_{t\ge 0}$ instead of x.

First, we give some results which show that $(X_t)_{t\ge 0}$ goes to $+\infty$ either with probability 1 or with positive probability, which is less then 1 under certain conditions.

Let t^* be a possible explosion time of $(X_t)_{t\ge 0}$, i.e.

$$t^* = \inf\{t \ge 0\colon X_t \notin (-\infty, +\infty)\}.$$

Moreover, we denote by $t^*_{(0,+\infty)}$ the first exit time from $(0, +\infty)$ for $(X_t)_{t\ge 0}$, i.e.

$$t^*_{(0,+\infty)} = \inf\{t \ge 0\colon X_t \notin (0, +\infty)\}.$$

By Theorem 7.1, equation (2.1) has a unique strong solution up to the explosion time t^*. Note that here and subsequently, we imply pathwise uniqueness of solutions.

For $x > 0$, we define

$$I_1 = \int_x^{+\infty} \exp\left\{-\frac{2}{b^2}\int_x^v \frac{p(u)}{u^2}\,du\right\} dv \quad \text{and} \quad I_2 = -\int_0^x \exp\left\{\frac{2}{b^2}\int_v^x \frac{p(u)}{u^2}\,du\right\} dv. \tag{2.2}$$

Proposition 2.1. *If $p(0) > 0$ and*

$$\limsup_{v\to+\infty}\left((1+\varepsilon)\ln v - \frac{2}{b^2}\int_x^v \frac{p(u)}{u^2}\,du\right) < +\infty \quad \textit{for some} \quad \varepsilon > 0, \tag{2.3}$$

then

$$\mathbb{P}\big[\lim\nolimits_{t\uparrow t^*_{(0,+\infty)}} X_t = +\infty\big] = 1.$$

Proof. Note that in this case $I_1 < +\infty$ and $I_2 = -\infty$ by Lemmas 6.1 and 6.2. Thus, the assertion of the proposition follows immediately from Theorem 7.2. □

Remark 2.2. If $p(0) = 0$, then I_2 may be finite. By Theorem 7.2, if $I_1 < +\infty$ and $I_2 > -\infty$, then $\lim_{t\uparrow t^*_{(0,+\infty)}} X_t$ exists a.s.,

$$0 < \mathbb{P}\big[\lim_{t\uparrow t^*_{(0,+\infty)}} X_t = +\infty\big] < 1,$$

and

$$\mathbb{P}\big[\lim_{t\uparrow t^*_{(0,+\infty)}} X_t = 0\big] = 1 - \mathbb{P}\big[\lim_{t\uparrow t^*_{(0,+\infty)}} X_t = +\infty\big].$$

Remark 2.3. Proposition 2.1 does not give us whether the exit time $t^*_{(0,+\infty)}$ is finite. It is well known that Feller's test for explosions (see, e.g., Theorem 5.29 in [13, p. 348] and [14]) gives precise conditions for whether or not a one-dimensional diffusion process explodes in finite time. This test is very useful when one wants to show that a diffusion process does not explode in finite time (see, e.g., [17]), but it does not solve our problem.

We now give a few examples.

Example 2.4. Let

$$p(u) = \begin{cases} p_1 u + p_0 & \text{if} \quad u \geq 0, \\ p_0 & \text{if} \quad u < 0. \end{cases}$$

The function $p(u)$ has the asserted properties provided that $p_0 \geq 0$ and $p_1 > 0$.

Since

$$I_1 = \int_x^{+\infty} \exp\left\{-\frac{2}{b^2}\int_x^v \frac{p_1 u + p_0}{u^2}\, du\right\} dv = \int_x^{+\infty} \left(\frac{x}{v}\right)^{2p_1/b^2} \cdot \exp\left\{\frac{2p_0}{b^2}\left(\frac{1}{v} - \frac{1}{x}\right)\right\} dv,$$

we have $I_1 = +\infty$ for $2p_1 \leq b^2$, and $I_1 < +\infty$ for $2p_1 > b^2$.

We first consider the case $p_0 > 0$. From Theorem 7.2 and Lemma 6.2 we conclude that

$$\mathbb{P}\big[t^*_{(0,+\infty)} = \infty\big] = 1 \quad \text{if} \quad 2p_1 \leq b^2,$$

and

$$\mathbb{P}\big[\lim_{t\uparrow t^*_{(0,+\infty)}} X_t = +\infty\big] = 1 \quad \text{if} \quad 2p_1 > b^2.$$

Consider now the case $p_0 = 0$. Since

$$I_2 = -\int_0^x \exp\left\{\frac{2}{b^2}\int_v^x \frac{p_1 u}{u^2}\, du\right\} dv = -\int_0^x \left(\frac{x}{v}\right)^{2p_1/b^2} dv,$$

we get $I_2 > -\infty$ for $2p_1 < b^2$, and $I_2 = -\infty$ for $2p_1 \geq b^2$. Theorem 7.2 yields $\mathbb{P}\big[\lim_{t\uparrow t^*_{(0,+\infty)}} X_t = 0\big] = 1$ if $2p_1 < b^2$, $\mathbb{P}\big[t^*_{(0,+\infty)} = \infty\big] = 1$ if $2p_1 = b^2$, and $\mathbb{P}\big[\lim_{t\uparrow t^*_{(0,+\infty)}} X_t = +\infty\big] = 1$ if $2p_1 > b^2$.

Example 2.5. Let

$$p(u) = \begin{cases} p_1(u+p_2)^\alpha & \text{if} \quad u \geq 0, \\ p_1 p_2^\alpha & \text{if} \quad u < 0. \end{cases}$$

We put the following restrictions on the parameters of $p(u)$: $\alpha > 1$, $p_1 > 0$, and $p_2 \geq 0$.

Since

$$\limsup_{v\to+\infty} \left((1+\varepsilon)\ln v - \frac{2}{b^2} \int_x^v \frac{p_1(u+p_2)^\alpha}{u^2}\, du \right)$$
$$\leq \limsup_{v\to+\infty} \left((1+\varepsilon)\ln v - \frac{2p_1}{b^2} \int_x^v u^{\alpha-2}\, du \right)$$
$$= \lim_{v\to+\infty} \left((1+\varepsilon)\ln v - \frac{2p_1\left(v^{\alpha-1} - x^{\alpha-1}\right)}{b^2(\alpha-1)} \right) = -\infty$$

for all $\varepsilon > 0$, Lemma 6.1 gives $I_1 < +\infty$.

If $p_2 > 0$, then $\mathbb{P}\big[\lim_{t\uparrow t^*_{(0,+\infty)}} X_t = +\infty\big] = 1$ by Proposition 2.1.

