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This paper is dedicated to Professor Leon Mikołajczyk
as a token of our gratitude and respect.

ON ∞-ENTROPY POINTS IN REAL ANALYSIS

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Abstract. We will consider ∞-entropy points in the context of the possibilities of approximation mappings by the functions having ∞-entropy points and belonging to essential (from the point of view of real analysis theory) classes of functions: almost continuous, Darboux Baire one and approximately continuous functions.

Keywords: topological entropy, Darboux function, almost continuity, Baire one function, approximately continuous function, pseudo fixed point, topology of uniform convergence, compact-open topology, ∞-entropy point.

Mathematics Subject Classification: 26A18, 37B40, 26A21, 54H25, 54C08, 54H20.

1. INTRODUCTION

In the theory of discrete dynamical systems, a particular role is played by the dynamics of functions mapping the unit interval into itself (see e.g. monographs [2, 3, 17]). In a natural way, this fact relates to the real functions theory. An intensive study of dynamics of functions being important from the point of view of real analysis began this century. These studies were initiated, among others, by the papers [9, 26, 27].

In this paper we will consider, among others, three families of functions being important from the point of view of real functions theory: almost continuous functions (in the sense of Stallings), Darboux Baire one functions and approximately continuous functions. The family of all almost continuous functions $f : X \to Y$ will be denoted by $\mathcal{A}(X,Y)$, the family of all Darboux Baire one functions – by $\mathfrak{DB}_1(X,Y)$, the family of all approximately continuous functions – by $\mathcal{C}_a(X,Y)$ and the family of all continuous functions – by $\mathcal{C}(X,Y)$. Moreover, we will write $\mathcal{A}(X)$, $\mathfrak{DB}_1(X)$, $\mathcal{C}_a(X)$, $\mathcal{C}(X)$, respectively, if $X = Y$ and $\mathcal{A}$, $\mathfrak{DB}_1$, $\mathcal{C}_a$, $\mathcal{C}$ if $X = Y = [0,1]$.

In the literature one can find different definitions of chaos. However, it is commonly accepted that entropy is some kind of "measure of chaos". There are two basic definitions of entropy for discrete dynamical systems: the "covery" concept introduced by Adler, Konheim and McAndrew in [1] and the Bowen-Dinaburg concept based on notions of a "separated set" or a "span set" ([4, 11]).

Topological entropy has been defined for continuous maps of a compact space, while metric entropy is defined for measurable functions which may be "strongly" discontinuous. However, it turned out that in many cases there is a need to consider discrete dynamical systems with respect to discontinuous functions. So the papers connected with this topic have appeared (e.g. [8,26,27]). In 2005, the paper [9] showing that Bowen-Dinaburg definition of entropy may be applied for discontinuous functions was published. In the literature it was mainly referred to Darboux-like functions. At the beginning these references were connected with generalizations of known results concerning dynamical systems of continuous functions, e.g. Sharkovskiĭ's theorem [27], or basic theorems regarding entropy [9]. Starting from the paper [22] (see also [21]), there is a new trend of considering results which are not the generalizations of properties known for continuous functions. The results presented in this paper correspond with this field of research.

Simultaneously, analysis of different examples of functions (not only of real variables – see [23]) has resulted in an interesting observation that entropy of a function may be focused at the point ([22, 23]). This enabled us to define the notion of the entropy point of a function ([23]) in quite a general spaces. Analysis of the obtained results shows that particularly interesting are ∞-entropy points (a formal definition of this notion is presented in a further part of the paper). That is why we have concentrated our investigations on such points, referring the considerations to the classes of functions mentioned at the beginning.

2. PRELIMINARIES

We will use standard definitions and notations. In order to avoid misunderstandings and to make the paper more readable, we will present some symbols, definitions and statements[1)] used in it.

We will use the letters $\mathbb{R}$ and $\mathbb{N}$ to denote the real numbers and positive integers, respectively. The unit interval $[0, 1]$ with the natural metric will be denoted by $\mathbb{I}$. The symbol $\mathrm{int}(A)$ will stand for the interior of a set $A \subset \mathbb{I}$ in natural topology. The cardinality of a set A will by denoted by $\mathrm{card}(A)$. Our discussion will be restricted to the functions mapping the unit interval into itself (in some proofs we will consider functions defined on subsets of $\mathbb{I}$, but it will always be emphasised).

The graph of a function f will be denoted by $\mathrm{Gr}(f)$. The symbols $\mathrm{Fix}(f)$ and $\mathcal{C}(f)$ will stand for the sets of all fixed points of f and of all continuity points of f, respectively. Moreover, if $A \subset \mathbb{I}$, then we will write $f \restriction A$ for the restriction of f to the set A.

Our considerations will be concentrated on the approximation of functions from a fixed family $\mathcal{F}$ by functions from another family $\mathcal{K}$, so the thing is to find a set

1) The paper deals with three theories: real analysis, discrete dynamical systems and topology (with some elements of fixed point theory). That is why it seems to be advisable to recall theorems used in our considerations.

$S \subset \mathcal{K}$ in an arbitrary neighborhood of $f \in \mathcal{F}$. For that purpose we will consider two topologies. The most common is the topology of uniform convergence generated by the metric $\rho_u(f,g) = \sup_{x\in\mathbb{I}} |f(x) - g(x)|$. As it will be shown later, it is not enough to use only this topology in our considerations. That is why we will also consider the compact-open topology. The choice of the compact-open topology for our considerations is not accidental. If we consider this topology and the topology of uniform convergence in the family of continuous functions $\mathcal{C}(X,Y)$, then, with some natural assumptions concerning X and Y, both these topologies coincide. That is why they may be treated as "close topologies". Since our discussion deals with wider classes than $\mathcal{C}(X,Y)$, distinguishing these topologies is fully justified. It is worth pointing out that from the topological point of view, the compact-open topology has more interesting properties than for example the topology of pointwise convergence, especially if we consider it not only in the context of the domain and the range of a function but also with respect to properties connected with some points of the domain of a function (with some natural assumptions about X and Y one can prove that in the set $\mathcal{C}(X,Y)$ the compact-open topology is the weakest one for which a function $T : \mathcal{C}(X,Y) \times X \to Y$ defined by $T(f,x) = f(x)$ is continuous).

