CHARACTERIZATIONS AND DECOMPOSITION OF STRONGLY WRIGHT-CONVEX FUNCTIONS OF HIGHER ORDER

Attila Gilányi, Nelson Merentes, Kazimierz Nikodem, and Zsolt Páles

Abstract. Motivated by results on strongly convex and strongly Jensen-convex functions by R. Ger and K. Nikodem in [Strongly convex functions of higher order, Nonlinear Anal. 74 (2011), 661–665] we investigate strongly Wright-convex functions of higher order and we prove decomposition and characterization theorems for them. Our decomposition theorem states that a function \( f \) is strongly Wright-convex of order \( n \) if and only if it is of the form \( f(x) = g(x) + p(x) + cx^{n+1} \), where \( g \) is a (continuous) \( n \)-convex function and \( p \) is a polynomial function of degree \( n \). This is a counterpart of Ng’s decomposition theorem for Wright-convex functions. We also characterize higher order strongly Wright-convex functions via generalized derivatives.

Keywords: generalized convex function, Wright-convex function of higher order, strongly convex function.

Mathematics Subject Classification: 26A51, 39B62.

1. INTRODUCTION

Let \( c \) be a positive constant and \( I \subseteq \mathbb{R} \) be an interval. A function \( f : I \to \mathbb{R} \) is called
- strongly convex with modulus \( c \) if
  \[
  f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2
  \]  \hspace{1cm} (1.1)
  for all \( x, y \in I \) and \( t \in [0,1] \);
- strongly Wright-convex with modulus \( c \) if
  \[
  f(tx + (1-t)y) + f((1-t)x + ty) \leq f(x) + f(y) - 2ct(1-t)(x-y)^2
  \]  \hspace{1cm} (1.2)
  for all \( x, y \in I \) and \( t \in [0,1] \);

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– strongly Jensen-convex with modulus c if

\[
f \left( \frac{x + y}{2} \right) \leq \frac{f(x) + f(y)}{2} - \frac{c}{4} (x - y)^2
\]

for all \( x, y \in I \).

Note that every strongly convex function is also strongly Wright-convex and every strongly Wright-convex function is also strongly Jensen-convex with the same modulus c, but the converse statement is not true (cf. [14, Example 1.1]). Strongly convex functions were introduced in the paper [19] by B. T. Polyak and they play an important role in optimization theory and mathematical economics. Several results on their behaviour and applications can be found in the literature (cf., e.g., [4,8,13,15,19,22–24]).

The concept of strongly Wright-convex functions was introduced by N. Merentes, K. Nikodem and S. Rivas in [14] (in connection with their study we also refer to [18]), while strongly Jensen-convex functions were considered, among others, in [1,4,17], and [24].

Obviously, the usual notion of convexity, Wright-convexity and Jensen-convexity can be obtained from the definition above in the case when \( c = 0 \). In [16], C. T. Ng proved that each Wright-convex function \( f \) can be represented as the sum of a convex and an additive function (cf. also [9]). A decomposition of real valued strongly Wright-convex functions \( f \) defined on an interval \( I \) of the form \( f(x) = h(x) + a(x) + cx^2 \), \( x \in I \), where \( h \) is a convex function and \( a \) is an additive function, was obtained in [14].

The aim of this note is to generalize this result to strongly Wright-convex functions of higher order. We also present a characterization of strongly Wright-convex functions of higher order via generalized derivatives.

We note that, throughout this paper, all of our considerations remain valid in the case when the constant \( c \) is negative. Then the results so formulated concern higher order approximate convexity, Wright-convexity, and Jensen-convexity, respectively.

2. NOTATION, TERMINOLOGY AND BASIC PROPERTIES

First we recall and also introduce the basic definitions that we shall use throughout this paper. Let \( n \) be a positive integer and let \( I \subseteq \mathbb{R} \) be an interval.

