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BOUNDED, ASYMPTOTICALLY STABLE, AND L^1 SOLUTIONS OF CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. The existence of bounded solutions, asymptotically stable solutions, and L^1 solutions of a Caputo fractional differential equation has been studied in this paper. The results are obtained from an equivalent Volterra integral equation which is derived by inverting the fractional differential equation. The kernel function of this integral equation is weakly singular and hence the standard techniques that are normally applied on Volterra integral equations do not apply here. This hurdle is overcomed using a resolvent equation and then applying some known properties of the resolvent. In the analysis Schauder's fixed point theorem and Liapunov's method have been employed. The existence of bounded solutions are obtained employing Schauder's theorem, and then it is shown that these solutions are asymptotically stable by a definition found in [C. Avramescu, C. Vladimirescu, *On the existence of asymptotically stable solution of certain integral equations*, Nonlinear Anal. 66 (2007), 472–483]. Finally, the L^1 properties of solutions are obtained using Liapunov's method.

Keywords: Caputo fractional differential equations, Volterra integral equations, weakly singular kernel, Schauder fixed point theorem, Liapunov's method.

Mathematics Subject Classification: 34K20, 45J05, 45D05.

1. INTRODUCTION

We consider the Caputo fractional differential equation of order q

$$^cD^q x(t) = f(t) - g(t, x(t)), \quad x(0) = x_0 \in \mathbb{R}, \quad 0 < q < 1, \tag{1.1}$$

where $f, g : [0, \infty) \to \mathbb{R}$ are continuous functions.

Equation (1.1) can be inverted into the equivalent Volterra integral equation

$$x(t) = x_0 - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[g(s, x(s)) - f(s)]ds \tag{1.2}$$

where Γ is the gamma function. The proof of this inversion can be found in ([8], p. 54), or ([6], pp. 78, 86, 103).

In this paper we prove the existence of bounded solutions, asymptotically stable solutions, and L^1 solutions of (1.2) when $g(t, x) = x + h(t, x)$. In the analysis we use the resolvent equation for a linear Volterra equation, Schauder fixed point theorem, and Liapunov's method. Schauder's fixed point theorem is used to obtain the existence of bounded solutions. Then it is shown that these solutions are asymptotically stable by a definition given in [2]. Finally, Liapunov's method is used to show the L^1 properties of solutions under certain conditions.

Let

$$C(t-s) = \frac{1}{\Gamma(q)}(t-s)^{q-1}, \quad 0 < q < 1. \tag{1.3}$$

Then, for $g(t, x) = x + h(t, x)$, equation (1.2) can be written as the familiar Volterra integral equation

$$x(t) = x_0 - \int_0^t C(t-s)[x(s) + h(s, x(s)) - f(s)]ds. \tag{1.4}$$

Let us present some known results regarding the Volterra equation (1.4) and the associated resolvent equation (cf. [10, pp. 189–193]). A function $x(t)$ is a solution of (1.4) if and only if $x(t)$ satisfies

$$x(t) = y(t) - \int_0^t R(t-s)[h(s, x(s)) - f(s)]ds, \tag{1.5}$$

where the function $y(t)$ is given by

$$y(t) = x_0 - \int_0^t R(t-s)x_0 ds, \tag{1.6}$$

and the function $R(t)$, known as the resolvent kernel of $C(t)$, is the solution of the resolvent equation

$$R(t) = C(t) - \int_0^t C(t-s)R(s)ds. \tag{1.7}$$

The function $C(t)$ defined in (1.3) is completely monotone on $[0, \infty)$ in the sense that $(-1)^m C^{(m)}(t) \geq 0$ for $m = 0, 1, 2, \ldots$ and $t \in (0, \infty)$. This $C(t)$ satisfies the conditions of Theorem 6.2 of [10], which states that the associated resolvent kernel $R(t)$ satisfies, for all $t \geq 0$,

$$0 \leq R(t) \leq C(t), \quad R(t) \to 0 \quad \text{as} \quad t \to \infty, \tag{1.8}$$

and that

$$C(t) \notin L^1[0,\infty) \Rightarrow \int_0^\infty R(t)dt = 1. \tag{1.9}$$

Also, it is stated in Theorem 7.2 of [10] that the resolvent $R(t)$ is completely monotone on $0 \leq t < \infty$.