Let $\mathcal{F}$ be a fixed family of functions. The compact-open topology[2)] in $\mathcal{F}$ is the topology generated by the base consisting of all sets

$$\{f \in \mathcal{F} : f(A_1) \subset U_1\} \cap \{f \in \mathcal{F} : f(A_2) \subset U_2\} \cap \ldots \cap \{f \in \mathcal{F} : f(A_m) \subset U_m\},$$

where A_i is a compact subset of $\mathbb{I}$ and U_i is an open subset of $\mathbb{I}$, for $i = 1, 2, \ldots, m$.

The topology of uniform convergence in the space $\mathcal{F}$ will be denoted by $T_u^{\mathcal{F}}$ and the compact-open topology by $T_k^{\mathcal{F}}$. If $\mathcal{F}$ is a family of functions, then symbol $\mathcal{F}_u$ ($\mathcal{F}_k$) will stand for the space $\mathcal{F}$ with the topology $T_u^{\mathcal{F}}$ ($T_k^{\mathcal{F}}$). By $B_u^{\mathcal{F}}(f,\varepsilon)$ we will denote an open ball in the space $\mathcal{F}$ with the metric ρ_u with centre at f and radius $\varepsilon > 0$.

In our considerations, one more topology in a space of functions will be useful (see [18]). Corresponding to each open set U in $\mathbb{I}^2$, let $\mathcal{F}_U = \{f \in \mathcal{F} : \mathrm{Gr}(f) \subset U\}$. The topology induced on $\mathcal{F}$ by a basis consisting of sets of the form $\mathcal{F}_U$ for each open set U in $\mathbb{I} \times \mathbb{I}$ is called the graph topology for $\mathcal{F}$. In our case of functions mapping the unit interval into itself one can note:

Statement 2.1 ([18]). *The graph topology coincides with the compact-open topology.*

In this paper we will consider different types of functions. One of them will be Cesàro type functions which have been introduced in [24]. A function $f : X \to Y$ is *of Cesàro type* iff there exist non-empty open sets $U \subset X$ and $V \subset Y$, such that $f^{-1}(y)$ is dense in U for each $y \in V$. Moreover, the Baire one functions i.e functions which are the pointwise limit of a sequence of continuous functions will play an important role in our considerations. Obviously, a Baire one function need not be continuous. However, it can not be everywhere discontinuous. More specifically, a set of discontinuity points of a Baire one function is a meager set ([6]).

2) Compact – open topology is commonly considered in the case of the family of continuous functions. The paper [18] is the first one where the compact – open topology is refers to almost continuous functions.

The next type of functions considered by us will be Darboux functions. A real valued function f defined on the interval $\mathbb{I}$ is said to be a Darboux function if it has *the intermediate value property* i.e. if whenever x and y belong to $\mathbb{I}$ and α is any number between $f(x)$ and $f(y)$, there is a number z between x and y such that $f(z) = \alpha$. This definition one can write in the following equivalent form: We say that f is a Darboux function if $f(C)$ is a connected set, for any connected set $C \subset \mathbb{I}$.

In our case, the local characterization of a Darboux function (see [7, 15]) will be very useful, so we shortly recall it.

Let us start with a *left- and right-range of f at x_0*(denoted by $R^-(f, x_0)$ and $R^+(f, x_0)$, respectively):

$$\alpha \in R^-(f, x_0) \text{ iff } \left(f^{-1}(\alpha) \cap (x_0 - \delta, x_0) \neq \emptyset\right) \text{ for any } \delta > 0,$$

$$\alpha \in R^+(f, x_0) \text{ iff } \left(f^{-1}(\alpha) \cap (x_0, x_0 + \delta) \neq \emptyset\right) \text{ for any } \delta > 0.$$

An element β is a *left-hand* (*right-hand*) *cluster number of f* at x_0 if there exists an increasing sequence $\{x_n\}_{n\in\mathbb{N}}$ (a decreasing sequence $\{y_n\}_{n\in\mathbb{N}}$) tending to x_0 such that $\lim_{n\to\infty} f(x_n) = \beta$ ($\lim_{n\to\infty} f(y_n) = \beta$). The set of all left-hand (right-hand) cluster numbers of f at x_0 will be denoted by $L^-(f, x_0)$ ($L^+(f, x_0)$).

We will say that x_0 is a *left-hand* (*right-hand*) *Darboux point of f* if for each left-hand (right-hand) cluster number β of f at x_0 different from $f(x_0)$ and each γ belonging to the open interval with endpoints at $f(x_0)$ and β we have $\gamma \in R^-(f, x_0)$ ($\gamma \in R^+(f, x_0)$). Of course, if we have $x_0 = 0$ or $x_0 = 1$, then we consider only one-side cluster numbers.

We shall say that x_0 is a *Daboux point of f* if it is simultaneously a right-hand and a left-hand Darboux point of f. Obviously, if $x_0 = 0$ ($x_0 = 1$), then x_0 is a Darboux point of f if x_0 is right- (left-) hand Darboux point of f.

It is well known that f is a Darboux function iff every point $x \in \mathbb{I}$ is a Darboux point of f ([7]).

Let f be a Darboux function. We will say that a point x_0 is an almost fixed point of f (denoted by $x_0 \in \mathrm{aFix}(f)$), if

$$x_0 \in \operatorname{int}\left(R^-(f, x_0)\right) \cup \operatorname{int}\left(R^+(f, x_0)\right).$$

Another class of functions that will be considered by us is a family of approximately continuous functions, which are connected with the notion of density point. If $A \subset \mathbb{R}$ is a Lebesgue measurable set and $x_0 \in \mathbb{R}$, then we say that a *density* (*right density, left density*) *of a set A at a point* x_0 is equal to α (α_r, α_l) if $\alpha = \lim_{h\to 0^+} \frac{m(A\cap[x_0-h, x_0+h])}{2h}$ ($\alpha_r = \lim_{h\to 0^+} \frac{m(A\cap[x_0, x_0+h])}{h}, \alpha_l = \lim_{h\to 0^+} \frac{m(A\cap[x_0-h, x_0])}{h}$). Moreover, if $\alpha = 1$, ($\alpha_r = 1, \alpha_l = 1$), then we say that x_0 is a *density* (*right density, left density*) *point of A*. The notion of approximately continuous function was introduced at the beginning of the 20th century ([10]). A function f is *approximately continuous* if for each point $x \in \mathbb{I}$ there exists a Lebesgue measurable set E_x such that $x \in E_x$, x is a density point of E_x and $f \restriction E_x$ is continuous at x. Obviously, if $x = 0$ ($x = 1$), then x has to be only a right (left) density point of the set E_x. Clearly, if $x \in \mathcal{C}(f)$, then f is approximately continuous at the point x. It is well known that $\mathcal{C}_a \subset \mathfrak{DB}_1$ (e.g. [6]).

