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HILDEBRANDT'S THEOREM FOR THE ESSENTIAL SPECTRUM

Janko Bračič and Cristina Diogo

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Abstract. We prove a variant of Hildebrandt's theorem which asserts that the convex hull of the essential spectrum of an operator A on a complex Hilbert space is equal to the intersection of the essential numerical ranges of operators which are similar to A. As a consequence, it is given a necessary and sufficient condition for zero not being in the convex hull of the essential spectrum of A.

Keywords: essential spectrum, essential numerical range, Hildebrandt's theorem.

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1. INTRODUCTION

The notion of the numerical range of an element in a normed algebra is well known and extensively studied during last five decades (see the standard references [2,3,5,6]). In this note we confine our self to unital C^*-algebras.

If $\mathcal{A}$ is a C^*-algebra with the identity 1, then let $\mathcal{A}^*$ denote its topological dual and let $\mathcal{P} = \{f \in \mathcal{A}^* : f(1) = 1 = \|f\|\}$ be the set of all normalized states on $\mathcal{A}$. The numerical range of an element $a \in \mathcal{A}$ is defined by

$$V(a) = \{f(a) : f \in \mathcal{P}\}. \tag{1.1}$$

This set is compact, convex and contains the spectrum $\sigma(a)$ (see [9, Theorem 1]). If $\mathcal{A}$ is the C^*-algebra $\mathcal{B}(\mathscr{H})$ of all bounded linear operators on a complex Hilbert space $\mathscr{H}$, then ([9, Corollary on p. 420])

$$V(T) = \overline{W(T)} \qquad \text{for any} \quad T \in \mathcal{B}(\mathscr{H}),$$

where

$$W(T) = \{\langle Tx, x\rangle : x \in \mathscr{H}, \|x\| = 1\}$$

is the usual numerical range.

As already mentioned, the spectrum of $a \in \mathcal{A}$ is contained in the numerical range of a. Since the numerical range is convex one has

$$\operatorname{conv}\bigl(\sigma(a)\bigr) = V(a).$$

Denote by $\operatorname{inv}(\mathcal{A})$ the set of all invertible elements in $\mathcal{A}$. In the case of the C^*-algebra $\mathcal{B}(\mathscr{H})$, Hildebrandt has proved the following result (see [7]).

Theorem 1.1 (Hildebrandt's Theorem)**.** *For every $A \in \mathcal{B}(\mathscr{H})$,*

$$\operatorname{conv}\bigl(\sigma(A)\bigr) = \bigcap_{S \in \operatorname{inv}(\mathcal{B}(\mathscr{H}))} \overline{W(SAS^{-1})}.$$

2. RESULTS

We include here a slightly more general version of Hildebrandt's theorem. The proof relies on the following lemma by Murphy and West [8]. For the sake of completeness we include its proof. We denote by $r(a)$ the spectral radius of $a \in \mathcal{A}$.

Lemma 2.1. *Let $\mathcal{A}$ be a C^*-algebra and $a \in \mathcal{A}$. For any $\varepsilon > 0$, there exists $s \in \operatorname{inv}(\mathcal{A})$ such that $\|sas^{-1}\| < r(a) + \varepsilon$.*

Proof. Let $b = \frac{1}{r(a)+\varepsilon} a$. Then $r(b) < 1$, i.e., by the Gelfand-Beurling formula, $\lim_{n\to\infty} \|b^n\|^{1/n} < 1$. It follows that the series $\sum_{n=0}^{\infty} \|(b^n)^* b^n\| = \sum_{n=0}^{\infty} \|b^n\|^2$ converges. Hence $c = \sum_{n=0}^{\infty} (b^n)^* b^n \in \mathcal{A}$ and $c \geq 1$. Let $s = \sqrt{c}$. Then $s \geq 1$, which means that it is invertible. Since $0 \leq 1 - s^{-2} \leq 1$, we have

$$\begin{aligned}
\|sbs^{-1}\|^2 = \|s^{-1} b^* s^2 b s^{-1}\| &= \Bigl\| s^{-1} \sum_{n=1}^{\infty} (b^n)^* b^n s^{-1} \Bigr\| \\
&= \|s^{-1}(s^2-1)s^{-1}\| = \|1 - s^{-2}\| = r(1-s^{-2}) < 1.
\end{aligned}$$

It is obvious now that $\|sas^{-1}\| < r(a) + \varepsilon$. □

Theorem 2.2. *Let $\mathcal{A}$ be a C^*-algebra and $a \in \mathcal{A}$. Then*

$$\operatorname{conv}\bigl(\sigma(a)\bigr) = \bigcap_{s \in \operatorname{inv}(\mathcal{A})} V(sas^{-1}). \tag{2.1}$$

Proof. Let $a \in \mathcal{A}$. Since $\sigma(a) \subseteq V(a)$ and, because of the convexity of the numerical range, one actually has $\operatorname{conv}\bigl(\sigma(a)\bigr) \subseteq V(a)$. Since the spectrum is preserved by similarities one has $\operatorname{conv}\bigl(\sigma(a)\bigr) \subseteq \bigcap_{s \in \operatorname{inv}(\mathcal{A})} V(sas^{-1})$.