The \( n^{th} \) order divided difference of a function \( f : I \rightarrow \mathbb{R} \) with respect to the pairwise distinct points \( x_0, \ldots, x_n \in I \) is defined by

\[
[x_0, \ldots, x_n; f] = \sum_{i=0}^{n} \frac{f(x_i)}{\prod_{j=0}^{n} (x_i - x_j)}.
\]

(2.1)

It is easy to prove that they satisfy the recursivity property

\[
[x_0, \ldots, x_n; f] = \frac{[x_1, \ldots, x_n; f] - [x_0, \ldots, x_{n-1}; f]}{x_n - x_0}
\]

(2.2)

for all positive integers \( n \) and \( x_0, \ldots, x_n \in I \), where \( [x_0; f] = f(x_0) \).
According to E. Hopf ([7]) and T. Popoviciu ([20,21]), a function $f : I \to \mathbb{R}$ is called \textit{convex of order} $n - 1$ on $I$ (or \textit{monotone of order} $n$) on $I$ if

$$[x_0, \ldots, x_n; f] \geq 0$$

holds for all $x_0 < \cdots < x_n \in I$. By the definition of R. Ger and K. Nikodem ([4]), if $c$ is a positive real number, a function $f : I \to \mathbb{R}$ is called \textit{strongly convex of order $n$ with modulus $c$} (or \textit{strongly $n$-convex with modulus $c$}) if

$$[x_0, \ldots, x_n; f] \geq c$$  \hspace{1cm} \text{(2.3)}

is valid for all $x_0 < \cdots < x_{n+1} \in I$.

The $\Delta_{h_1, \ldots, h_n}$ \textit{difference} of $f : I \to \mathbb{R}$ with increments $h_1, \ldots, h_n$ is defined recursively by

$$\Delta_{h_1, \ldots, h_n} f(x) = f(x + h_1) - f(x),$$

$$\Delta_{h_1, \ldots, h_n} f(x) = \Delta_{h_1, \ldots, h_{n-1}} f(x + h_n) - \Delta_{h_1, \ldots, h_{n-1}} f(x)$$

for each $x \in I$ and $h_1, \ldots, h_n > 0$ such that $x + h_1 + \cdots + h_n \in I$. In the case when $h = h_1 = \cdots = h_n$, we also use the notation $\Delta^n h$ instead of $\Delta_{h_1, \ldots, h_n}$.

Also based on Hopf’s ([7]) and Popoviciu’s ([20,21]) definition, a function $f : I \to \mathbb{R}$ is said to be \textit{Jensen-convex of order $n$} (or \textit{n-Jensen-convex}) if it satisfies the inequality

$$\Delta^{n+1}_h f(x) \geq 0$$

for all $x \in I$, $h > 0$ such that $x + (n + 1)h \in I$. If $c$ is a positive real number, $f$ is called \textit{strongly Jensen-convex of order $n$ with modulus $c$} (or \textit{strongly $n$-Jensen-convex with modulus $c$}) if it fulfills

$$\Delta^{n+1}_h f(x) \geq c(n + 1)! h^{n+1}$$  \hspace{1cm} \text{(2.4)}

for all $x \in I$, $h > 0$ such that $x + (n + 1)h \in I$ (cf. [4]).

The function $f$ is said to be \textit{Wright-convex of order $n$} (or \textit{n-Wright-convex}) if

$$\Delta_{h_1, \ldots, h_{n+1}} f(x) \geq 0$$

for all $x \in I$, $h_1, \ldots, h_{n+1} > 0$ such that $x + h_1 + \cdots + h_{n+1} \in I$. We call $f$ \textit{strongly Wright-convex of order $n$ with modulus $c$} (or \textit{strongly $n$-Wright-convex with modulus $c$}) if

$$\Delta_{h_1, \ldots, h_{n+1}} f(x) \geq c(n + 1)! h_1 \cdots h_{n+1}$$  \hspace{1cm} \text{(2.5)}

holds for all $x \in I$, $h_1, \ldots, h_{n+1} > 0$ such that $x + h_1 + \cdots + h_{n+1} \in I$.

\textbf{Remark 2.1.} It is easy to see that the definitions of strongly $n$-convex functions, strongly $n$-Wright-convex functions and strongly $n$-Jensen-convex functions, with $c = 0$, give the concepts of $n$-convex, $n$-Wright-convex and $n$-Jensen-convex functions, respectively.
We will use the following property of the difference operator in the sequel.

