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CHARACTERIZATIONS AND DECOMPOSITION OF STRONGLY WRIGHT-CONVEX FUNCTIONS OF HIGHER ORDER

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Abstract. Motivated by results on strongly convex and strongly Jensen-convex functions by R. Ger and K. Nikodem in [*Strongly convex functions of higher order*, Nonlinear Anal. 74 (2011), 661–665] we investigate strongly Wright-convex functions of higher order and we prove decomposition and characterization theorems for them. Our decomposition theorem states that a function f is strongly Wright-convex of order n if and only if it is of the form $f(x) = g(x)+p(x)+cx^{n+1}$, where g is a (continuous) n-convex function and p is a polynomial function of degree n. This is a counterpart of Ng's decomposition theorem for Wright-convex functions. We also characterize higher order strongly Wright-convex functions via generalized derivatives.

Keywords: generalized convex function, Wright-convex function of higher order, strongly convex function.

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1. INTRODUCTION

Let c be a positive constant and $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is called

- *strongly convex with modulus* c if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2 \tag{1.1}$$

  for all $x, y \in I$ and $t \in [0,1]$;
- *strongly Wright-convex with modulus* c if

$$f(tx + (1-t)y) + f((1-t)x + ty) \leq f(x) + f(y) - 2ct(1-t)(x-y)^2 \tag{1.2}$$

  for all $x, y \in I$ and $t \in [0,1]$;

– *strongly Jensen-convex with modulus* c if

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} - \frac{c}{4}(x-y)^2 \tag{1.3}$$

for all $x, y \in I$.

Note that every strongly convex function is also strongly Wright-convex and every strongly Wright-convex function is also strongly Jensen-convex with the same modulus c, but the converse statement is not true (cf. [14, Example 1.1]). Strongly convex functions were introduced in the paper [19] by B. T. Polyak and they play an important role in optimization theory and mathematical economics. Several results on their behaviour and applications can be found in the literature (cf., e.g., [4,8,13,15,19,22–24]). The concept of strongly Wright-convex functions was introduced by N. Merentes, K. Nikodem and S. Rivas in [14] (in connection with their study we also refer to [18]), while strongly Jensen-convex functions were considered, among others, in [1, 4, 17], and [24].

Obviously, the usual notion of convexity, Wright-convexity and Jensen-convexity can be obtained from the definition above in the case when $c = 0$. In [16], C. T. Ng proved that each Wright-convex function f can be represented as the sum of a convex and an additive function (cf. also [9]). A decomposition of real valued strongly Wright-convex functions f defined on an interval I of the form $f(x) = h(x)+a(x)+cx^2$, $(x \in I)$, where h is a convex function and a is an additive function, was obtained in [14]. The aim of this note is to generalize this result to strongly Wright-convex functions of higher order. We also present a characterization of strongly Wright-convex functions of higher order via generalized derivatives.

We note that, throughout this paper, all of our considerations remain valid in the case when the constant c is negative. Then the results so formulated concern higher order approximate convexity, Wright-convexity, and Jensen-convexity, respectively.

2. NOTATION, TERMINOLOGY AND BASIC PROPERTIES

First we recall and also introduce the basic definitions that we shall use throughout this paper. Let n be a positive integer and let $I \subseteq \mathbb{R}$ be an interval.

The n^{th} *order divided difference* of a function $f : I \to \mathbb{R}$ with respect to the pairwise distinct points $x_0, \dots, x_n \in I$ is defined by

$$[x_0, \dots, x_n; f] = \sum_{i=0}^{n} \frac{f(x_i)}{\prod_{\substack{j=0 \\ j\neq i}}^{n} (x_i - x_j)}. \tag{2.1}$$

It is easy to prove that they satisfy the recursivity property

$$[x_0, \dots, x_n; f] = \frac{[x_1, \dots, x_n; f] - [x_0, \dots, x_{n-1}; f]}{x_n - x_0} \tag{2.2}$$

for all positive integers n and $x_0, \dots, x_n \in I$, where $[x_0; f] = f(x_0)$.