The information presented above is found in [4], which contains a considerable amount of work on the use of resolvent in the study of Caputo fractional differential equation (1.1).

From (1.9) we see that $R(t) \in L^1[0,\infty)$. Therefore, $\int_0^t R(t-s)ds$ is continuous in t for all $t \geq 0$, and for $t_2 \geq t_1$, the resolvent R satisfies the following property.

$$\int_0^{t_1} [R(t_2 - s) - R(t_1 - s)]ds \to 0, \tag{1.10}$$

as $|t_1 - t_2| \to 0$.

We remark that the function $C(t,s) = C(t-s)$ defined in (1.3) is weakly singular on $0 \leq s \leq t < \infty$ by the following definition. This definition is obtained from [3]. It can also be found in ([4, p. 25]).

Definition 1.1. Let $\Omega_T := \{(t,s) : 0 \leq s \leq t \leq T\}$. Function $C(t,s)$ is weakly singular on the set Ω_T if it is discontinuous in Ω_T; but for each $t \in [0,T]$, $C(t,s)$ has at most finitely many discrete discontinuities in the interval $0 \leq s \leq t$ and for every continuous function $\phi : [0,T] \to \mathbb{R}$

$$\int_0^t C(t,s)\phi(s)ds$$

and

$$\int_0^t |C(t,s)|ds$$

both exist and are continuous on $[0,T]$. If $C(t,s)$ is weakly singular on Ω_T for every $T > 0$, then it is weakly singular on the set $\Omega_T := \{(t,s) : 0 \leq s \leq t < \infty\}$.

We observe that the kernels of Volterra integral equations obtained from Caputo fractional differential equations with $0 < q < 1$ are weakly singular. Caputo fractional differential equations arise in many applications. Even the single value of $q = \frac{1}{2}$ is found in a number of important real-world problems (cf. [7, 9–12]).

In recent years, many researchers have focused on discrete fractional calculus, which includes fractional difference equations. We refer to [1] and the references therein for some studies on discrete fractional calculus.

Let $\mathbb{R}_+ := [0, \infty)$ and

$$BC := \{x : \mathbb{R}_+ \to \mathbb{R}, x \text{ is bounded and continuous}\}.$$

Then BC is a Banach space with the norm $\|x\| = \sup_{t\geq 0} |x(t)|$.

Definition 1.2 ([2]). A function x is said to be asymptotically stable solution of equation (1.2) if for every $\epsilon > 0$, there exists a $T = T(\epsilon)$ such that for every $t \geq T$, and for every other solution y of (1.2), $|x(t) - y(t)| \leq \epsilon$.

Define the space $C_l \subset BC$ by

$$C_l := \{x \in BC, \lim_{t\to\infty} x(t) \in \mathbb{R} \text{ exists}\}.$$

Definition 1.3. A family $\mathcal{A} \subset C_l$ is called equiconvergent if for every $\epsilon > 0$, there exists a $T(\epsilon) > 0$, such that for all $x \in \mathcal{A}$, and for all $t_1, t_2 \geq T$, $|x(t_1) - x(t_2)| \leq \epsilon$.

On the space C_l the following compactness criterion holds (see [2]).

Lemma 1.4. *A family $\mathcal{A} \subset C_l$ is relatively compact if and only if*

(a) *$\mathcal{A}$ is uniformly bounded,*
(b) *$\mathcal{A}$ is equicontinuous on compact subsets of $\mathbb{R}_+$,*
(c) *$\mathcal{A}$ is equiconvergent.*

We now present a lemma that will be used in this paper later.