Moreover $f \in \mathcal{C}_a$ iff for any $a \in \mathbb{R}$ the sets $\{x : f(x) < a\}$ and $\{x : f(x) > a\}$ belong to the density topology ([28]).

We finish this section with some information about almost continuous functions, which were introduced in the fifties by J. Stallings. We say that a function f is *almost continuous (in the sense of Stallings)* if every open set $U \subset \mathbb{I}^2$ containing the graph of f contains the graph of some continuous function.

The family of almost continuous functions has become an object of interest of many mathematicians because it contains many important classes of functions e.g. $\mathfrak{DB}_1$ and thus also $\mathcal{C}_a$ and the family of all derivatives (a derivative is a function being a derivative of some function). Moreover, with some additional simple assumptions imposed on X, having a fixed point by all continuous functions (we say then that X has the fixed point property) is equivalent to having a fixed point by all almost continuous functions. We will formulate it as the following statement:

Statement 2.2 ([25])**.** *If a nondegenerate Hausdorff space X has a fixed point property, then each almost continuous function $g : X \to X$ has a fixed point.*

We say that a function f is *connectivity*, if $\mathrm{Gr}(f \restriction C)$ is a connected set, for each connected set C contained in the domain of f. Of course, each connectivity function is a Darboux function.

Statement 2.3 ([25])**.** *Each almost continuous function is a connectivity function, so it is also a Darboux function.*

Now we will present some technical statements which will be used in the further part of the paper. They will be formulated for a real valued function f defined on some subset of $\mathbb{I}$, although they are mostly true for the wider classes of spaces and functions.

Statement 2.4 ([19])**.** *Let an interval $J \subset \mathbb{I}$ be a union of countably many closed intervals I_n such that $\mathrm{int}(I_n) \cap \mathrm{int}(I_m) = \emptyset$ for $m \neq n$ and $I_n \cap I_{n+1} \neq \emptyset$ for each integer n. For any function $f : J \to \mathbb{R}$, f is almost continuous iff $f \restriction I_n$ is almost continuous for each n.*

Statement 2.5 ([13])**.** *If $a, b \in \mathbb{I}$ and $f : (a, b) \to \mathbb{R}$ is almost continuous, $y \in L^+(f, a)$, $z \in L^-(f, b)$, then the functions $f_1 : [a, b) \to \mathbb{R}$, $f_2 : (a, b] \to \mathbb{R}$, $f_3 : [a, b] \to \mathbb{R}$ such that $f_1(x) = f_2(x) = f_3(x) = f(x)$ for $x \in (a, b)$, $f_1(a) = f_3(a) = y$ and $f_2(b) = f_3(b) = z$ are almost continuous.*

Statement 2.6 ([20])**.** *If $f : \mathbb{I} \to \mathbb{R}$ is almost continuous and A is a subset of $\mathbb{I}$, then $f \restriction A$ is almost continuous.*

Let us introduce one more notation. If $\mathcal{F}$, $\mathcal{K}$ are families of real functions defined on a topological space X, then the symbol $\mathcal{M}_a(\mathcal{F}, \mathcal{K})$ will stand for maximal additive class of $\mathcal{F}$ with respect to $\mathcal{K}$, i.e.

$$\mathcal{M}_a(\mathcal{F}, \mathcal{K}) = \{f \in \mathcal{F} : f + g \in \mathcal{K} \text{ for each } g \in \mathcal{F}\}.$$

We shall write $\mathcal{M}_a(\mathcal{F})$ if $\mathcal{K} = \mathcal{F}$ and call this family the maximal additive class of $\mathcal{F}$. Moreover, we shall denote the family $\{f \in \mathcal{F} : \max\{f, g\} \in \mathcal{K} \text{ for all } g \in \mathcal{F}\}$

($\{f \in \mathcal{F} : \min\{f, g\} \in \mathcal{K}$ for all $g \in \mathcal{F}\}$) by $\mathcal{M}_{\max}(\mathcal{F}, \mathcal{K})$ ($\mathcal{M}_{\min}(\mathcal{F}, \mathcal{K})$). If $\mathcal{F} = \mathcal{K}$, then we shall write $\mathcal{M}_{\max}(\mathcal{F})$ and $\mathcal{M}_{\min}(\mathcal{F})$. We have the following properties:

Statement 2.7 ([19])**.** *$\mathcal{M}_a(\mathcal{A}(\mathbb{R})) = \mathcal{C}(\mathbb{R})$, $\mathcal{C}(\mathbb{R}) \subset \mathcal{M}_{\min}(\mathcal{A}(\mathbb{R}))$ and $\mathcal{C}(\mathbb{R}) \subset \mathcal{M}_{\max}(\mathcal{A}(\mathbb{R}))$.*

Moreover, it is worth adding that the maximal classes for almost continuous functions with respect to maximum and minimum were characterized in [16]: $\mathcal{M}_{\min}(\mathcal{A}(\mathbb{R}))$ is the family of all Darboux lower semicontinuous functions and $\mathcal{M}_{\max}(\mathcal{A}(\mathbb{R}))$ is the family of all Darboux upper semicontinuous functions.