For $p_2 = 0$, we have

$$I_2 = -\int_0^x \exp\left\{ \frac{2p_1}{b^2} \int_x^v u^{\alpha-2}\, du \right\} dv = -\int_0^x \exp\left\{ \frac{2p_1\left(v^{\alpha-1} - x^{\alpha-1}\right)}{b^2(\alpha-1)} \right\} dv > -\infty.$$

Hence, in this case $\lim_{t\uparrow t^*_{(0,+\infty)}} X_t$ exists a.s.,

$$0 < \mathbb{P}\big[\lim_{t\uparrow t^*_{(0,+\infty)}} X_t = +\infty\big] < 1,$$

and

$$\mathbb{P}\big[\lim_{t\uparrow t^*_{(0,+\infty)}} X_t = 0\big] = 1 - \mathbb{P}\big[\lim_{t\uparrow t^*_{(0,+\infty)}} X_t = +\infty\big]$$

by Theorem 7.2.

Example 2.6. Let

$$p(u) = \begin{cases} p_2 u^2 + p_1 u + p_0 & \text{if} \quad u \geq 0, \\ p_0 & \text{if} \quad u < 0. \end{cases} \tag{2.4}$$

If $p_0 \geq 0$, $p_1 \geq 0$, and $p_2 > 0$, then $p(u)$ has all the properties required.

For all $\varepsilon > 0$, we have

$$\limsup_{v\to+\infty} \left((1+\varepsilon)\ln v - \frac{2}{b^2} \int_x^v \frac{p_2u^2 + p_1u + p_0}{u^2}\, du \right)$$
$$\leq \limsup_{v\to+\infty} \left((1+\varepsilon)\ln v - \frac{2}{b^2} \int_x^v p_2\, du \right)$$
$$= \lim_{v\to+\infty} \left((1+\varepsilon)\ln v - \frac{2p_2(v-x)}{b^2} \right) = -\infty.$$

Hence, $I_1 < +\infty$ by Lemma 6.1.

If $p_0 > 0$, then $\mathbb{P}\big[\lim_{t\uparrow t^*_{(0,+\infty)}} X_t = +\infty\big] = 1$ by Proposition 2.1.

For $p_0 = 0$, we get

$$I_2 = -\int_0^x \exp\left\{\frac{2}{b^2}\int_v^x \frac{p_2u^2 + p_1u}{u^2}\,du\right\} dv = -\int_0^x \left(\frac{x}{v}\right)^{2p_1/b^2} \cdot \exp\left\{\frac{2p_2(x-v)}{b^2}\right\} dv.$$

This gives $I_2 > -\infty$ for $2p_1 < b^2$, and $I_2 = -\infty$ for $2p_1 \geq b^2$. Consequently, if $2p_1 < b^2$, then $\lim_{t\uparrow t^*_{(0,+\infty)}} X_t$ exists a.s.,

$$0 < \mathbb{P}\big[\lim_{t\uparrow t^*_{(0,+\infty)}} X_t = +\infty\big] < 1,$$

and

$$\mathbb{P}\big[\lim_{t\uparrow t^*_{(0,+\infty)}} X_t = 0\big] = 1 - \mathbb{P}\big[\lim_{t\uparrow t^*_{(0,+\infty)}} X_t = +\infty\big];$$

if $2p_1 \geq b^2$, then

$$\mathbb{P}\big[\lim_{t\uparrow t^*_{(0,+\infty)}} X_t = +\infty\big] = 1.$$

One question which is still unanswered is whether $t^*_{(0,+\infty)}$ is finite. We now study it under the conditions of Example 2.6.

Theorem 2.7. *Let $(X_t)_{t\geq 0}$ be a strong solution of* (2.1) *and $p(u)$ be defined by* (2.4) *with $p_0 \geq 0$, $p_1 \geq 0$, and $p_2 > 0$. If $p_0 = 0$ and $\frac{2p_1}{b^2} < 1$, then*

$$\mathbb{P}\big[t^*_{(0,+\infty)} < \infty,\ \lim_{t\uparrow t^*_{(0,+\infty)}} X_t = +\infty\big] = \frac{\int_0^x v^{-2p_1/b^2} \cdot \exp\left\{-\frac{2p_2 v}{b^2}\right\} dv}{\int_0^{+\infty} v^{-2p_1/b^2} \cdot \exp\left\{-\frac{2p_2 v}{b^2}\right\} dv}. \tag{2.5}$$

If either $p_0 = 0$ and $\frac{2p_1}{b^2} \geq 1$ or $p_0 > 0$, then

$$\mathbb{P}\big[t^*_{(0,+\infty)} < \infty,\ \lim_{t\uparrow t^*_{(0,+\infty)}} X_t = +\infty\big] = 1. \tag{2.6}$$

Proof. Let $n_0 = \min\{n \in \mathbb{N}\colon 1/n < x\}$. For all integer n such that $n \geq n_0$, we denote by $t^*_{(1/n,+\infty)}$ the first exit time from $(1/n, +\infty)$ for $(X_t)_{t\geq 0}$, i.e.

$$t^*_{(1/n,+\infty)} = \inf\{t \geq 0\colon X_t \notin (1/n, +\infty)\}.$$

Note that the sequence of events

$$\Big(\big\{\omega \in \Omega\colon t^*_{(1/n,+\infty)}(\omega) < \infty,\ \lim_{t\uparrow t^*_{(1/n,+\infty)}} X_t(\omega) = +\infty\big\}\Big)_{n\geq n_0}$$

is monotone nondecreasing. Hence,

$$\begin{aligned}&\lim_{n\to\infty}\big\{\omega \in \Omega\colon t^*_{(1/n,+\infty)}(\omega) < \infty,\ \lim_{t\uparrow t^*_{(1/n,+\infty)}} X_t(\omega) = +\infty\big\}\\ &= \bigcup_{n=n_0}^{\infty} \big\{\omega \in \Omega\colon t^*_{(1/n,+\infty)}(\omega) < \infty,\ \lim_{t\uparrow t^*_{(1/n,+\infty)}} X_t(\omega) = +\infty\big\}.\end{aligned}$$

Furthermore,

$$\bigcup_{n=n_0}^{\infty} \big\{\omega \in \Omega \colon t^*_{(1/n,+\infty)}(\omega) < \infty,\ \lim_{t\uparrow t^*_{(1/n,+\infty)}} X_t(\omega) = +\infty\big\}$$
$$= \big\{\omega \in \Omega \colon t^*_{(0,+\infty)}(\omega) < \infty,\ \lim_{t\uparrow t^*_{(0,+\infty)}} X_t(\omega) = +\infty\big\}.$$