Lemma 1.5. *Suppose, for a function $K(t, s)$, $0 \leq s \leq t < \infty$, the following hypotheses hold:*

(H1) *there exists $M > 0$ such that*

$$\int_0^t |K(t,s)|ds \leq M \quad \textit{for all} \quad t \in \mathbb{R}_+,$$

(H2) *for all $T > 0$, one has*

$$\lim_{t\to\infty} \int_0^T K(t,s)ds = 0,$$

(H3)

$$\lim_{t\to\infty} \int_0^t K(t,s)ds = 1.$$

Then for every $x \in C_l$,

$$\lim_{t\to\infty} \int_0^t K(t,s)x(s)ds = \lim_{t\to\infty} x(t).$$

A proof of Lemma 1.5 is available in [2].

2. BOUNDED SOLUTIONS AND ASYMPTOTICALLY STABLE SOLUTIONS

Let $\rho > 0$ be a constant, and let $B_\rho := \{x \in \mathbb{R}, |x| \le \rho\}$. In this section we prove the existence of bounded solutions and asymptotically stable solutions for continuous $h : \mathbb{R}_+ \times B_\rho \to \mathbb{R}$.

For $g(t,x) = x + h(t,x)$, equation (1.4) is the equivalent integral equation of (1.1). Since (1.4) and (1.5) are equivalent, we show the existence of bounded and asymptotically stable solutions of (1.1) by showing the same properties for the solutions of (1.5).

Let

$$m_\rho := \sup\{|h(t,x) - f(t)|, t \in \mathbb{R}_+, x \in B_\rho\} < \infty. \tag{2.1}$$

Suppose

(A1) there exists a $\theta \in \mathbb{R}$, such that $\lim_{t\to\infty}(h(t,x)-f(t)) = \theta$ uniformly with respect to $x \in B_\rho$.

Theorem 2.1. *Suppose assumption* (A1) *holds. Also, suppose there exists a $\rho > 0$ such that*

$$\sup\{|y(t)|, t \in \mathbb{R}_+\} + m_\rho < \rho. \tag{2.2}$$

Then equation (1.5) *has at least one solution in S_ρ, where*

$$S_\rho := \{x \in C_l, \|x\| \le \rho\}.$$

Moreover, every solution in S_ρ is asymptotically stable.

Proof. For $x \in S_\rho$ define H by

$$Hx(t) = y(t) - \int_0^t R(t-s)[h(s,x(s)) - f(s)]ds, \tag{2.3}$$

where $y(t)$ satisfies (1.6).

The function $R(t-s)$ satisfies the hypotheses of Lemma 1.5. Clearly, one can see from (1.9), that R satisfies (H1) and (H3). It is easy to verify that the kernel C defined in (1.3) satisfies (H2). Therefore, by (1.8), the resolvent R satisfies $(H2)$.

Therefore, by Lemma 1.5,

$$\lim_{t\to\infty} \int_0^t R(t-s)[h(s,x(s)) - f(s)]ds = \theta, \tag{2.4}$$

the limit being uniform with respect to $x \in S_\rho$.

It follows from (1.6), (1.8) and (1.9) that $y(t)$ is bounded, and that $\lim_{t\to\infty} y(t) = 0$. Therefore $HS_\rho \subset C_l$. In addition, using (2.1), (2.2), along with (1.8) and (1.9) one obtains from (2.3),

$$|Hx(t)| \leq |y(t)| + \int_0^t |R(t-s)||h(s,x(s)) - f(s)|ds$$

$$\leq |y(t)| + m_\rho \int_0^t R(s)ds$$

$$\leq |y(t)| + m_\rho < \rho.$$

This shows that $HS_\rho \subset S_\rho$, which means HS_ρ is uniformly bounded.