Statement 2.8 ([25])**.** *Let X, Y, Z be nondegenerate intervals. For each $f \in \mathcal{A}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$ the composed map $g \circ f$ is almost continuous.*

Statement 2.9 ([12])**.** *Let X, Y, Z be nondegenerate intervals. If $h : X \to Y$ is a homeomorphism and $f : Y \to Z$ is almost continuous, then the composition $f \circ h$ is almost continuous.*

In hitherto considerations concerning entropy points, a special role was played by fixed points of a function. However, in many situations it is advantageous to consider points "close to fixed points" (in the sense that fixed points of the function lie arbitrarily close to a given point, e.g. almost fixed points and Theorem 3.2 [22]). It should be noted that the consideration of almost fixed points makes sense only for discontinuous functions. The following definition of the pseudo-fixed point aims to find a uniform approach to the idea of this type of points, in such a way that they could be considered both for continuous, as well as discontinuous functions.

We say that a point x_0 is *a pseudo-fixed point of a function f* (denoted by $x_0 \in \mathrm{pFix}(f)$) if every neighborhood of x_0 (i.e. an open set containing x_0) contains a fixed point of f.

Using the properties of almost fixed points (see [22]) we get immediately that for any $f \in \mathfrak{DB}_1$ if $x_0 \in \mathrm{aFix}(f)$, then $x_0 \in \mathrm{pFix}(f)$. We can prove even more.

Lemma 2.10. *If $f \in \mathcal{A}$, then* $\mathrm{aFix}(f) \subset \mathrm{pFix}(f)$.

Proof. Let $f \in \mathcal{A}$, $x_0 \in \mathrm{aFix}(f)$ and $\Delta = \{(x, x) : x \in \mathbb{I}\}$. Thus $x_0 \in \mathrm{int}\,(R^-(f, x_0)) \cup \mathrm{int}\,(R^+(f, x_0))$. Fix $\varepsilon > 0$ and assume, that $x_0 \in \mathrm{int}\,(R^-(f, x_0))$ (for $x_0 \in \mathrm{int}\,(R^+(f, x_0))$ the proof runs in the similar manner). There are two possibilities: $x_0 \in (0, 1)$ or $x_0 = 1$. In both cases one can choose $\varepsilon_0 \in (0, \varepsilon)$ such that $(x_0 - \varepsilon_0, x_0 + \varepsilon_0) \cap (0, 1] \subset R^-(f, x_0)$. Fix $y \in (x_0 - \varepsilon_0, x_0) \cap (0, 1)$ and $z \in [x_0, x_0 + \varepsilon_0) \cap (0, 1]$. Clearly, there exist $s \in (y, x_0)$ and $t \in (s, x_0)$ such that $f(s) = y < s$ and $f(t) = z \geq x_0 > t$. From Statement 2.3 it may be concluded that $\mathrm{Gr}(f \restriction [s, t]) \cap \Delta \neq \emptyset$. Since $[s, t] \subset (x_0 - \varepsilon, x_0 + \varepsilon_0)$, it follows that $\mathrm{Fix}(f) \cap (x_0 - \varepsilon, x_0 + \varepsilon_0) \neq \emptyset$. Thus $x_0 \in \mathrm{pFix}(f)$. □

It is easy to see that $\mathrm{pFix}(f) = \mathrm{Fix}(f)$ for any continuous function f. For other functions the equality can not be true. What is more:

Proposition 2.11. *In the space $\mathcal{A}_u$ the family Ω of all functions ξ from $\mathcal{A}$ such that* $\mathrm{pFix}(\xi) \setminus \mathrm{Fix}(\xi) \neq \emptyset$ *is a dense set.*

Proof. To prove the theorem we have to show that:

$$B_u^{\mathcal{A}}(\xi, \delta) \cap \Omega \neq \emptyset \text{ for any function } \xi \in \mathcal{A} \text{ and any } \delta > 0. \tag{2.1}$$

Fix $\xi \in \mathcal{A}$ and $\delta > 0$. Let $x_0 \in \mathrm{Fix}(\xi)$. For the simplicity of considerations we assume that $x_0 \in (0, 1)$ (the proof in the case $x_0 \in \{0, 1\}$ runs analogously). The following two cases are possible:

1. $x_0 \in \mathcal{C}(\xi)$, so there exists $\sigma > 0$ such that $[x_0 - \sigma, x_0 + \sigma] \subset (0, 1)$ and $\xi([x_0 - \sigma, x_0 + \sigma]) \subset (x_0 - \frac{\delta}{3}, x_0 + \frac{\delta}{3})$. Putting $c_n = x_0 + \frac{\sigma}{2^n}$ for $n \in \mathbb{N}$ we obtain that $\lim_{n\to\infty} c_n = x_0$ and $c_n \in (x_0, x_0 + \sigma)$ for $n \in \mathbb{N}$. We define the function $\eta : \mathbb{I} \to \mathbb{I}$ in the following way: $\eta(x) = \xi(x)$ for $x \in [0, x_0 - \sigma] \cup [x_0 + \sigma, 1]$, $\eta(x_0) = x_0 + \frac{\delta}{8}$, $\eta(c_{2k-1}) = x_0 + \frac{\delta}{4}$, $\eta(c_{2k}) = x_0 - \frac{\delta}{4}$ for $k \in \mathbb{N}$ and η is linear on intervals $[x_0 - \sigma, x_0]$, $[c_1, x_0 + \sigma]$, $[c_{2k+1}, c_{2k}]$ and $[c_{2k}, c_{2k-1}]$ for $k \in \mathbb{N}$. From Statements 2.5 and 2.4 it may be concluded that $\eta \in \mathcal{A}$. Moreover, it is obvious that $x_0 \in \mathrm{pFix}(\eta) \setminus \mathrm{Fix}(\eta)$. According to the definition of η we obtain that $\eta \in B_u^{\mathcal{A}}(\xi, \delta) \cap \Omega$.
2. $x_0 \notin \mathcal{C}(\xi)$. For simplicity of the notation, we assume that x_0 is a right-discontinuity point of ξ (if x_0 is a left-discontinuity point of ξ, the proof runs in the similar way). Obviously, x_0 is a right-hand Darboux point of ξ. Thus there exists $\beta \in L^+(\xi, x_0)$ such that $0 < |\beta - \xi(x_0)| < \delta$. There is no loss of generality in assuming that $\beta > \xi(x_0)$. Put $\mu = \frac{\beta - \xi(x_0)}{2}$. Now, we define the function $\nu : \mathbb{I} \to \mathbb{I}$ as follows: $\nu(x) = \min\{\xi(x) + \mu, 1\}$ for $x \in [0, x_0]$ and $\nu(x) = \max\{\xi(x) - \frac{\mu}{2}, 0\}$ for $x \in (x_0, 1]$. Statements 2.5, 2.6, 2.4 and 2.7 imply that $\nu \in \mathcal{A}$. Moreover,