Therefore, by the continuity of probability measures, we conclude that

$$\begin{aligned} &\mathbb{P}\big[t^*_{(0,+\infty)} < \infty,\ \lim_{t\uparrow t^*_{(0,+\infty)}} X_t = +\infty\big] \\ &= \mathbb{P}\Big[\lim_{n\to\infty} \big\{t^*_{(1/n,+\infty)} < \infty,\ \lim_{t\uparrow t^*_{(1/n,+\infty)}} X_t = +\infty\big\}\Big] \\ &= \lim_{n\to\infty} \mathbb{P}\big[t^*_{(1/n,+\infty)} < \infty,\ \lim_{t\uparrow t^*_{(1/n,+\infty)}} X_t = +\infty\big]. \end{aligned} \tag{2.7}$$

From [13, pp. 343–344] it follows that $\mathbb{E}[t^*_{(1/n,+\infty)}] = M_n(x)$ for all $n \ge n_0$, where $M_n(x)$ is a solution of the boundary value problem

$$\frac{1}{2} b^2 x^2 M_n''(x) + (p_2 x^2 + p_1 x + p_0) M_n'(x) = -1, \quad M_n\left(\frac{1}{n}\right) = 0,\ M_n(+\infty) = 0, \tag{2.8}$$

which can be solved by the usual technique (see, e.g., [1]). Here and subsequently, the value of a function at $+\infty$ stands for its limit as the value of the argument tends to $+\infty$.

Boundary value problem (2.8) has the unique solution

$$M_n(x) = \frac{2m_n(x)}{b^2 m_n(+\infty)} \int_{1/n}^{+\infty} \frac{m_n(+\infty) - m_n(z)}{z^2 m_n'(z)}\, dz - \frac{2}{b^2} \int_{1/n}^{x} \frac{m_n(x) - m_n(z)}{z^2 m_n'(z)}\, dz,$$

where

$$m_n(x) = \int_{1/n}^{x} \exp\left\{-\frac{2}{b^2} \int_{1/n}^{v} \frac{p_2 u^2 + p_1 u + p_0}{u^2}\, du\right\} dv.$$

Note that $m_n(+\infty) < +\infty$. Furthermore, since

$$\lim_{z\to+\infty} \frac{m_n(+\infty) - m_n(z)}{m_n'(z)} = \lim_{z\to+\infty} \frac{\int_z^{+\infty} \exp\left\{-\frac{2}{b^2} \int_{1/n}^{v} \frac{p_2 u^2 + p_1 u + p_0}{u^2}\, du\right\} dv}{\exp\left\{-\frac{2}{b^2} \int_{1/n}^{z} \frac{p_2 u^2 + p_1 u + p_0}{u^2}\, du\right\}} = \frac{b^2}{2p_2} < +\infty$$

(here we applied L'Hopital's rule) and $\int_{1/n}^{+\infty} \frac{1}{z^2}\, dz < +\infty$, we get

$$\int_{1/n}^{+\infty} \frac{m_n(+\infty) - m_n(z)}{z^2 m_n'(z)}\, dz < +\infty.$$

Thus, $\mathbb{E}[t^*_{(1/n,+\infty)}] < \infty$ for all $n \geq n_0$. This gives $\mathbb{P}[t^*_{(1/n,+\infty)} < \infty] = 1$ for all $n \geq n_0$. Moreover, by [13, pp. 343–344], we have

$$\mathbb{P}\big[\lim_{t\uparrow t^*_{(1/n,+\infty)}} X_t = +\infty\big] = \frac{\int\limits_{1/n}^{x} \exp\left\{-\frac{2}{b^2}\int\limits_{1/n}^{v} \frac{p_2u^2+p_1u+p_0}{u^2}\,du\right\} dv}{\int\limits_{1/n}^{+\infty} \exp\left\{-\frac{2}{b^2}\int\limits_{1/n}^{v} \frac{p_2u^2+p_1u+p_0}{u^2}\,du\right\} dv} = \frac{\int\limits_{1/n}^{x} v^{-2p_1/b^2}\cdot \exp\left\{\frac{2p_0}{b^2v} - \frac{2p_2v}{b^2}\right\} dv}{\int\limits_{1/n}^{+\infty} v^{-2p_1/b^2}\cdot \exp\left\{\frac{2p_0}{b^2v} - \frac{2p_2v}{b^2}\right\} dv}. \tag{2.9}$$

Consequently, (2.7) and (2.9) yield

$$\mathbb{P}\big[t^*_{(0,+\infty)} < \infty,\, \lim_{t\uparrow t^*_{(0,+\infty)}} X_t = +\infty\big] = \lim_{n\to\infty} \frac{\int\limits_{1/n}^{x} v^{-2p_1/b^2}\cdot \exp\left\{\frac{2p_0}{b^2v} - \frac{2p_2v}{b^2}\right\} dv}{\int\limits_{1/n}^{+\infty} v^{-2p_1/b^2}\cdot \exp\left\{\frac{2p_0}{b^2v} - \frac{2p_2v}{b^2}\right\} dv}. \tag{2.10}$$

Consider now two cases.

Case 1. If $p_0 = 0$ and $\frac{2p_1}{b^2} < 1$, then both of the integrals in the right-hand side of (2.10) are finite as $n \to \infty$. This yields (2.5). Note that in this case

$$0 < \mathbb{P}\big[t^*_{(0,+\infty)} < \infty,\, \lim_{t\uparrow t^*_{(0,+\infty)}} X_t = +\infty\big] < 1.$$

Case 2. If either $p_0 = 0$ and $\frac{2p_1}{b^2} \geq 1$ or $p_0 > 0$, then both of the integrals in the right-hand side of (2.10) are infinite as $n \to \infty$. Applying L'Hopital's rule we obtain (2.6).

The theorem is proved. □

Remark 2.8. Since $c(u)$ is positive by our assumption, the surplus of the insurance company becomes infinitely large in finite time a.s. if the premium intensity is a quadratic function and the claims do not arrive. Note that the time interval between two successive claims can be large enough with positive probability. Hence, the process $\big(X_t(x)\big)_{t\geq 0}$ that follows (1.3) goes to $+\infty$ with positive probability. It is clear that the ruin does not occur in this case. Consequently, from now on we can consider $\big(X_t(x)\big)_{t\geq 0}$ up to the minimum from the ruin time and its possible explosion.