Let us define the operators: $U : S_\rho \to C_l$ and $V : S_\rho \to C_l$, by

$$(Ux)(t) = \int_0^t R(t-s)x(s)ds,$$

and

$$(Vx)(t) = h(t, x(t)) - f(t),$$

for all $t \in \mathbb{R}_+$.

Clearly, U is a linear operator, and hence is continuous. The operator V is continuous because the function h is continuous in x. Therefore, the operator H is continuous, because $Hx = y + (U \circ V)x$, for all $x \in S_\rho$.

From (1.6) and (1.9), it follows that $\lim_{t\to\infty} y(t) = 0$. Therefore, from (2.3) and (2.4) we see that $\lim_{t\to\infty}(Hx)(t) = \theta$ uniformly with respect to $x \in S_\rho$. This implies that HS_ρ is equiconvergent.

We have already shown that the set HS_ρ is uniformly bounded. Now, we show that HS_ρ is equicontinuous on compact subsets of $\mathbb{R}_+$. For this, it is sufficient to show that HS_ρ is equicontinuous on interval $[0, \gamma]$, for any $\gamma > 0$. From (1.9), we see $R \in L^1(\mathbb{R}_+)$. Since the convolution of a continuous function and an L^1 function is continuous, the function $y(t)$ defined in (1.6) is continuous for $t \geq 0$. Therefore, $y(t)$ is uniformly continuous on $[0, \gamma]$.

Let $\epsilon > 0$ be arbitrary. Then there exists a $\delta > 0$ such that $t_1, t_2 \in [0, \gamma]$, with $|t_1 - t_2| < \delta$ implies $|y(t_1) - y(t_2)| < \frac{\epsilon}{3}$, $\int_0^{t_1} |R(t_1 - s) - R(t_2 - s)|ds < \frac{\epsilon}{3m_\rho}$. The second property follows from (1.10). Also, we can say that $\int_{t_1}^{t_2} |R(t_2, s)|ds < \frac{\epsilon}{3m_\rho}$ since $R \in L^1[0, \infty)$. Therefore, for $x \in S_\rho$, and $t_1, t_2 \in [0, \gamma]$,

$$
\begin{aligned}
|(Hx)(t_1) - (Hx)(t_2)| \le |y(t_1) - y(t_2)| & \\
&+ \int_0^{t_1} |R(t_1 - s) - R(t_2 - s)||h(s, x(s)) - f(s)|ds \\
&+ \int_{t_1}^{t_2} |R(t_2, s)||h(s, x(s)) - f(s)|ds \\
&< \frac{\epsilon}{3} + m_\rho \frac{\epsilon}{3m_\rho} + m_\rho \frac{\epsilon}{3m_\rho} = \epsilon.
\end{aligned}
$$

This shows that HS_ρ is equicontinuous on compact subsets of $\mathbb{R}_+$. Therefore, by Lemma 1.4, the set HS_ρ is relatively compact. By Schauder's fixed point theorem there exists at least one solution of (1.5) in S_ρ.

Now we show that all solutions of (1.5) in S_ρ are asymptotically stable.

Let

$$\phi(t) = \sup\{|h(t, x) - f(t) - \theta|, x \in B_\rho\}$$

for all $t \in \mathbb{R}_+$, where θ is defined in assumption (A1). Then $\lim_{t\to\infty} \phi(t) = 0$ uniformly with respect to $x \in B_\rho$. Since $R(t) \in L^1[0, \infty)$, it follows from a known result ([4, p. 74, Convolution Lemma]) that

$$\lim_{t\to\infty} \int_0^t R(t - s)\phi(s)ds = 0. \tag{2.5}$$

Let $x_1, x_2 \in S_\rho$ be two solutions of (1.5). Then $x_1(t) = (Hx_1)(t)$, and $x_2(t) = (Hx_2)(t)$. Then for all $t \ge 0$,

$$
\begin{aligned}
|x_1(t) - x_2(t)| &\le \int_0^t |R(t - s)||[h(s, x_1(s)) - f(s) - \theta]|ds \\
&\quad + \int_0^t |R(t - s)||[h(s, x_2(s)) - f(s) - \theta]|ds \\
&\le 2 \int_0^t |R(t - s)|\phi(s)ds.
\end{aligned}
$$