**Lemma 2.2** ([6, Lemma 5.1]). Let \( n \) be a positive integer, \( I \subseteq \mathbb{R} \) be an interval and \( f : I \to \mathbb{R} \) be a function. Then the equation

\[
\Delta_{h_1, \ldots, h_n} f(x) = h_1 \cdots h_n \sum_{(i_1, \ldots, i_n)} [x, x + h_{i_1}, \ldots, x + h_{i_1} + \cdots + h_{i_n}; f]
\]

is valid for all \( x \in I, h_1, \ldots, h_n > 0 \) with \( x + h_1 + \cdots + h_n \in I \), where the summation is for all permutations \((i_1, \ldots, i_n)\) of the integers \(\{1, \ldots, n\}\).

**Remark 2.3.** It is a consequence of the statement above that every function \( f : I \to \mathbb{R} \) which is strongly \( n \)-convex with modulus \( c \), is also strongly \( n \)-Wright-convex with modulus \( c \). Indeed, if \( f \) is \( n \)-convex with modulus \( c \), then

\[
[x, x + h_{i_1}, \ldots, x + h_{i_1} + \cdots + h_{i_{n+1}}; f] \geq c
\]

for all \( x \in I \) and \( h_1, \ldots, h_{n+1} > 0 \), such that \( x + h_1 + \cdots + h_{n+1} \in I \), where \((i_1, \ldots, i_n)\) is an arbitrary permutation of the integers \(\{1, \ldots, n\}\). By Lemma 2.2, we have

\[
\Delta_{h_1, \ldots, h_{n+1}} f(x) = h_1 \cdots h_{n+1} \sum_{(i_1, \ldots, i_{n+1})} [x, x + h_{i_1}, \ldots, x + h_{i_1} + \cdots + h_{i_{n+1}}; f]
\]

\[
\geq c(n + 1)! h_1 \cdots h_{n+1},
\]

which means that \( f \) is strongly \( n \)-Wright-convex with modulus \( c \).

It is also easy to see that a strongly \( n \)-Wright-convex function with modulus \( c \) is also \( n \)-Jensen-convex with modulus \( c \).

**Remark 2.4.** In the case when \( n = 1 \), inequality (2.5) reduces to

\[
\Delta_{h_1, h_2} f(x) \geq 2ch_1 h_2,
\]

that is,

\[
f(x + h_1 + h_2) - f(x + h_1) - f(x + h_2) + f(x) \geq 2ch_1 h_2.
\]

Putting \( u = x, \ v = x + h_1 + h_2 \) and \( t = \frac{h_2}{h_1 + h_2} \), we get \( x + h_1 = tu + (1 - t)v \), \( x + h_2 = (1 - t)u + tv \) and \( h_1h_2 = t(1 - t)(u - v)^2 \). Thus, property (2.6) gives

\[
f((tu + (1 - t)v) + f((1 - t)u + tv) \leq f(u) + f(v) + 2ct(1 - t)(u - v)^2,
\]

which means that \( f \) is strongly Wright-convex with modulus \( c \). Note that, if \( n = 1 \), also (2.3) and (2.4) reduces to (1.1) and (1.3), respectively.

3. MAIN RESULTS

Before formulating our main results, we present two lemmas. They can be proved by a simple calculation (cf. also [10] and [11, Chapter 15]).
Lemma 3.1. The operator \( \Delta_{h_1,\ldots,h_n} \) is linear, that is, if \( n \) is a positive integer, \( h_1,\ldots,h_n \) and \( a,b \) are real numbers, \( I \subseteq \mathbb{R} \) is an interval and \( f,g : I \to \mathbb{R} \) are arbitrary functions, then
\[
\Delta_{h_1,\ldots,h_n}(af + bg) = a\Delta_{h_1,\ldots,h_n}f + b\Delta_{h_1,\ldots,h_n}g.
\]

Lemma 3.2. Let \( n \) be a positive integer and let \( h_1,\ldots,h_n \) be real numbers. Then
\[
\Delta_{h_1,\ldots,h_n}x^n = nh_1 \cdots h_n.
\]

Now, we characterize higher order strongly Wright-convex functions via Wright-convex functions of higher order.