3. EXISTENCE AND UNIQUENESS THEOREM

Consider now equation (1.3). Let $t^*(x)$ be a possible explosion time of $\big(X_t(x)\big)_{t\geq 0}$, i.e.

$$t^*(x) = \inf\{t \geq 0\colon X_t(x) \notin (-\infty, +\infty)\}.$$

To shorten notation, we let t^* stand for $t^*(x)$.

Theorem 3.1. *If $c(u)$ is a locally Lipschitz continuous function on $\mathbb{R}$, then* (1.3) *has a unique strong solution up to the time $\tau \wedge t^*$.*

Proof. Since the process $(N_t)_{t\ge 0}$ is homogeneous, it has only a finite number of jumps on any finite time interval a.s. To prove the theorem, we study (1.3) between two successive jumps of $(N_t)_{t\ge 0}$.

Let us first consider (1.3) on the time interval $[\tau_0, \tau_1)$. It can be rewritten as

$$X_t = X_{\tau_0} + \int_{\tau_0}^{t} \big(c(X_s) + aX_s\big)\, ds + b\int_{\tau_0}^{t} X_s\, dW_s, \quad \tau_0 \le t < \tau_1. \tag{3.1}$$

By Theorem 7.1, the locally Lipschitz continuity of $c(u) + au$ and bu on $\mathbb{R}$ implies the existence of a unique strong solution of (3.1) on $[\tau_0, \tau_1 \wedge t^*)$. Moreover, the comparison theorem (see, e.g., Theorem 1.1 in [11, pp. 437–438]) shows that this solution is not less then the solution of

$$X_t = X_{\tau_0} + a\int_{\tau_0}^{t} X_s\, ds + b\int_{\tau_0}^{t} X_s\, dW_s, \quad \tau_0 \le t \le \tau_1 \wedge t^*, \tag{3.2}$$

a.s. Since the solution of (3.2) is positive, so is the solution of (3.1) on $[\tau_0, \tau_1 \wedge t^*)$. Hence, $\lim_{t\uparrow t^*} X_t = +\infty$ if $t^* \le \tau_1$. Thus, the ruin does not occur up to the time $\tau_1 \wedge t^*$.

If $t^* \le \tau_1$, then the theorem follows. Otherwise $X_{\tau_{1-}} < +\infty$ and we set $X_{\tau_1} = X_{\tau_{1-}} - Y_1$. Next, if $X_{\tau_1} < 0$, then $\tau = \tau_1$, which completes the proof. Otherwise we consider (1.3) on the time interval $[\tau_1, \tau_2)$. We rewrite it as

$$X_t = X_{\tau_1} + \int_{\tau_1}^{t} \big(c(X_s) + aX_s\big)\, ds + b\int_{\tau_1}^{t} X_s\, dW_s, \quad \tau_1 \le t < \tau_2. \tag{3.3}$$

Repeating the same arguments, we conclude that (3.3) has a unique strong solution on $[\tau_1, \tau_2 \wedge t^*)$ and the ruin does not occur up to time $\tau_2 \wedge t^*$.

Thus, we have proved that (1.3) has a unique strong solution on $[0, \tau_2 \wedge t^*)$, which is our assertion if $t^* \le \tau_2$. For the case $t^* > \tau_2$, we set $X_{\tau_2} = X_{\tau_{2-}} - Y_2$. Next, if $X_{\tau_2} < 0$, then $\tau = \tau_2$, which proves the theorem. Otherwise we continue in this fashion and prove the theorem by induction. □

Remark 3.2. Note that if $t^* < \infty$, then the proof of Theorem 3.1 implies $\lim_{t\uparrow t^*} X_t = +\infty$ and (1.3) also holds for $t = t^*$ provided that we let both of its sides be formally equal to $+\infty$. In this case we formally set $X_{t^*} = +\infty$. In addition, if $\tau < \infty$, then we set $X_\tau = X_{\tau_{i-}} - Y_i$, where i is the number of the claim that caused the ruin, and (1.3) also holds for $t = \tau$.

4. SUPERMARTINGALE PROPERTY FOR THE EXPONENTIAL PROCESS

Let the stopped process $\big(\tilde{X}_t(x)\big)_{t\geq 0}$ be defined by $\tilde{X}_t(x) = X_{t\wedge\tau\wedge t^*}(x)$. Note that $\big(\tilde{X}_t(x)\big)_{t\geq 0}$ is a solution of (1.3) provided that $\big(X_t(x)\big)_{0\leq t<\tau\wedge t^*}$ is. In particular, if $t^* < \infty$, we formally let $\tilde{X}_t(x) = +\infty$ for all $t \geq t^*$ and (1.3) hold.

For all $r \geq 0$, we define the processes $\big(U_t(x,r)\big)_{t\geq 0}$ and $\big(V_t(x,r)\big)_{t\geq 0}$ by

$$U_t(x,r) = -r\tilde{X}_t(x) \quad \text{and} \quad V_t(x,r) = \mathrm{e}^{U_t(x,r)}.$$

In what follows, we write $\tilde{X}_t$, U_t, and V_t instead of $\tilde{X}_t(x)$, $U_t(x,r)$, and $V_t(x,r)$, respectively, when no confusion can arise. Recall that $\lambda > 0$ is the intensity of the Poisson process $(N_t)_{t\geq 0}$. The function $h(r)$ and the constant $r_\infty \in (0,+\infty]$, which are used below, were defined in the Introduction.

Theorem 4.1. *If* (1.3) *has a unique strong solution up to the time* $\tau \wedge t^*$ *and there exists* $\hat{r} \in (0, r_\infty)$ *such that*

$$\frac{\hat{r}^2 b^2}{2} u^2 - \hat{r}\big(c(u) + au\big) + \lambda h(\hat{r}) \leq 0 \quad \textit{for all} \quad u \geq 0, \tag{4.1}$$

then $\big(V_t(x,r)\big)_{t\geq 0}$ *is an* $(\mathfrak{F}_t)$*-supermartingale.*

Proof. Since $\big(\tilde{X}_t\big)_{t\geq 0}$ is a solution of (1.3), we have

$$U_t = -rx - r\int_0^{t\wedge\tau\wedge t^*} \big(c(X_s) + aX_s\big)\,ds - rb\int_0^{t\wedge\tau\wedge t^*} X_s\,dW_s + r\sum_{i=1}^{N_{t\wedge\tau\wedge t^*}} Y_i, \quad t \geq 0. \tag{4.2}$$