Then by (2.5), we have $|x_1(t) - x_2(t)| \to 0$ as $t \to \infty$, showing that every solution of (1.5) in S_ρ is asymptotically stable. This concludes the proof of Theorem 2.1. □

3. SOLUTIONS WITH L^1 PROPERTY

In this section we study the L^1 property of solutions of (1.5) when $x_0 = 0$. In that case $y(t) = 0$ by (1.6). Then equation (1.5) becomes

$$x(t) = -\int_0^t R(t-s)[h(s,x(s)) - f(s)]ds. \tag{3.1}$$

Let

$$a(t) = \int_0^t R(t-s)f(s)ds. \tag{3.2}$$

Then (3.1) becomes

$$x(t) = a(t) - \int_0^t R(t-s)h(s,x(s))ds. \tag{3.3}$$

Theorem 3.1. *Suppose f is bounded, and there exists a $k < 1$ such that $|h(t,x)| \le k|x|$ for all $x \in \mathbb{R}$, $t \ge 0$. Then any bounded solution function $x(t)$ of (3.3) is in $L^1(\mathbb{R}_+)$.*

Proof. Since f is bounded and R satisfies the properties (1.8) and (1.9), the function $a(t)$ of (3.2) is in $L^1(\mathbb{R}_+)$. From (3.3)

$$\begin{aligned}|x(t)| &\le |a(t)| + \int_0^t R(t-s)|h(s,x(s))|ds \\ &\le |a(t)| + k\int_0^t R(t-s)|x(s)|ds.\end{aligned}$$

This implies

$$-k\int_0^t R(t-s)|x(s)|ds \le |a(t)| - |x(t)|. \tag{3.4}$$

Define a Liapunov function V by

$$V(t) = k\int_0^t \int_{t-s}^{\infty} R(u)du|x(s)|ds, \tag{3.5}$$

where x is a bounded solution of (3.3). Then differentiating $V(t)$ and using (3.4) along with (1.8) and (1.9) yields

$$\begin{aligned} V'(t) &= k\int_0^\infty R(u)du|x(t)| - k\int_0^t R(t-s)|x(s)|ds \\ &\le k|x(t)| + |a(t)| - |x(t)| \\ &= (k-1)|x(t)| + |a(t)|. \end{aligned}$$

Integrating both sides of the above inequality from 0 to t we obtain,

$$V(t) - V(0) \le (k-1)\int_0^t |x(s)|ds + \int_0^t |a(s)|ds.$$

Since $V(t) \ge 0$, $V(0) = 0$ and $(k-1) < 0$,

$$(1-k)\int_0^t |x(s)|ds \le \int_0^t |a(s)|ds.$$

This shows that $x \in L^1(\mathbb{R}_+)$ because $a \in L^1(\mathbb{R}_+)$. This concludes the proof of Theorem 3.1. □

We refer the interested readers to [5] for many results on L^p solutions of fractional differential equations of Caputo type.

The property defined in (1.9) about C and R seems a bit unusual. Here is an example on that property.

Example 3.2. For $q = \frac{1}{2}$, the kernel C defined in (1.3) is

$$C(t-s) = \frac{1}{\Gamma(\frac{1}{2})}(t-s)^{-\frac{1}{2}}.$$

Clearly, $C \notin L^1[0,\infty)$. Employing the Laplace transform method, one can solve the associated resolvent equation

$$R(t) = C(t) - \int_0^t C(t-s)R(s)ds,$$

and obtains

$$R(t) = \frac{1}{\sqrt{\pi t}} - e^t\, erfc(\sqrt{t}).$$

This R satisfies $\int_0^\infty R(t)dt = 1$.

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