$$x_0 \in \mathrm{pFix}(\nu) \setminus \mathrm{Fix}(\nu). \tag{2.2}$$

Indeed, obviously $\nu(x_0) = \xi(x_0) + \mu = x_0 + \mu \neq x_0$, so $x_0 \notin \mathrm{Fix}(\nu)$. Furthermore, if $y \in (\max\{0, x_0 - \frac{\mu}{4}\}, x_0 + \frac{\mu}{2})$, then it is easy to see that $\xi(x_0) < y + \frac{\mu}{2} < \beta$. Thus $y + \frac{\mu}{2} \in R^+(\xi, x_0)$. Let $\sigma > 0$. There exists $x_\sigma \in (x_0, x_0 + \sigma)$ such that $\xi(x_\sigma) = y + \frac{\mu}{2}$. Clearly, $\xi(x_\sigma) - \frac{\mu}{2} = y > 0$, so $\nu(x_\sigma) = y$. It means that $y \in R^+(\nu, x_0)$. Since y was arbitrary, we obtain that $(\max\{0, x_0 - \frac{\mu}{4}\}, x_0 + \frac{\mu}{2}) \subset \mathrm{int}(R^+(\nu, x_0))$ and, in consequence, $x_0 \in \mathrm{aFix}(\nu)$. Hence, by Lemma 2.10, we obtain (2.2).
Obviously, $\nu \in B_u^{\mathcal{A}}(\xi, \delta)$. Therefore $B_u^{\mathcal{A}}(\xi, \delta) \cap \Omega \neq \emptyset$. □

By making certain modifications in the above proof we can show that Proposition 2.11 is also true for the classes $\mathfrak{DB}_1$ and $\mathcal{C}_a$.

3. MAIN RESULTS

Existing papers concerning the approximation of functions (both continuous as well as discontinuous) by functions having an entropy point, observe that they are strictly connected with fixed points of functions (or with points in some sense "close" to fixed points) and they deal with entropy points for which entropy is "close to infinity". These facts have prompted us to distinguish so called ∞-entropy points.

For completness of explanation, we now recall the definition of bundle-entropy, modelling on the Bowen-Dinaburg version of the definition of entropy.

Let f be a function, $\mathcal{L}$ be a family of pairwise disjoint (nonsingletons) continuums in $\mathbb{I}$ and $J \subset \mathbb{I}$ be a connected set such that $J \subset f(A)$ for any $A \in \mathcal{L}$. A pair $B_f = (\mathcal{L}, J)$ is called an f-bundle and J is said to be a fibre of bundle. Moreover, if we additionally assume that $A \subset J$ for all $A \in \mathcal{L}$, then such an f-bundle will be called an f-bundle with dominating fibre.

Let $\varepsilon > 0$, $n \in \mathbb{N}$, $B_f = (\mathcal{L}, J)$ be an f-bundle and $M \subset \bigcup \mathcal{L}$. We shall say that M is (B_f, n, ε)-separated if for each $x, y \in M$, $x \neq y$ one can find $i \in \{0, 1, \ldots, n-1\}$ such that $f^i(x), f^i(y) \in J$ and $|f^i(x) - f^i(y)| > \varepsilon$.

Moreover, we put

$$\operatorname{maxsep}[B_f, n, \varepsilon] = \max\{\operatorname{card}(M) : M \subset [0,1] \text{ is } (B_f, n, \varepsilon)\text{-separated set}\}.$$

The entropy of an f-bundle B_f is the number

$$h(B_f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \left[\frac{1}{n} \log \left(\operatorname{maxsep}[B_f, n, \varepsilon] \right) \right].$$

For our consideration it is important to note:

Statement 3.1 ([23])**.** *Let f be an arbitrary function and $B_f = (\mathcal{L}, J)$ be an f-bundle with dominating fibre. Then $h(B_f) \geq \log(\operatorname{card}(\mathcal{L}))$ whenever $\mathcal{L}$ is finite and $h(B_f) = +\infty$ whenever $\mathcal{L}$ is infinite.*

We shall say that a sequence of f-bundles $B_f^k = (\mathcal{L}_k, J_k)$ converges to a point x_0, if for any $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that $\bigcup \mathcal{L}_k \subset (x_0 - \varepsilon, x_0 + \varepsilon)$ and $(f(x_0) - \varepsilon, f(x_0) + \varepsilon) \cap J_k \neq \emptyset$ for any $k \geq k_0$.

Putting

$$E_f(x) = \left\{ \limsup_{n \to \infty} h(B_f^n) : B_f^n \underset{n \to \infty}{\longrightarrow} x \right\}$$

we obtain a multifunction $E_f : \mathbb{I} \multimap \mathbb{R} \cup \{+\infty\}$.

In [23], the basic properties of a multifunction E_f were examined. It allows us to adopt the following definition.

We shall say that a point x_0 is an *∞-entropy point of f* if $\infty \in E_f(x_0)$ and $x_0 \in \operatorname{pFix}(f)$. The set of all ∞-entropy points of f will be denoted by $\mathfrak{e}(f)$. Obviously, we have that each ∞-entropy point of f is also an entropy point of f, in the sense of the definition assumed in [23]. Moreover, we say that a nonempty family of functions $\mathcal{F}$ has *the ∞-entropy point property* if $\bigcap_{f \in \mathcal{F}} \mathfrak{e}(f) \neq \emptyset$.

In existing papers concerning the issue of approximations, single functions having entropy points were being searched for in the neighborhood of a fixed function. Observation of concrete examples shows that "arbitrarily close" to any function it is possible to find a "big" set of functions having entropy points. However, the description of these observations is hindered by the following, easy to prove, facts.

Proposition 3.2.