The process $(\tilde{X}_t)_{t\geq 0}$ is a sum of local martingales and càdlàg processes of locally bounded variation. Indeed, since

$$\mathbb{E}\left[\left|\int_0^{t\wedge\tau\wedge t^*\wedge T_n} X_s\,dW_s\right|\right] < +\infty$$

for all $t \geq 0$, the process $\Big(\int_0^{t\wedge\tau\wedge t^*} X_s\,dW_s\Big)_{t\geq 0}$ is a local $(\mathfrak{F}_t)$-martingale with the localizing sequence $(T_n)_{n\geq 1}$, where

$$T_n = \inf\{t \geq 0\colon X_t \geq n\} \wedge n.$$

Similarly, $\Big(\int_0^{t\wedge\tau\wedge t^*} X_s\,ds\Big)_{t\geq 0}$ and $\Big(\int_0^{t\wedge\tau\wedge t^*} c(X_s)\,ds\Big)_{t\geq 0}$ are càdlàg processes of locally bounded variation with the localizing sequence $(T_n)_{n\geq 1}$. Next, the process

$$\left(\sum_{i=1}^{N_{t\wedge\tau\wedge t^*}} Y_i - \lambda\mu(t\wedge\tau\wedge t^*)\right)_{t\geq 0}$$

is a compensated process with independent increments. Hence, it is an $(\mathfrak{F}_t)$-martingale.

Thus, $(U_t)_{t\ge 0}$ is an $(\mathfrak{F}_t)$-semimartingale and so is $(V_t)_{t\ge 0}$. Applying Itô's formula

$$g(U_t)-g(U_0)=\int_{0_+}^{t} g'(U_{s_-})\,dU_s+\frac{1}{2}\int_{0_+}^{t} g''(U_{s_-})\,d\langle U^c,U^c\rangle_s + \sum_{0<s\le t}\big(g(U_s)-g(U_{s_-})-g'(U_{s_-})(U_s-U_{s_-})\big),\quad t\ge 0,$$

where $(U_t)_{t\ge 0}$ is a semimartingale, $(U_t^c)_{t\ge 0}$ is a continuous component of the local martingale in the decomposition of $(U_t)_{t\ge 0}$, and $g\in C^2(\mathbb{R})$, we get

$$V_t=\mathrm{e}^{-rx}+\int_{0_+}^{t\wedge\tau\wedge t^*}\mathrm{e}^{U_{s-}}\,dU_s+\frac{1}{2}\int_{0_+}^{t\wedge\tau\wedge t^*}\mathrm{e}^{U_{s-}}\,d\langle U^c,U^c\rangle_s + \sum_{0<s\le t\wedge\tau\wedge t^*}\big(\mathrm{e}^{U_s}-\mathrm{e}^{U_{s-}}-\mathrm{e}^{U_{s-}}(U_s-U_{s_-})\big),\quad t\ge 0, \tag{4.3}$$

where

$$U_{t_-}=-rx-r\int_0^{t\wedge\tau\wedge t^*}\big(c(X_s)+aX_s\big)\,ds-rb\int_0^{t\wedge\tau\wedge t^*}X_s\,dW_s+r\sum_{0<s\le t_-\wedge\tau\wedge t^*}Y_{N_s}\mathbb{I}_{\{\Delta N_s\ne 0\}},$$

$$dU_s=-r\big(c(\tilde X_s)+a\tilde X_s\big)\,ds-rb\tilde X_s\,dW_s+rY_{N_s}\mathbb{I}_{\{\Delta N_s\ne 0\}},$$
$$d\langle U^c,U^c\rangle_s=r^2b^2\tilde X_s^2,$$
$$\mathrm{e}^{U_s}-\mathrm{e}^{U_{s-}}=\mathrm{e}^{U_{s-}}\big(\mathrm{e}^{rY_{N_s}\mathbb{I}_{\{\Delta N_s\ne 0\}}}-1\big),$$
$$U_s-U_{s_-}=rY_{N_s}\mathbb{I}_{\{\Delta N_s\ne 0\}},$$
$$\Delta N_s=N_s-N_{s_-}.$$

Substituting all the above equalities into (4.3) yields

$$\begin{aligned}V_t=\mathrm{e}^{-rx}&-r\int_{0_+}^{t\wedge\tau\wedge t^*}\mathrm{e}^{U_{s-}}\big(c(X_s)+aX_s\big)\,ds-rb\int_{0_+}^{t\wedge\tau\wedge t^*}\mathrm{e}^{U_{s-}}X_s\,dW_s\\ &+r\sum_{0<s\le t\wedge\tau\wedge t^*}\mathrm{e}^{U_{s-}}Y_{N_s}\mathbb{I}_{\{\Delta N_s\ne 0\}}+\frac{1}{2}r^2b^2\int_{0_+}^{t\wedge\tau\wedge t^*}\mathrm{e}^{U_{s-}}X_s^2\,ds\\ &+\sum_{0<s\le t\wedge\tau\wedge t^*}\mathrm{e}^{U_{s-}}\big(\mathrm{e}^{rY_{N_s}\mathbb{I}_{\{\Delta N_s\ne 0\}}}-1-rY_{N_s}\mathbb{I}_{\{\Delta N_s\ne 0\}}\big),\quad t\ge 0.\end{aligned} \tag{4.4}$$

Simplifying (4.4) gives

$$V_t = e^{-rx} + \int_{0_+}^{t\wedge\tau\wedge t^*} e^{U_{s-}} \left(\frac{1}{2}r^2b^2X_s^2 - r\big(c(X_s) + aX_s\big)\right) ds - rb\int_{0_+}^{t\wedge\tau\wedge t^*} e^{U_{s-}} X_s\, dW_s + \sum_{0<s\le t\wedge\tau\wedge t^*} e^{U_{s-}} \big(e^{rY_{N_s}\mathbb{I}_{\{\Delta N_s\neq 0\}}} - 1\big), \quad t\ge 0. \tag{4.5}$$

Next, the process

$$\left(\sum_{0<s\le t\wedge\tau\wedge t^*} e^{U_{s-}} \big(e^{rY_{N_s}\mathbb{I}_{\{\Delta N_s\neq 0\}}} - 1\big)\right)_{t\ge 0}$$

is nondecreasing and can be written in integral form

$$\sum_{0<s\le t\wedge\tau\wedge t^*} e^{U_{s-}} \big(e^{rY_{N_s}\mathbb{I}_{\{\Delta N_s\neq 0\}}} - 1\big) = \int_{0_+}^{t\wedge\tau\wedge t^*} e^{U_{s-}}\, dQ_s, \quad t\ge 0,$$

where

$$Q_t = \sum_{0<s\le t\wedge\tau\wedge t^*} \big(e^{rY_{N_s}\mathbb{I}_{\{\Delta N_s\neq 0\}}} - 1\big).$$

By Wald's identity, $\mathbb{E}[Q_t] = \lambda t h(r)$. Hence, $\mathbb{E}[Q_t] < +\infty$ for all $t\ge 0$ and $r < r_\infty$. Furthermore, since $(Q_t)_{t\ge 0}$ is a process with independent increments, the compensated process $\big(Q_t - \mathbb{E}[Q_t]\big)_{t\ge 0}$ is an $(\mathfrak{F}_t)$-martingale. Thus,

$$\left(\sum_{0<s\le t\wedge\tau\wedge t^*} \big(e^{rY_{N_s}\mathbb{I}_{\{\Delta N_s\neq 0\}}} - 1\big) - \lambda h(r) \int_{0_+}^{t\wedge\tau\wedge t^*} e^{U_{s-}}\, ds\right)_{t\ge 0}$$

is a local $(\mathfrak{F}_t)$-martingale with the localizing sequence $(T_n)_{n\ge 1}$ defined above.