To prove the other inclusion, let $\lambda \in \mathbb{C} \setminus \operatorname{conv}\bigl(\sigma(a)\bigr)$. Since $\operatorname{conv}\bigl(\sigma(a)\bigr)$ is a compact convex set there exists a disk $\mathbb{D}(\mu,\rho)$ such that $\lambda \notin \overline{\mathbb{D}(\mu,\rho)}$ and $\operatorname{conv}\bigl(\sigma(a)\bigr) \subseteq \mathbb{D}(\mu,\rho)$. Hence $\lambda - \mu \notin \overline{\mathbb{D}(0,\rho)}$ and $\operatorname{conv}\bigl(\sigma(a-\mu)\bigr) \subseteq \mathbb{D}(0,\rho)$, which means that $r(a-\mu) < \rho$. Let $\varepsilon > 0$ be such that $r(a-\mu) + \varepsilon < \rho$. By Lemma 2.1, there exists an invertible element s such that $\|s(a-\mu)s^{-1}\| < r(a-\mu) + \varepsilon < \rho$. It follows that $\lambda - \mu \notin \overline{W(s(a-\mu)s^{-1})}$ and consequently $\lambda \notin \overline{W(sas^{-1})}$. □

Let $\mathcal{K}(\mathscr{H})$ be the ideal of all compact operators on a complex Hilbert space $\mathscr{H}$ and $\mathcal{C}(\mathscr{H}) = \mathcal{B}(\mathscr{H})/\mathcal{K}(\mathscr{H})$ be the Calkin algebra. For $A \in \mathcal{B}(\mathscr{H})$, let $[A]$ denote the equivalence class $A + \mathcal{K}(\mathscr{H})$. If $[A] \in \operatorname{inv}(\mathcal{C}(\mathscr{H}))$, then A is said to be a Fredholm operator (see [4, Definition 5.14]). The set of all Fredholm operators in $\mathcal{B}(\mathscr{H})$ is denoted by $\Phi(\mathscr{H})$.

The essential spectrum of $A \in \mathcal{B}(\mathscr{H})$ is defined by

$$\sigma_{ess}(A) = \{\lambda \in \mathbb{C} : \ A - \lambda I \notin \Phi(\mathscr{H})\},$$

i.e., $\sigma_{ess}(A) = \sigma([A])$, which means that $\lambda \in \mathbb{C}$ is in the essential spectrum of A if and only if the element $[A - \lambda I]$ is not invertible in the Calkin algebra (see [1]). The essential spectrum $\sigma_{ess}(A)$ is a non-empty compact subset of $\sigma(A)$. The essential numerical range $W_{ess}(A)$ of $A \in \mathcal{B}(\mathscr{H})$ is defined analogously, i.e., $W_{ess}(A)$ is equal to $V([A])$, the numerical range of $[A]$ in the Calkin algebra. Note that $W_{ess}(A)$ is a non-empty compact subset of complex numbers.

In the case of the Calkin algebra, (2.1) reads as

$$\operatorname{conv}\big(\sigma([A])\big) = \bigcap_{[S] \in \operatorname{inv}(\mathcal{C}(\mathscr{H}))} V([S][A][S]^{-1}),$$

that is,

$$\operatorname{conv}\big(\sigma_{ess}(A)\big) = \bigcap_{S \in \Phi(\mathscr{H})} V([S][A][S]^{-1}).$$

Since,

$$\bigcap_{S \in \Phi(\mathscr{H})} V([S][A][S]^{-1}) \subseteq \bigcap_{S \in \operatorname{inv}(\mathcal{B}(\mathscr{H}))} V([SAS^{-1}]) = \bigcap_{S \in \operatorname{inv}(\mathcal{B}(\mathscr{H}))} W_{ess}(SAS^{-1}),$$

we have

$$\operatorname{conv}\big(\sigma_{ess}(A)\big) \subseteq \bigcap_{S \in \operatorname{inv}(\mathcal{B}(\mathscr{H}))} W_{ess}(SAS^{-1}). \tag{2.2}$$

The goal of this paper is to show that (2.2) is actually an equality, see Theorem 2.3.