(a) *In the space $\mathcal{C}_u$ there is no nonempty open set having the ∞-entropy point property.*

(b) *In the space $\mathcal{F}_k$, where $\mathcal{F} \in \{\mathcal{A}, \mathfrak{DB}_1, \mathcal{C}_a\}$ there is no nonempty open set having the ∞-entropy point property.* □

The next theorem shows that if we consider a topology of uniform convergence in $\mathcal{A}$, then there exists an open set with the ∞-entropy point property.

Problems analyzed in the next theorem could be formulated in the language of bitopological spaces (information about such spaces one can find for example in [14]). In order to avoid overly complicated notations in the next theorem we use the methods of these spaces without going into their structure.

Theorem 3.3. *Let $\mathcal{F}, \mathcal{K} \in \{\mathcal{A}, \mathfrak{DB}_1, \mathcal{C}_a\}$ be families of functions such that $\mathcal{K} \subset \mathcal{F}$. Then for any nonempty, $T_k^{\mathcal{F}}$-open set U there exists $B_u^{\mathcal{K}}(\phi, \delta_0) \subset U$ such that $B_u^{\mathcal{K}}(\phi, \delta_0)$ has the ∞-entropy points property. Moreover, if $\mathcal{H} \in \{\mathfrak{DB}_1, \mathcal{C}_a\}$ and $\mathcal{H} \subsetneq \mathcal{K}$, then we can additionally demand that $B_u^{\mathcal{K}}(\phi, \delta_0) \cap \mathcal{H} = \emptyset$.*

Proof. We first assume that $\mathcal{F} = \mathcal{A}$. Let $U \in T_k^{\mathcal{A}} \setminus \{\emptyset\}$ be a basis set in $T_k^{\mathcal{A}}$ and $\zeta \in U$. Statement 2.1 implies that there exists an open set $W_U \subset \mathbb{I}^2$, such that $U = \{f \in \mathcal{A} : \mathrm{Gr}(f) \subset W_U\}$. Then one can choose a continuous function ψ such that

$$\psi \in U \text{ and } \mathrm{Fix}(\psi) \cap (0,1) \neq \emptyset.$$

Indeed, since $\zeta \in \mathcal{A}$, so we obtain that there is a continuous function ψ_0 such that $\psi_0 \in U$. If $\mathrm{Fix}(\psi_0) \cap (0,1) \neq \emptyset$, then simply put $\psi = \psi_0$. Otherwise, $\emptyset \neq \mathrm{Fix}(\psi_0) \subset \{0,1\}$. If $1 \in \mathrm{Fix}(\psi_0)$, then there exists $w \in (0,1)$ such that $[w,1] \times [w,1] \subset W_U$. Clearly, there exists $v^* \in (w,1)$ such that $\psi_0(v^*) \in [w,1]$. Fix $v \in (v^*,1)$ and define ψ as follows: $\psi(x) = \psi_0(x)$ for $x \in [0,v^*]$, $\psi(x) = x$ for $x \in [v,1]$ and ψ is linear on $[v^*,v]$. Obviously, ψ has the required property. If $\mathrm{Fix}(\psi_0) \subset \{0\}$, we construct the function ψ in a similar way.

Put

$$Z = \mathbb{I}^2 \setminus W_U.$$

It is sufficient to consider the case $Z \neq \emptyset$. Let $d : \mathbb{I}^2 \to \mathbb{I}$ be the function defined in the following way: $d((x,y)) = d_2((x,y), Z)$ for $(x,y) \in \mathbb{I}^2$, where $d_2((x,y), Z)$ denotes the distance between the point (x,y) and the set Z. Obviously, d is a continuous function and $\mathrm{Gr}(\psi)$ is a compact set. Therefore

$$\varepsilon_0 = \inf\{d((x,y)) : (x,y) \in \mathrm{Gr}(\psi)\} > 0$$

and consequently

$$d_2((z_1,z_2), \mathrm{Gr}(\psi)) \geq \varepsilon_0 \text{ for any } (z_1,z_2) \in Z. \tag{3.1}$$

Now, we show that

$$B_u^{\mathcal{A}}(\psi, \varepsilon_0) \subset U. \tag{3.2}$$

Let $g \in B_u^{\mathcal{A}}(\psi, \varepsilon_0)$. We first prove that

$$(x, g(x)) \notin Z \text{ for } x \in \mathbb{I}. \tag{3.3}$$

So, let $x \in \mathbb{I}$. Obviously, $d_2((x, g(x)), \mathrm{Gr}(\psi)) \leq \rho_2((x, g(x)), (x, \psi(x))) < \varepsilon_0$ (where ρ_2 is the Euclidean metric in $\mathbb{I}^2$). This, together with (3.1) gives (3.3) and, in consequence, we have that $(x, g(x)) \in W_U$. From the arbitrariness of x, we obtain that $\mathrm{Gr}(g) \subset W_U$, so $g \in U$, and the proof of (3.2) is complete.

Now, let $x_0 \in \mathrm{Fix}(\psi) \cap (0, 1)$. Since ψ is continuous, so one can find $\delta \in (0, \frac{\varepsilon_0}{4})$ such that $[x_0 - \delta, x_0 + \delta] \subset (0, 1)$ and

$$\psi([x_0 - \delta, x_0 + \delta]) \subset \Big(x_0 - \frac{\varepsilon_0}{8}, x_0 + \frac{\varepsilon_0}{8}\Big). \tag{3.4}$$

Then

$$P = [x_0 - \delta, x_0 + \delta] \times [x_0 - \delta, x_0 + \delta] \subset W_U. \tag{3.5}$$

Indeed, let $t_0 = (t_1, t_2) \in P$. Obviously, $t_2 \in (x_0 - \frac{\varepsilon_0}{4}, x_0 + \frac{\varepsilon_0}{4})$. Moreover, from (3.4) we infer that $d_2(t_0, \mathrm{Gr}(\psi)) \leq |t_2 - \psi(t_1)| < \varepsilon_0$. From this and (3.1) we conclude that $t_0 \notin Z$, so $t_0 \in W_U$, and (3.5) is proved.

Now, we construct a function $\phi : \mathbb{I} \to \mathbb{I}$ for $\mathcal{K} = \mathcal{A}$.