Since we have already justified at the beginning of the proof that

$$\left(-rb\int_{0_+}^{t\wedge\tau\wedge t^*} e^{U_{s-}} X_s\, dW_s\right)_{t\ge 0}$$

is also a local $(\mathfrak{F}_t)$-martingale with the localizing sequence $(T_n)_{n\ge 1}$, so is

$$\left(-rb\int_{0_+}^{t\wedge\tau\wedge t^*} e^{U_{s-}} X_s\, dW_s + \sum_{0<s\le t\wedge\tau\wedge t^*} \big(e^{rY_{N_s}\mathbb{I}_{\{\Delta N_s\neq 0\}}} - 1\big) - \lambda h(r) \int_{0_+}^{t\wedge\tau\wedge t^*} e^{U_{s-}}\, ds\right)_{t\ge 0}.$$

We define the process $(R_t)_{t\ge 0}$ by

$$R_t = V_t - V_0 + rb \int\limits_{0_+}^{t\wedge\tau\wedge t^*} \mathrm{e}^{U_{s-}} X_s\, dW_s - \sum_{0<s\le t\wedge\tau\wedge t^*} \big(\mathrm{e}^{rY_{N_s}\mathbb{I}_{\{\Delta N_s\neq 0\}}} - 1\big) + \lambda h(r) \int\limits_{0_+}^{t\wedge\tau\wedge t^*} \mathrm{e}^{U_{s-}}\, ds.$$

Substituting V_t from (4.5) we obtain

$$R_t = \int\limits_{0_+}^{t\wedge\tau\wedge t^*} \mathrm{e}^{-rX_{s-}} \left(\frac{1}{2} r^2 b^2 X_s^2 - r\big(c(X_s) + aX_s\big) + \lambda h(r)\right) ds, \quad t\ge 0.$$

Note $(V_t)_{t\ge 0}$ is a local $(\mathfrak{F}_t)$-supermartingale with the localizing sequence $(T_n)_{n\ge 1}$ provided that $(R_t)_{t\ge 0}$ is a measurable nonincreasing process, i.e.

$$\int\limits_{t_1\wedge\tau\wedge t^*}^{t_2\wedge\tau\wedge t^*} \mathrm{e}^{-rX_{s-}} \left(\frac{1}{2} r^2 b^2 X_s^2 - r\big(c(X_s) + aX_s\big) + \lambda h(r)\right) ds \le 0 \quad \text{for all} \quad t_2\ge t_1\ge 0. \tag{4.6}$$

By the assumption of the theorem, there exists $\hat r \in (0, r_\infty)$ such that (4.1) holds. Therefore, (4.6) is true with $r=\hat r$ and $(V_t(x,\hat r))_{t\ge 0}$ is a nonnegative local $(\mathfrak{F}_t)$-supermartingale with the localizing sequence $(T_n)_{n\ge 1}$.

By Fatou's lemma, for all $t_2\ge t_1\ge 0$, we get

$$\begin{aligned} 0 &\le \mathbb{E}\big[V_{t_2}(x,\hat r) \,/\, \mathfrak{F}_{t_1}\big] = \mathbb{E}\Big[\lim_{n\to\infty} V_{t_2\wedge T_n}(x,\hat r) \,/\, \mathfrak{F}_{t_1}\Big] = \mathbb{E}\Big[\liminf_{n\to\infty} V_{t_2\wedge T_n}(x,\hat r) \,/\, \mathfrak{F}_{t_1}\Big] \\ &\le \liminf_{n\to\infty} \mathbb{E}\big[V_{t_2\wedge T_n}(x,\hat r) \,/\, \mathfrak{F}_{t_1}\big] \le \liminf_{n\to\infty} V_{t_1\wedge T_n}(x,\hat r) = V_{t_1}(x,\hat r). \end{aligned}$$

Hence, $(V_t(x,\hat r))_{t\ge 0}$ is an $(\mathfrak{F}_t)$-supermartingale, which completes the proof. □

Theorem 4.1 allows us to get an exponential bound for the ruin probability under certain conditions.

5. EXPONENTIAL BOUND FOR THE RUIN PROBABILITY

Let the premium intensity $c(u)$ be a quadratic function for $u\ge 0$, i.e.

$$c(u) = \begin{cases} c_2u^2 + c_1u + c_0 & \text{if} \quad u\ge 0, \\ c_0 & \text{if} \quad u<0, \end{cases} \tag{5.1}$$

where $c_2\neq 0$. The function $c(u)$ is strictly increasing and positive on $[0,+\infty)$ if and only if $c_0>0$, $c_1\ge 0$, and $c_2>0$. This model implies that the premium intensity grows rapidly with increasing surplus. Recall that the infinite-horizon ruin probability is given by $\psi(x) = \mathbb{P}\big[\inf_{t\ge 0} X_t(x) < 0\big]$, which is equivalent to $\psi(x) = \mathbb{P}[\tau(x)<\infty]$.

Theorem 5.1. *Let the surplus process $(X_t(x))_{t\geq 0}$ follow* (1.3) *under the above assumptions, the premium intensity $c(u)$ be defined by* (5.1) *with $c_0 > 0$, $c_1 \geq 0$, and $c_2 > 0$. Moreover, let $a + c_1 \geq 0$ and at least one of the following two conditions hold:*

1) $\frac{2c_2}{b^2} < r_\infty$ *and* $h\left(\frac{2c_2}{b^2}\right) \leq \frac{2c_0c_2}{b^2\lambda}$,
2) $\lambda\mu < c_0$.