According to E. Hopf ([7]) and T. Popoviciu ([20, 21]), a function $f : I \to \mathbb{R}$ is called *convex of order* $n-1$ on I (or *monotone of order* n*)* on I if

$$[x_0, \dots, x_n; f] \geq 0$$

holds for all $x_0 < \cdots < x_n \in I$. By the definition of R. Ger and K. Nikodem ([4]), if c is a positive real number, a function $f : I \to \mathbb{R}$ is called *strongly convex of order* n *with modulus* c (or *strongly* n*-convex with modulus* c) if

$$[x_0, \dots, x_{n+1}; f] \geq c \tag{2.3}$$

is valid for all $x_0 < \cdots < x_{n+1}$ in I.

The $\Delta_{h_1,\dots,h_n}$ *difference* of $f : I \to \mathbb{R}$ with increments $h_1, \dots, h_n$ is defined recursively by

$$\Delta_{h_1} f(x) = f(x + h_1) - f(x),$$

$$\Delta_{h_1,\dots,h_n} f(x) = \Delta_{h_1,\dots,h_{n-1}} f(x + h_n) - \Delta_{h_1,\dots,h_{n-1}} f(x)$$

for each $x \in I$ and $h_1, \dots, h_n > 0$ such that $x + h_1 + \cdots + h_n \in I$. In the case when $h = h_1 = \cdots = h_n$, we also use the notation Δ_h^n instead of $\Delta_{h_1,\dots,h_n}$.

Also based on Hopf's ([7]) and Popoviciu's ([20,21]) definition, a function $f : I \to \mathbb{R}$ is said to be *Jensen-convex of order* n (or n-Jensen-convex) if it satisfies the inequality

$$\Delta_h^{n+1} f(x) \geq 0$$

for all $x \in I$, $h > 0$ such that $x + (n+1)h \in I$. If c is a positive real number, f is called *strongly Jensen-convex of order* n *with modulus* c (or strongly n-Jensen-convex with modulus c) if it fulfills

$$\Delta_h^{n+1} f(x) \geq c(n+1)!h^{n+1} \tag{2.4}$$

for all $x \in I$, $h > 0$ such that $x + (n+1)h \in I$ (cf. [4]).

The function f is said to be *Wright-convex of order* n (or n*-Wright-convex*) if

$$\Delta_{h_1,\dots,h_{n+1}} f(x) \geq 0$$

for all $x \in I$, $h_1, \dots, h_{n+1} > 0$ such that $x + h_1 + \cdots + h_{n+1} \in I$. We call f *strongly Wright-convex of order* n *with modulus* c (or *strongly* n*-Wright-convex with modulus* c) if

$$\Delta_{h_1,\dots,h_{n+1}} f(x) \geq c(n+1)!h_1 \cdots h_{n+1} \tag{2.5}$$

holds for all $x \in I$, $h_1, \dots, h_{n+1} > 0$ such that $x + h_1 + \cdots + h_{n+1} \in I$.

Remark 2.1. It is easy to see that the definitions of strongly n-convex functions, strongly n-Wright-convex functions and strongly n-Jensen-convex functions, with $c = 0$, give the concepts of n-convex, n-Wright-convex and n-Jensen-convex functions, respectively.

We will use the following property of the difference operator in the sequel.

Lemma 2.2 ([6, Lemma 5.1]). *Let n be a positive integer, $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a function. Then the equation*

$$\Delta_{h_1,\dots,h_n} f(x) = h_1 \cdots h_n \sum_{(i_1,\dots,i_n)} [x, x+h_{i_1}, \dots, x+h_{i_1}+\cdots+h_{i_n}; f]$$

is valid for all $x \in I$, $h_1, \dots, h_n > 0$ with $x + h_1 + \cdots + h_n \in I$, where the summation is for all permutations $(i_1, \dots, i_n)$ of the integers $\{1, \dots, n\}$.