We need the notion of the Weyl spectrum. Recall that by the Atkinson Theorem (see [4, Theorem 5.17]), $A \in \mathcal{B}(\mathscr{H})$ is a Fredholm operator if and only if its range $A\mathscr{H}$ is closed and the kernels $\ker(A)$ and $\ker(A^*)$ are finite dimensional. The index of $A \in \Phi(\mathscr{H})$ is then defined as $\operatorname{ind}(A) = \dim(\ker(A)) - \dim(\ker(A^*))$. Let $\Phi_0(\mathscr{H})$ be the set of all Fredholm operators with index 0. The Weyl spectrum of $A \in \mathcal{B}(\mathscr{H})$ is

$$\sigma_w(A) = \{\lambda \in \mathbb{C} : \ A - \lambda I \notin \Phi_0(\mathscr{H})\}.$$

Note that every invertible operator is a Fredholm operator with index 0, i.e., $\mathrm{inv}(\mathcal{B}(\mathscr{H})) \subseteq \Phi_0(\mathscr{H})$. Since $\Phi_0(\mathscr{H}) \subseteq \Phi(\mathscr{H})$, we have the following inclusions

$$\sigma_{ess}(A) \subseteq \sigma_w(A) \subseteq \sigma(A). \tag{2.3}$$

Schechter proved (see [1, Theorem 2.5]) that, for any operator $A \in \mathcal{B}(\mathscr{H})$,

$$\sigma_w(A) = \bigcap_{K\in\mathcal{K}(\mathscr{H})} \sigma(A+K). \tag{2.4}$$

It follows from (2.3) and (2.4) that $\sigma_w(A)$ is a non-empty compact subset of $\sigma(A)$. By [9, Theorem 9] one has

$$W_{ess}(A) = \bigcap_{K\in\mathcal{K}(\mathscr{H})} \overline{W(A+K)}, \quad \text{for any } A \in \mathcal{B}(\mathscr{H}). \tag{2.5}$$

Now, since $\sigma(A+K) \subseteq \overline{W(A+K)}$ for every $K \in \mathcal{K}(\mathscr{H})$, we have

$$\bigcap_{K\in\mathcal{K}(\mathscr{H})} \sigma(A+K) \subseteq \bigcap_{K\in\mathcal{K}(\mathscr{H})} \overline{W(A+K)}. \tag{2.6}$$

However, by (2.4), the left hand side of (2.6) is $\sigma_w(A)$, and by (2.5), the right hand side of (2.6) is $W_{ess}(T)$. Hence, by the convexity of the essential numerical range, $\mathrm{conv}(\sigma_w(A)) \subseteq W_{ess}(A)$. Since the convex hulls of the essential spectrum and the Weyl spectrum coincide we conclude that

$$\mathrm{conv}(\sigma_{ess}(A)) = \mathrm{conv}(\sigma_w(A)) \subseteq W_{ess}(A). \tag{2.7}$$

Now we are able to prove the remaining part of our main result.

Theorem 2.3. *For every $A \in \mathcal{B}(\mathscr{H})$,*

$$\mathrm{conv}\big(\sigma_{ess}(A)\big) = \bigcap_{S\in\mathrm{inv}(\mathcal{B}(\mathscr{H}))} W_{ess}(SAS^{-1}).$$

Proof. We have to prove the inclusion

$$\mathrm{conv}\big(\sigma_{ess}(A)\big) \supseteq \bigcap_{S\in\mathrm{inv}(\mathcal{B}(\mathscr{H}))} W_{ess}(SAS^{-1}).$$

By [9], there exists $K_0 \in \mathcal{K}(\mathscr{H})$ such that $\sigma_w(A) = \sigma(A+K_0)$. Therefore, by (2.7),

$$\mathrm{conv}\big(\sigma_{ess}(A)\big) = \mathrm{conv}\big(\sigma(A+K_0)\big). \tag{2.8}$$

By Hildebrandt's theorem (Theorem 1.1), we have

$$\mathrm{conv}\big(\sigma(A+K_0)\big) = \bigcap_{S\in\mathrm{inv}(\mathcal{B}(\mathscr{H}))} \overline{W(S(A+K_0)S^{-1})},$$

that is,

$$\operatorname{conv}\bigl(\sigma(A+K_0)\bigr)=\bigcap_{S\in\operatorname{inv}(\mathcal{B}(\mathscr{H}))}\overline{W(SAS^{-1}+K_s)},$$

where $K_s=SK_0S^{-1}$. It follows that

$$\operatorname{conv}\bigl(\sigma(A+K_0)\bigr)\supseteq\bigcap_{S\in\operatorname{inv}(\mathcal{B}(\mathscr{H}))}\ \bigcap_{K'\in\mathcal{K}(\mathscr{H})}\overline{W(SAS^{-1}+K')}. \tag{2.9}$$

By (2.5), the right-hand side of (2.9) is $\bigcap_{S\in\operatorname{inv}(\mathcal{B}(\mathscr{H}))}W_{ess}(SAS^{-1})$ and therefore, because of (2.8), we may conclude that

$$\operatorname{conv}\bigl(\sigma_{ess}(A)\bigr)\supseteq\bigcap_{S\in\operatorname{inv}(\mathcal{B}(\mathscr{H}))}W_{ess}(SAS^{-1}). \qquad \square$$

We conclude the paper with the following corollary of Theorem 2.3.