Then for all $x \geq 0$, we have

$$\psi(x) \leq \mathrm{e}^{-\hat{r}x}, \tag{5.2}$$

where $\hat{r} = \frac{2c_2}{b^2}$ if condition 1) *holds, and $\hat{r} = \min\left\{r_0, \frac{2c_2}{b^2}\right\}$ if condition* 1) *holds. Here r_0 stands for the unique positive solution of*

$$h(r) = \frac{c_0 r}{\lambda}. \tag{5.3}$$

Lemma 5.2. *Let the conditions of Theorem 5.1 hold. Then the process $\big(V_t(x,\hat{r})\big)_{t\geq 0}$ is an $(\mathfrak{F}_t)$-supermartingale, where $\hat{r} = \frac{2c_2}{b^2}$ if condition* 1) *of Theorem 5.1 is true, and $\hat{r}$ is an arbitrary number from $(0, r_0]$ for $r_0 < \frac{2c_2}{b^2}$ or from $\left(0, \frac{2c_2}{b^2}\right)$ for $r_0 \geq \frac{2c_2}{b^2}$ if condition* 2) *of Theorem 5.1 is true. Here r_0 stands for the unique positive solution of* (5.3).

Proof of Lemma 5.2. Since $c(u)$ defined by (5.1) is a locally Lipschitz continuous function on $\mathbb{R}$, equation (1.3) has a unique strong solution up to the time $\tau \wedge t^*$ by Theorem 3.1. According to Theorem 4.1, if there exists $\hat{r} \in (0, r_\infty)$ such that

$$\left(\frac{\hat{r}^2 b^2}{2} - \hat{r}c_2\right)u^2 - \hat{r}(a + c_1)u - \hat{r}c_0 + \lambda h(\hat{r}) \leq 0 \quad \text{for all} \quad u \geq 0, \tag{5.4}$$

then $\big(V_t(x,\hat{r})\big)_{t\geq 0}$ is an $(\mathfrak{F}_t)$-supermartingale.

Condition (5.4) holds in one of the two following cases.

Case 1. The coefficient of u^2 is equal to 0, i.e. $\hat{r} = \frac{2c_2}{b^2}$. Then (5.4) is true if and only if

$$\frac{2c_2}{b^2} < r_\infty \quad \text{and} \quad -\frac{2c_2}{b^2}c_0 + \lambda h\left(\frac{2c_2}{b^2}\right) \leq 0,$$

which coincides with condition 1) of the theorem.

Case 2. The coefficient of u^2 is negative, i.e. $\hat{r} \in \left(0, \frac{2c_2}{b^2}\right)$. Since $u = \frac{a+c_1}{\hat{r}b^2 - 2c_2}$, which is negative, maximizes the left-hand side of (5.4), the last one is true if and only if

$$\hat{r} \in \left(0, \min\left\{\frac{2c_2}{b^2}, r_\infty\right\}\right) \tag{5.5}$$

and

$$\lambda h(\hat{r}) \leq c_0 \hat{r}. \tag{5.6}$$

Consider the functions $g_1(r) = \lambda h(r)$ and $g_2(r) = c_0 r$ on $[0, r_\infty)$. Note that $g_1(0) = 0$, $g_2(0) = 0$, $g_1'(0) = \lambda\mu$, and $g_2'(0) = c_0$. Moreover, since $h(r)$ is increasing and convex, so is $g_1(r)$. Thus, we get the following.

If $\lambda\mu \geq c_0$, then $g_2(r) < g_1(r)$ for all $r \in (0, r_\infty)$. Hence, for no $\hat{r} \in (0, r_\infty)$ does (5.4) hold.

If $\lambda\mu < c_0$, then the equation $g_1(r) = g_2(r)$ has a unique solution $r_0 \in (0, r_\infty)$. Therefore, (5.3) has a unique positive solution and (5.6) is true for all $\hat{r} \in (0, r_0]$. Taking into account the condition (5.5) we conclude that (5.4) holds for all $\hat{r} \in (0, r_0]$ if $r_0 < \frac{2c_2}{b^2}$, and for all $\hat{r} \in \big(0, \frac{2c_2}{b^2}\big)$ if $r_0 \geq \frac{2c_2}{b^2}$.

The lemma is proved. □

Proof of Theorem 5.1. Let $\hat{r}$ be defined in the assertion of Lemma 5.2. Then $\big(V_t(x, \hat{r})\big)_{t\geq 0}$ is an $(\mathfrak{F}_t)$-supermartingale by this lemma. Therefore, for all $t \geq 0$, we get

$$\begin{aligned} e^{-\hat{r}x} &= V_0(x, \hat{r}) \geq \mathbb{E}\big[V_t(x, \hat{r}) \,/\, \mathfrak{F}_0\big] = \mathbb{E}\big[e^{-\hat{r}X_{t\wedge\tau\wedge t^*}(x)}\big] \\ &= \mathbb{E}\big[e^{-\hat{r}X_\tau(x)} \cdot \mathbb{I}_{\{\tau(x)<t\wedge t^*\}}\big] + \mathbb{E}\big[e^{-\hat{r}X_{t\wedge t^*}(x)} \cdot \mathbb{I}_{\{\tau(x)\geq t\wedge t^*\}}\big] \\ &\geq \mathbb{E}\big[e^{-\hat{r}X_\tau(x)} \cdot \mathbb{I}_{\{\tau(x)<t\wedge t^*\}}\big]. \end{aligned} \tag{5.7}$$

Letting $t \to \infty$ in (5.7) gives

$$\mathbb{E}\big[e^{-\hat{r}X_\tau(x)} \cdot \mathbb{I}_{\{\tau(x)<t^*\}}\big] \leq e^{-\hat{r}x}. \tag{5.8}$$

Since the surplus becomes infinitely large at the explosion time, the ruin does not occur after t^*. Hence,

$$\big\{\omega \in \Omega \colon \tau(x, \omega) < t^*(x, \omega)\big\} = \big\{\omega \in \Omega \colon \tau(x, \omega) < \infty\big\}$$

and (5.8) can be rewritten as

$$\mathbb{E}\big[e^{-\hat{r}X_\tau(x)} \cdot \mathbb{I}_{\{\tau(x)<\infty\}}\big] \leq e^{-\hat{r}x}. \tag{5.9}$$

Furthermore,

$$\mathbb{E}\big[e^{-\hat{r}X_\tau(x)} \cdot \mathbb{I}_{\{\tau(x)<\infty\}}\big] = \mathbb{E}\big[e^{-\hat{r}X_\tau(x)} \,/\, \tau(x) < \infty\big] \cdot \mathbb{P}[\tau(x) < \infty],$$

and

$$\mathbb{E}\big[e^{-\hat{r}X_\tau(x)} \cdot \mathbb{I}_{\{\tau(x)<\infty\}}\big] \geq 1$$

by the definition of the ruin time. Therefore, from (5.9) we conclude that

$$\mathbb{P}[\tau(x) < \infty] \leq \frac{e^{-\hat{r}x}}{\mathbb{E}\big[e^{-\hat{r}X_\tau(x)} \,/\, \tau(x) < \infty\big]} \leq e^{-\hat{r}x},$$

which yields (5.2).