Remark 2.3. It is a consequence of the statement above that every function $f : I \to \mathbb{R}$ which is strongly n-convex with modulus c, is also strongly n-Wright-convex with modulus c. Indeed, if f is n-convex with modulus c, then

$$\big[x, x+h_{i_1}, \dots, x+h_{i_1}+\cdots+h_{i_{n+1}}; f\big] \geq c$$

for all $x \in I$ and $h_1, \dots, h_{n+1} > 0$, such that $x + h_1 + \cdots + h_{n+1} \in I$, where $(i_1, \dots, i_n)$ is an arbitrary permutation of the integers $\{1, \dots, n\}$. By Lemma 2.2, we have

$$\begin{aligned}\Delta_{h_1,\dots,h_{n+1}} f(x) &= h_1 \cdots h_{n+1} \sum_{(i_1,\dots,i_{n+1})} \big[x, x+h_{i_1}, \dots, x+h_{i_1}+\cdots+h_{i_{n+1}}; f\big]\\ &\geq c(n+1)! h_1 \cdots h_{n+1},\end{aligned}$$

which means that f is strongly n-Wright-convex with modulus c.

It is also easy to see that a strongly n-Wright-convex function with modulus c is also n-Jensen-convex with modulus c.

Remark 2.4. In the case when $n = 1$, inequality (2.5) reduces to

$$\Delta_{h_1,h_2} f(x) \geq 2ch_1h_2,$$

that is,

$$f(x+h_1+h_2) - f(x+h_1) - f(x+h_2) + f(x) \geq 2ch_1h_2. \tag{2.6}$$

Putting $u = x$, $v = x + h_1 + h_2$ and $t = \frac{h_2}{h_1+h_2}$, we get $x + h_1 = tu + (1-t)v$, $x + h_2 = (1-t)u + tv$ and $h_1h_2 = t(1-t)(u-v)^2$. Thus, property (2.6) gives

$$f(tu + (1-t)v) + f((1-t)u + tv) \leq f(u) + f(v) + 2ct(1-t)(u-v)^2,$$

which means that f is strongly Wright-convex with modulus c. Note that, if $n = 1$, also (2.3) and (2.4) reduces to (1.1) and (1.3), respectively.

3. MAIN RESULTS

Before formulating our main results, we present two lemmas. They can be proved by a simple calculation (cf. also [10] and [11, Chapter 15]).

Lemma 3.1. *The operator $\Delta_{h_1,\dots,h_n}$ is linear, that is, if n is a positive integer, $h_1,\dots,h_n$ and a,b are real numbers, $I\subseteq\mathbb{R}$ is an interval and $f,g:I\to\mathbb{R}$ are arbitrary functions, then*

$$\Delta_{h_1,\dots,h_n}(af+bg)=a\Delta_{h_1,\dots,h_n}f+b\Delta_{h_1,\dots,h_n}g.$$

Lemma 3.2. *Let n be a positive integer and let $h_1,\dots,h_n$ be real numbers. Then*

$$\Delta_{h_1,\dots,h_n}x^n=n!h_1\cdots h_n.$$

Now, we characterize higher order strongly Wright-convex functions via Wright-convex functions of higher order.