Let $h : \mathbb{I} \to [x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}]$ and $\pi : [x_0, x_0 + \frac{\delta}{4}] \to \mathbb{I}$ be homeomorphisms such that $h(0) = x_0 - \frac{\delta}{2}$ and $\pi(x_0) = 0$. Moreover, let $\phi_1 : \mathbb{I} \to \mathbb{I}$ be the function defined by $\phi_1(x) = \limsup_{m\to\infty} \frac{a_1+\ldots+a_m}{m}$, where a_n for $n \in \mathbb{N}$ are given by the unique nonterminating binary expansion of the number $x = (0.a_1, a_2, \ldots)_2$. Then ϕ_1 (called the Cesàro-Vietoris function) is an almost continuous function of Cesàro type, $\phi_1(J) = \mathbb{I}$ for any nonempty open set $J \subset \mathbb{I}$ and $\phi_1(0) = 0$ (see [5, 20]).

Next, we define $\phi^* : [x_0, x_0 + \frac{\delta}{4}] \to [x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}]$ to be $\phi^* = h \circ \phi_1 \circ \pi$. From Statements 2.8 and 2.9, we conclude that $\phi^* \in \mathcal{A}$. Furthermore, it is easy to check that ϕ^* is of Cesàro type.

Finally, we can afford to give the definition of the function ϕ in the following way: $\phi(x) = \psi(x)$ for $x \in [0, x_0] \cup [x_0 + \frac{\delta}{2}, 1]$, $\phi(x) = \phi^*(x)$ for $x \in (x_0, x_0 + \frac{\delta}{4}]$ and ϕ is linear on $[x_0 + \frac{\delta}{4}, x_0 + \frac{\delta}{2}]$. Statements 2.6, 2.5 and 2.4 imply that $\phi \in \mathcal{A}$. Moreover, it is clear that $\rho_u(\psi, \phi) < \frac{\varepsilon_0}{2}$.

Now, we construct a function $\phi : \mathbb{I} \to \mathbb{I}$ for $\mathcal{K} = \mathfrak{DB}_1$.

Let $\{[a_n, b_n]\}_{n\in\mathbb{N}}$ be a sequence of closed intervals such that $x_0 < \ldots < a_n < b_n < \ldots < b_2 < a_1 < b_1 < x_0 + \frac{\delta}{6}$, $\lim_{n\to\infty} b_n = x_0$ and the right density of sets $\bigcup_{i=1}^{\infty}[a_{2i}, b_{2i}]$ and $\bigcup_{i=0}^{\infty}[a_{2i+1}, b_{2i+1}]$ at the point x_0 is greater than 0.

We define the function ϕ as follows: $\phi(x) = \psi(x)$ for $x \in [0, x_0] \cup [x_0 + \delta, 1]$, $\phi(x) = x_0 - \frac{\delta}{2}$ for $x \in \bigcup_{i=1}^{\infty}[a_{2i}, b_{2i}]$, $\phi(x) = x_0 + \frac{\delta}{2}$ for $x \in \bigcup_{i=0}^{\infty}[a_{2i+1}, b_{2i+1}]$ and ϕ is linear on each interval $[b_{n+1}, a_n]$ (for $n \in \mathbb{N}$) and $[b_1, x_0 + \delta]$. It is easy to see that $\phi \in \mathfrak{DB}_1$ and $\rho_u(\psi, \phi) < \frac{\varepsilon_0}{2}$.

Now, we construct a function $\phi : \mathbb{I} \to \mathbb{I}$ for $\mathcal{K} = \mathcal{C}_a$.

As in the previous case, let us consider a sequence $\{[c_n, d_n]\}_{n\in\mathbb{N}}$ of closed intervals such that $x_0 < \ldots < c_n < d_n < \ldots < d_2 < c_1 < d_1 = x_0 + \frac{\delta}{6}$ and $\lim_{n\to\infty} d_n = x_0$. However, in this case we will require that x_0 is a right density point of the set $\bigcup_{i=1}^{\infty}[c_i, d_i]$.

Consider the function ϕ defined in the following way: $\phi(x) = \psi(x)$ for $x \in [0, x_0] \cup [x_0 + \delta, 1]$, $\phi(x) = x_0$ for $x \in \bigcup_{i=1}^{\infty}[c_i, d_i]$, ϕ is linear on $[d_1, x_0 + \delta]$ and ϕ is a continuous function on $[d_{n+1}, c_n]$ and $\phi([d_{n+1}, c_n]) = [x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}]$ for each $n \in \mathbb{N}$. Obviously, $\phi \in \mathcal{C}_a$ and $\rho_u(\psi, \phi) < \frac{\varepsilon_0}{2}$.

For all cases discussed above put $\delta_0 = \frac{\delta}{6}$. Obviously, if $\mathcal{K} \in \{\mathcal{A}, \mathfrak{DB}_1, \mathcal{C}_a\}$ and $f \in B_u^{\mathcal{K}}(\phi, \delta_0)$, then $f \in \mathcal{A}$ and $\rho_u(f, \psi) < \varepsilon_0$. From this inequality and (3.2) we conclude that

$$B_u^{\mathcal{K}}(\phi, \delta_0) \subset U. \tag{3.6}$$

Now, we show that

$$B_u^{\mathcal{K}}(\phi, \delta_0) \cap \mathcal{H} = \emptyset, \tag{3.7}$$

for any $\mathcal{K} \in \{\mathcal{A}, \mathfrak{DB}_1, \mathcal{C}_a\}$ and each $\mathcal{H} \in \{\mathfrak{DB}_1, \mathcal{C}_a\}$ such that $\mathcal{H} \subsetneq \mathcal{K}$.

If $\mathcal{K} = \mathcal{A}$, then

$$\Big[x_0 - \frac{\delta}{3}, x_0 + \frac{\delta}{3}\Big] \subset \xi(V) \text{ for any } \xi \in B_u^{\mathcal{A}}(\phi, \delta_0) \text{ and any nonempty open set } V \subset \Big[x_0, x_0 + \frac{\delta}{4}\Big]. \tag{3.8}$$

Indeed, let $\xi \in B_u^{\mathcal{A}}(\phi, \delta_0)$ and $V \subset [x_0, x_0 + \frac{\delta}{4}]$ be a nonempty open set. Obviously, there exists an open interval $V_1 \subset V$. According to properties of ϕ_1 one can conclude that $\phi(V_1) = [x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}]$, so $x_0 - \frac{\delta}{3}, x_0 + \frac{\delta}{3} \in \xi(V_1)$. As ξ is a Darboux function we obtain (3.8). Hence, we obtain immediately that ξ is not Baire one function and, in consequence, we have (3.7).