Theorem 3.3. Let \( n \) be a positive integer, \( c \) be a positive real number, and \( I \subseteq \mathbb{R} \) be an interval. A function \( f : I \to \mathbb{R} \) is strongly \( n \)-Wright-convex with modulus \( c \) if and only if the function \( g : I \to \mathbb{R} \), \( g(x) = f(x) - cx^{n+1} \), \( x \in I \) is \( n \)-Wright-convex.

Proof. Suppose first that \( f \) is strongly \( n \)-Wright-convex with modulus \( c \) and let \( g(x) = f(x) - cx^{n+1} \). Then, by Lemmas 3.1 and 3.2, we have
\[
\Delta_{h_1,\ldots,h_{n+1}}g(x) = \Delta_{h_1,\ldots,h_{n+1}}f(x) - \Delta_{h_1,\ldots,h_{n+1}}cx^{n+1} \\
\geq c(n+1)h_1 \cdots h_{n+1}c(n+1)!h_1 \cdots h_{n+1} = 0,
\]
which implies that \( g \) is \( n \)-Wright-convex. Let us assume now that \( g \) is \( n \)-Wright-convex. For \( f(x) = g(x) + cx^{n+1} \), using Lemmas 3.1 and 3.2 again, we obtain
\[
\Delta_{h_1,\ldots,h_{n+1}}f(x) = \Delta_{h_1,\ldots,h_{n+1}}g(x) + \Delta_{h_1,\ldots,h_{n+1}}cx^{n+1} \\
\geq 0 + c(n+1)h_1 \cdots h_{n+1} = c(n+1)!h_1 \cdots h_{n+1},
\]
which gives the strong \( n \)-Wright-convexity of \( f \) with modulus \( c \).

In the decomposition of \( n \)-Wright-convex and strongly \( n \)-Wright-convex functions, polynomial functions are used. A function \( f : I \to \mathbb{R} \) is said to be a polynomial function of degree \( n \) if it satisfies the equation
\[
\Delta_{h}^{n+1}f(x) = 0
\]
for all \( x \in I \), \( h > 0 \) such that \( x + (n+1)h \in I \).

The following generalization of Ng's Theorem for Wright-convex functions of higher order was proved by Gy. Maksa and Zs. Páles.

Theorem 3.4 ([12]). Let \( n \) be a positive integer and \( I \subseteq \mathbb{R} \) be an open interval. A function \( f : I \to \mathbb{R} \) is \( n \)-Wright-convex if and only if it is of the form
\[
f(x) = h(x) + p(x) \quad (x \in I),
\]
where \( h : I \to \mathbb{R} \) is an \( n \)-convex function and \( p : \mathbb{R} \to \mathbb{R} \) is a polynomial function of degree \( n \) with \( p(\mathbb{Q}) = \{0\} \). Furthermore, the decomposition in (3.1) is unique.
The following theorem is a counterpart of the statement above for strongly Wright-convex functions of higher order. Note that the above result was proved for open intervals, therefore, the next result is stated also in this setting.

**Theorem 3.5.** Let \( n \) be a positive integer, \( c \) be a positive real number, and \( I \subseteq \mathbb{R} \) be an open interval. A function \( f : I \to \mathbb{R} \) is strongly \( n \)-Wright-convex with modulus \( c \) if and only if it is of the form

\[
f(x) = h(x) + p(x) + cx^{n+1} \quad (x \in I),
\]

where \( h : I \to \mathbb{R} \) is an \( n \)-convex function and \( p : \mathbb{R} \to \mathbb{R} \) is a polynomial function of degree \( n \) with \( p(0) = 0 \). Furthermore, the decomposition in (3.2) is unique.

**Proof.** The statement can be obtained as a combination of Theorems 3.4 and 3.3. \( \square \)

In the last part of the paper, we give a characterization of higher order Wright-convex functions via a generalized derivative introduced by Zs. Páles and A. Gilányi in [5].