Theorem 3.3. *Let n be a positive integer, c be a positive real number, and $I\subseteq\mathbb{R}$ be an interval. A function $f:I\to\mathbb{R}$ is strongly n-Wright-convex with modulus c if and only if the function $g:I\to\mathbb{R}$, $g(x)=f(x)-cx^{n+1}$, $(x\in I)$ is n-Wright-convex.*

Proof. Suppose first that f is strongly n-Wright-convex with modulus c and let $g(x)=f(x)-cx^{n+1}$. Then, by Lemmas 3.1 and 3.2, we have

$$\begin{aligned}\Delta_{h_1,\dots,h_{n+1}}g(x)&=\Delta_{h_1,\dots,h_{n+1}}f(x)-\Delta_{h_1,\dots,h_{n+1}}cx^{n+1}\\&\geq c(n+1)!h_1\cdots h_{n+1}-c(n+1)!h_1\cdots h_{n+1}=0,\end{aligned}$$

which implies that g is n-Wright-convex. Let us assume now that g is n-Wright-convex. For $f(x)=g(x)+cx^{n+1}$, using Lemmas 3.1 and 3.2 again, we obtain

$$\begin{aligned}\Delta_{h_1,\dots,h_{n+1}}f(x)&=\Delta_{h_1,\dots,h_{n+1}}g(x)+\Delta_{h_1,\dots,h_{n+1}}cx^{n+1}\\&\geq 0+c(n+1)!h_1\cdots h_{n+1}=c(n+1)!h_1\cdots h_{n+1},\end{aligned}$$

which gives the strong n-Wright-convexity of f with modulus c. □

In the decomposition of n-Wright-convex and strongly n-Wright-convex functions, polynomial functions are used. A function $f:I\to\mathbb{R}$ is said to be a *polynomial function of degree n* if it satisfies the equation

$$\Delta_h^{n+1}f(x)=0$$

for all $x\in I$, $h>0$ such that $x+(n+1)h\in I$.

The following generalization of Ng's Theorem for Wright-convex functions of higher order was proved by Gy. Maksa and Zs. Páles.

Theorem 3.4 ([12])**.** *Let n be a positive integer and $I\subseteq\mathbb{R}$ be an open interval. A function $f:I\to\mathbb{R}$ is n-Wright-convex if and only if it is of the form*

$$f(x)=h(x)+p(x)\qquad(x\in I),\tag{3.1}$$

where $h:I\to\mathbb{R}$ is an n-convex function and $p:\mathbb{R}\to\mathbb{R}$ is a polynomial function of degree n with $p(\mathbb{Q})=\{0\}$. Furthermore, the decomposition in (3.1) *is unique.*

The following theorem is a counterpart of the statement above for strongly Wright-convex functions of higher order. Note that the above result was proved for open intervals, therefore, the next result is stated also in this setting.

Theorem 3.5. *Let n be a positive integer, c be a positive real number, and $I \subseteq \mathbb{R}$ be an open interval. A function $f : I \to \mathbb{R}$ is strongly n-Wright-convex with modulus c if and only if it is of the form*

$$f(x) = h(x) + p(x) + cx^{n+1} \qquad (x \in I), \tag{3.2}$$

where $h : I \to \mathbb{R}$ is an n-convex function and $p : \mathbb{R} \to \mathbb{R}$ is a polynomial function of degree n with $p(\mathbb{Q}) = \{0\}$. Furthermore, the decomposition in (3.2) is unique.

Proof. The statement can be obtained as a combination of Theorems 3.4 and 3.3. □

In the last part of the paper, we give a characterization of higher order Wright-convex functions via a generalized derivative introduced by Zs. Páles and A. Gilányi in [5].

If n is a positive integer, $I \subseteq \mathbb{R}$ is an interval then the n^{th} *order lower generalized Dinghas interval derivative* of a function $f : I \to \mathbb{R}$ at a point $\xi \in I$ is defined by

$$\underline{\mathrm{D}}^n f(\xi) = \liminf_{\substack{(x \to \xi,\, h_1 \searrow 0, \ldots, h_n \searrow 0) \\ x \le \xi \le x + (h_1 + \cdots + h_n)}} \frac{\Delta_{h_1,\ldots,h_n} f(x)}{h_1 \cdots h_n}.$$

We note that the operator $\underline{\mathrm{D}}^n$ is superlinear, i.e., superadditive and positively homogeneous.