If $\mathcal{K} = \mathfrak{DB}_1$, then it is easily seen that any function ξ from $B_u^{\mathfrak{DB}_1}(\phi, \delta_0)$ is not approximately continuous. Thus (3.7) is true if $\mathcal{K} = \mathfrak{DB}_1$.

The proof in the case $\mathcal{F} = \mathcal{A}$ will be completed if we show that $B_u^{\mathcal{K}}(\phi, \delta_0)$ has the ∞-entropy point property for any $\mathcal{K} \in \{\mathcal{A}, \mathfrak{DB}_1, \mathcal{C}_a\}$. For this purpose, fix $\eta \in B_u^{\mathcal{K}}(\phi, \delta_0)$.

Assume first that $\mathcal{K} = \mathcal{A}$. If $r \in (0, \frac{\delta}{4})$, then we conclude from (3.8) that $x_0 - \frac{\delta}{3}$, $x_0 + \frac{\delta}{3} \in \eta([x_0, x_0 + r))$. Moreover, Statement 2.3 implies that η is a connectivity function. Thus $\mathrm{Fix}(\eta) \cap [x_0, x_0 + r) \neq \emptyset$ and, as a consequence, we have that $x_0 \in \mathrm{pFix}(\eta)$.

Consider a sequence $\{[s_n, t_n]\}_{n\in\mathbb{N}}$ of intervals such that $x_0 < \ldots < s_n < t_n < \ldots < t_2 < s_1 < t_1 < x_0 + \frac{\delta}{4}$ and $\lim_{n\to\infty} t_n = x_0$. From (3.8) we see that $[x_0 - \frac{\delta}{3}, x_0 + \frac{\delta}{3}] \subset \eta([s_n, t_n])$ for $n \in \mathbb{N}$. Putting $J = [x_0 - \frac{\delta}{3}, x_0 + \frac{\delta}{3}]$ and $\mathcal{W}_n = \{[s_i, t_i] : i = n, n+1, \ldots\}$ for $n \in \mathbb{N}$ we obtain that $\{\mathcal{B}_\eta^n\}_{n\in\mathbb{N}}$, where $\mathcal{B}_\eta^n = (\mathcal{W}_n, J)$, is a sequence of η-bundles with dominating fibre converging to the point x_0. Statement 3.1 implies that $h(\mathcal{B}_\eta^n) = +\infty$, so $+\infty \in E_\eta(x_0)$. Finally, we get that $x_0 \in \mathfrak{e}(\eta)$. Since η was arbitrary, so $B_u^{\mathcal{A}}(\phi, \delta_0)$ has the ∞-entropy point property.

Now, assume that $\mathcal{K} = \mathcal{C}_a$. Since $\rho_u(\eta, \phi) < \frac{\delta}{6}$ and η is a Darboux function, we obtain immediately that $[x_0 - \frac{\delta}{6}, x_0 + \frac{\delta}{6}] \subset \eta([d_{n+1}, c_n])$ for $n \in N$. Thus $[x_0 - \frac{\delta}{6}, x_0 + \frac{\delta}{6}] \subset R^+(\eta, x_0)$, so $x_0 \in \mathrm{aFix}(\eta)$. From Lemma 2.10 we conclude that $x_0 \in \mathrm{pFix}(\eta)$. If we put $J = [x_0 - \frac{\delta}{6}, x_0 + \frac{\delta}{6}]$, $\mathcal{V}_n = \{[d_{i+1}, c_i] : i = n, n+1, \ldots\}$ and $\mathcal{B}_\eta^n = (\mathcal{V}_n, J)$, then $\{\mathcal{B}_\eta^n\}_{n\in\mathbb{N}}$ is a sequence of η-bundles with dominating fibre converging to the point x_0. Thus, by Statement 3.1, $h(\mathcal{B}_\eta^n) = +\infty$. Hence $+\infty \in E_\eta(x_0)$ and, in consequence, $x_0 \in \mathfrak{e}(\eta)$. Since η was arbitrary, so $B_u^{\mathcal{C}_a}(\phi, \delta_0)$ has the ∞-entropy point property.

Analysis similar to the above shows that $B_u^{\mathfrak{DB}_1}(\phi, \delta_0)$ has the ∞-entropy point property.

Consider finally the situation when $\mathcal{F} \in \{\mathfrak{DB}_1, \mathcal{C}_a\}$ and $\mathcal{K} \subset \mathcal{F}$. Let $U_1 \in T_k^{\mathcal{F}} \setminus \{\emptyset\}$ be a basis set in $T_k^{\mathcal{F}}$. From Statement 2.1 we obtain that there exists an open set $W_{U_1} \subset \mathbb{I}^2$ such that

$$U_1 = \{f \in \mathcal{F} : \mathrm{Gr}(f) \subset W_{U_1}\}.$$

Putting $U = \{f \in \mathcal{A} : \mathrm{Gr}(f) \subset W_{U_1}\}$ we obtain that $U \in T_k^{\mathcal{A}} \setminus \{\emptyset\}$. Clearly, $\mathcal{F} \subset \mathcal{A}$. From what has already been proved, it may be concluded that there exist $\phi \in \mathcal{K}$ and $\delta_0 > 0$ such that $B_u^{\mathcal{K}}(\phi, \delta_0)$ has the ∞-entropy point property, $B_u^{\mathcal{K}}(\phi, \delta_0) \subset U$ and $B_u^{\mathcal{K}}(\phi, \delta_0) \cap \mathcal{H} = \emptyset$ whenever $\mathcal{H} \in \{\mathfrak{DB}_1, \mathcal{C}_a\}$ and $\mathcal{H} \subsetneq \mathcal{K}$.

It is easy to see that $B_u^{\mathcal{K}}(\phi, \delta_0) \subset U_1$, and the proof is complete. □

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