If \( n \) is a positive integer, \( I \subseteq \mathbb{R} \) is an interval then the \( n \)th order lower generalized Dinghas interval derivative of a function \( f : I \to \mathbb{R} \) at a point \( \xi \in I \) is defined by

\[
D^n f(\xi) = \liminf_{(x \to \xi, h_1 \searrow 0, \ldots, h_n \searrow 0, x \leq \xi \leq x + (h_1 + \cdots + h_n))} \frac{\Delta_{h_1, \ldots, h_n} f(x)}{h_1 \cdots h_n}.
\]

We note that the operator \( D^n \) is superlinear, i.e., superadditive and positively homogeneous.

If the limit

\[
\lim_{(x \to \xi, h_1 \searrow 0, \ldots, h_n \searrow 0, x \leq \xi \leq x + (h_1 + \cdots + h_n))} \frac{\Delta_{h_1, \ldots, h_n} f(x)}{h_1 \cdots h_n}
\]

exists, we call it the \( n \)th order generalized Dinghas interval derivative of \( f \) at \( \xi \) and we denote it by \( D^n f(\xi) \).

**Remark 3.6.** It is easy to see that, in the case when \( f \) is \( n \) times differentiable at \( \xi \), then \( D^n f(\xi) = f^{(n)}(\xi) \), that is, \( D^n \) is a generalized derivative. We also note that, in the case when \( h_1 = \cdots = h_n \) and the limit in (3.3) exists, the definition above gives the so called Dinghas interval derivative, introduced by A. Dinghas in [2] (cf. also [3,25] and [5]).

The following theorem is a simple consequence of Corollary 1 proved in [5].

**Theorem 3.7.** Let \( n \) be a positive integer and \( I \subseteq \mathbb{R} \) be an interval. A function \( f : I \to \mathbb{R} \) is \( n \)-Wright-convex on \( I \) if and only if

\[
D^{n+1} f(\xi) \geq 0
\]

for all \( \xi \in I \).
Finally, we present the characterization theorem for strongly $n$-Wright-convex functions via the generalized derivative above and we formulate its consequence for $n + 1$ times differentiable functions.

**Theorem 3.8.** Let $n$ be a positive integer, $c$ be a positive real number, and $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is strongly $n$-Wright-convex with modulus $c$ if and only if
\[
D^{n+1} f(\xi) \geq c(n+1)!
\]  
for all $\xi \in I$.

**Proof.** Let first $f$ be an $n$-Wright-convex function with modulus $c$. Then, by theorem 3.3, the function $g : I \to \mathbb{R}$, $g(x) = f(x) - cx^{n+1}$, $(x \in I)$ is $n$-Wright-convex. Using Theorem 3.7 and Lemmas 3.1 and 3.2, we obtain that
\[
D^{n+1} f(\xi) = D^{n+1} (g(\xi) + c\xi^{n+1}) \geq D^{n+1} g(\xi) + D^{n+1} c\xi^{n+1} \geq 0 + c(n+1)! = c(n+1)!
\]  
for all $\xi \in I$, which gives the first part of the statement. Assume now that $f$ satisfies inequality (3.4) with a $c > 0$ for all $\xi \in I$. Let us consider the function $g : I \to \mathbb{R}$, $g(x) = f(x) - cx^{n+1}$, $(x \in I)$. It is easy to see that, by (3.4) and Lemmas 3.1 and 3.2,
\[
D^{n+1} g(\xi) = D^{n+1} (f(\xi) - c\xi^{n+1}) \geq D^{n+1} f(\xi) + D^{n+1} (-c\xi^{n+1})
\]  
\[
\geq c(n+1)! - c(n+1)! = 0
\]  
for all $\xi \in I$, which, combined with Theorem 3.7, implies our statement.

**Corollary 3.9.** Let $n$ be a positive integer, $c$ be a positive real number, $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ be a function and suppose that $f$ is $n+1$ times differentiable on $I$. Then $f$ is strongly $n$-Wright-convex with modulus $c$ if and only if $f^{(n+1)}(\xi) \geq c(n+1)!$ for all $\xi \in I$.

**Proof.** The statement is a consequence of Theorem 3.8 and Remark 3.6.

**Remark 3.10.** We note, that the corollary above can also be obtained as a consequence of a characterization of strong convex functions of higher order via derivatives given in Theorem 6 in [4], and the fact that in the case of continuous functions, the classes of $n$-Wright-convex functions and $n$-convex functions coincide.

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REFERENCES


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