If the limit

$$\lim_{\substack{(x \to \xi,\, h_1 \searrow 0, \ldots, h_n \searrow 0) \\ x \le \xi \le x + (h_1 + \cdots + h_n)}} \frac{\Delta_{h_1,\ldots,h_n} f(x)}{h_1 \cdots h_n} \tag{3.3}$$

exists, we call it the n^{th} *order generalized Dinghas interval derivative* of f at ξ and we denote it by $\mathrm{D}^n f(\xi)$.

Remark 3.6. It is easy to see that, in the case when f is n times differentiable at ξ, then $\underline{\mathrm{D}}^n f(\xi) = f^{(n)}(\xi)$, that is, $\underline{\mathrm{D}}$ is a generalized derivative. We also note that, in the case when $h_1 = \cdots = h_n$ and the limit in (3.3) exists, the definition above gives the so called Dinghas interval derivative, introduced by A. Dinghas in [2] (cf. also [3, 25] and [5]).

The following theorem is a simple consequence of Corollary 1 proved in [5].

Theorem 3.7. *Let n be a positive integer and $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is n-Wright-convex on I if and only if*

$$\underline{\mathrm{D}}^{n+1} f(\xi) \ge 0$$

for all $\xi \in I$.

Finally, we present the characterization theorem for strongly n-Wright-convex functions via the generalized derivative above and we formulate its consequence for $n+1$ times differentiable functions.

Theorem 3.8. *Let n be a positive integer, c be a positive real number, and $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is strongly n-Wright-convex with modulus c if and only if*

$$\underline{\mathrm{D}}^{n+1} f(\xi) \geq c(n+1)! \tag{3.4}$$

for all $\xi \in I$.

Proof. Let first f be an n-Wright-convex function with modulus c. Then, by theorem 3.3, the function $g : I \to \mathbb{R}$, $g(x) = f(x) - cx^{n+1}$, $(x \in I)$ is n-Wright-convex. Using Theorem 3.7 and Lemmas 3.1 and 3.2, we obtain that

$$\underline{\mathrm{D}}^{n+1} f(\xi) = \underline{\mathrm{D}}^{n+1} \left(g(\xi) + c\xi^{n+1}\right) \geq \underline{\mathrm{D}}^{n+1} g(\xi) + \underline{\mathrm{D}}^{n+1} c\xi^{n+1} \geq 0 + c(n+1)! = c(n+1)!$$

for all $\xi \in I$, which gives the first part of the statement. Assume now that f satisfies inequality (3.4) with a $c > 0$ for all $\xi \in I$. Let us consider the function $g : I \to \mathbb{R}$, $g(x) = f(x) - cx^{n+1}$, $(x \in I)$. It is easy to see that, by (3.4) and Lemmas 3.1 and 3.2,

$$\begin{aligned} \underline{\mathrm{D}}^{n+1} g(\xi) &= \underline{\mathrm{D}}^{n+1} \left(f(\xi) - c\xi^{n+1}\right) \geq \underline{\mathrm{D}}^{n+1} f(\xi) + \underline{\mathrm{D}}^{n+1}(-c\xi^{n+1}) \\ &\geq c(n+1)! - c(n+1)! = 0 \end{aligned}$$

for all $\xi \in I$, which, combined with Theorem 3.7, implies our statement. □

Corollary 3.9. *Let n be a positive integer, c be a positive real number, $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ be a function and suppose that f is $n+1$ times differentiable on I. Then f is strongly n-Wright-convex with modulus c if and only if $f^{(n+1)}(\xi) \geq c(n+1)!$ for all $\xi \in I$.*

Proof. The statement is a consequence of Theorem 3.8 and Remark 3.6. □

Remark 3.10. We note, that the corollary above can also be obtained as a consequence of a characterization of strong convex functions of higher order via derivatives given in Theorem 6 in [4], and the fact that in the case of continuous functions, the classes of n-Wright-convex functions and n-convex functions coincide.

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