What is left is to note that the larger $\hat{r}$ we choose, the better bound in (5.2) we get. Thus, if condition 2) of the theorem holds and $r_0 < \frac{2c_2}{b^2}$, then we set $\hat{r} = r_0$. If condition 2) of the theorem holds and $r_0 \geq \frac{2c_2}{b^2}$, then (5.2) is true for all $\hat{r} \in \big(0, \frac{2c_2}{b^2}\big)$; hence, it is also true for $\hat{r} = \frac{2c_2}{b^2}$. This completes the proof. □

6. SUFFICIENT CONDITIONS FOR I_1 BEING FINITE AND I_2 BEING INFINITE

Consider now equation (2.1). Let I_1 and I_2 be defined by (2.2). The following lemmas provide sufficient conditions for I_1 being finite and I_2 being infinite.

Lemma 6.1. *If condition* (2.3) *holds, then* $I_1 < +\infty$.

Proof. Since $\int_x^{+\infty} \frac{1}{v^{1+\varepsilon}}\, dv < +\infty$ for all $\varepsilon > 0$, it suffices to show that

$$\limsup_{v\to+\infty} \frac{\exp\left\{-\frac{2}{b^2}\int_x^v \frac{p(u)}{u^2}\, du\right\}}{\exp\left\{-(1+\varepsilon)\ln v\right\}} < +\infty \quad \text{for some} \quad \varepsilon > 0 \tag{6.1}$$

in order to get $I_1 < +\infty$.

We can rewrite (6.1) as

$$\limsup_{v\to+\infty} \exp\left\{(1+\varepsilon)\ln v - \frac{2}{b^2}\int_x^v \frac{p(u)}{u^2}\, du\right\} < +\infty \quad \text{for some} \quad \varepsilon > 0,$$

which gives (2.3). □

Lemma 6.2. *If* $p(0) > 0$, *then* $I_2 = -\infty$.

Proof. It is easily seen that

$$\begin{aligned} -I_2 &\geq \int_0^x \exp\left\{\frac{2p(0)}{b^2}\int_v^x \frac{1}{u^2}\, du\right\} dv = \exp\left\{-\frac{2p(0)}{b^2 x}\right\}\cdot \int_0^x \exp\left\{\frac{2p(0)}{b^2 v}\right\} dv \\ &= \exp\left\{-\frac{2p(0)}{b^2 x}\right\}\cdot \int_{1/x}^{+\infty} \frac{1}{u^2}\exp\left\{\frac{2p(0)u}{b^2}\right\} du = +\infty, \end{aligned}$$

which proves the lemma. □

7. AUXILIARY THEOREMS

Consider the following stochastic differential equation

$$X_t = x + \int_0^t p(X_s)\, ds + \int_0^t b(X_s)\, dW_s, \quad t \geq 0, \tag{7.1}$$

where $x \in \mathbb{R}$, $(W_t)_{t\geq 0}$ is a standard Brownian motion, $p\colon \mathbb{R}\to\mathbb{R}$, and $b\colon \mathbb{R}\to\mathbb{R}$.

Theorem 7.1 ([11, pp. 178–179], Theorem 3.1)**.** *If the functions $p(l)$ and $b(l)$ are locally Lipschitz continuous on $\mathbb{R}$, then* (7.1) *has a unique strong solution up to the time $t^* = \inf\{t \ge 0\colon X_t \notin (-\infty, +\infty)\}$.*

Theorem 7.2 ([11, p. 447], Theorem 3.1)**.** *Let $(X_t)_{t\ge 0}$ follow* (7.1)*, the functions $p(l)$ and $b(l)$ are continuously differentiable on (l_1, l_2), and $b^2(l) > 0$ on (l_1, l_2), where l_1 and l_2 are such that $-\infty \le l_1 < x < l_2 \le +\infty$. For all $l \in (l_1, l_2)$, we define*

$$I(l) = \int\limits_x^l \exp\left\{ -\int\limits_x^v \frac{2p(u)}{b^2(u)}\, du \right\} dv.$$

Moreover, let $I(l_1) = \lim_{l\downarrow l_1} I(l)$, $I(l_2) = \lim_{l\uparrow l_2} I(l)$, and

$$t^*_{(l_1,l_2)} = \inf\{t \ge 0\colon X_t \notin (l_1, l_2)\}.$$

1. *If $I(l_1) = -\infty$ and $I(l_2) = +\infty$, then*
$$\mathbb{P}\big[t^*_{(l_1,l_2)} = \infty\big] = \mathbb{P}\big[\limsup_{t\uparrow\infty} X_t = l_2\big] = \mathbb{P}\big[\liminf_{t\uparrow\infty} X_t = l_1\big] = 1.$$
2. *If $I(l_1) > -\infty$ and $I(l_2) = +\infty$, then $\lim_{t\uparrow t^*_{(l_1,l_2)}} X_t$ exists a.s. and*
$$\mathbb{P}\big[\lim_{t\uparrow t^*_{(l_1,l_2)}} X_t = l_1\big] = \mathbb{P}\big[\sup_{t< t^*_{(l_1,l_2)}} X_t < l_2\big] = 1.$$
3. *If $I(l_1) = -\infty$ and $I(l_2) < +\infty$, then $\lim_{t\uparrow t^*_{(l_1,l_2)}} X_t$ exists a.s. and*
$$\mathbb{P}\big[\lim_{t\uparrow t^*_{(l_1,l_2)}} X_t = l_2\big] = \mathbb{P}\big[\inf_{t< t^*_{(l_1,l_2)}} X_t > l_1\big] = 1.$$
4. *If $I(l_1) > -\infty$ and $I(l_2) < +\infty$, then $\lim_{t\uparrow t^*_{(l_1,l_2)}} X_t$ exists a.s. and*
$$\mathbb{P}\big[\lim_{t\uparrow t^*_{(l_1,l_2)}} X_t = l_1\big] = 1 - \mathbb{P}\big[\lim_{t\uparrow t^*_{(l_1,l_2)}} X_t = l_2\big] = \frac{I(l_2)}{I(l_2) - I(l_1)}.$$

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