Corollary 2.4. *Let $A\in\mathcal{B}(\mathscr{H})$. Then $0\notin\operatorname{conv}\bigl(\sigma_{ess}(A)\bigr)$ if and only if there exists a positive definite operator $P\in\mathcal{B}(\mathscr{H})$ such that $0\notin W_{ess}(PA)$.*

Proof. First we will show that, for any invertible $S\in\mathcal{B}(\mathscr{H})$, zero is in $W_{ess}(SAS^{-1})$ if and only if zero is in $W_{ess}(S^*SA)$. Let $S\in\operatorname{inv}(\mathcal{B}(\mathscr{H}))$ be arbitrary and assume that $0\in W_{ess}(SAS^{-1})$. By (2.5), $0\in\overline{W(SAS^{-1}+K)}$ for every operator $K\in\mathcal{K}(\mathscr{H})$. Let K be fixed. Then there exists a sequence $(x_n)_{n=1}^{\infty}\subseteq\mathscr{H}$ of unit vectors such that

$$\langle(SAS^{-1}+K)x_n,x_n\rangle\longrightarrow 0. \tag{2.10}$$

Denote $y_n=\|S^{-1}x_n\|^{-1}S^{-1}x_n$ ($n\in\mathbb{N}$). Since $1=\|x_n\|\le\|S\|\|S^{-1}x_n\|$ one has $\|S^{-1}x_n\|^{-2}\le\|S\|^2$. Thus, because of (2.10), the sequence

$$\langle(S^*SA+S^*KS)y_n,y_n\rangle=\|S^{-1}x_n\|^{-2}\langle(SAS^{-1}+K)x_n,x_n\rangle$$

converges to 0, i.e., $0\in\overline{W(S^*SA+S^*KS)}$. Since K is an arbitrary compact operator and S is invertible we may conclude that $0\in W_{ess}(S^*SA)$.

For the opposite implication assume that $0\in W_{ess}(S^*SA)$, i.e., $0\in\overline{W(S^*SA+K)}$ for every $K\in\mathcal{K}(\mathscr{H})$. Let K be fixed and $(x_n)_{n=1}^{\infty}\subseteq\mathscr{H}$ a sequence of unit vectors such that

$$\langle(S^*SA+K)x_n,x_n\rangle\longrightarrow 0.$$

We denote $y_n=\|Sx_n\|^{-1}Sx_n$ ($n\in\mathbb{N}$). Since $\|Sx_n\|^{-2}\le\|S^{-1}\|^2$ for any n we have

$$\langle(SAS^{-1}+(S^*)^{-1}KS^{-1})y_n,y_n\rangle=\|Sx_n\|^{-2}\langle(S^*SA+K)x_n,x_n\rangle\longrightarrow 0.$$

As before we conclude that $0\in W_{ess}(SAS^{-1})$.

To finish the proof assume that $0\notin\operatorname{conv}\bigl(\sigma_{ess}(A)\bigr)$. Then, by Theorem 2.3, there exists $S\in\operatorname{inv}(\mathcal{B}(\mathscr{H}))$ such that $0\notin W_{ess}(SAS^{-1})$, which gives $0\notin W_{ess}(PA)$ for the positive definite operator $P=S^*S$. On the other hand, if $0\notin W_{ess}(PA)$ for a positive definite operator P, then $0\notin W_{ess}(SAS^{-1})$, where S is an arbitrary invertible operator such that $P=S^*S$. By Theorem 2.3, $0\notin\operatorname{conv}\bigl(\sigma_{ess}(A)\bigr)$. $\square$

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Janko Bračič
janko.bracic@fmf.uni-lj.si

University of Ljubljana, IMFM
Jadranska ul. 19
SI-1000 Ljubljana
Slovenia

Cristina Diogo
cristina.diogo@iscte.pt

Instituto Universitário de Lisboa
Departamento de Matemática
Av. das Forças Armadas
1649-026 Lisboa, Portugal

Center for Mathematical Analysis, Geometry, and Dynamical Systems
Mathematics Department
Instituto Superior Técnico, Universidade de Lisboa
Av. Rovisco Pais, 1049-001 Lisboa, Portugal

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