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This work is dedicated to Professor Leon Mikołajczyk
on the occasion of his 85th birthday.

ON SOME SUBCLASSES OF THE FAMILY OF DARBOUX BAIRE 1 FUNCTIONS

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Abstract. We introduce a subclass of the family of Darboux Baire 1 functions $f : \mathbb{R} \to \mathbb{R}$ modifying the Darboux property analogously as it was done by Z. Grande in [*On a subclass of the family of Darboux functions*, Colloq. Math. 17 (2009), 95–104], and replacing approximate continuity with $\mathcal{I}$-approximate continuity, i.e. continuity with respect to the $\mathcal{I}$-density topology. We prove that the family of all Darboux quasi-continuous functions from the first Baire class is a strongly porous set in the space $\mathcal{DB}_1$ of Darboux Baire 1 functions, equipped with the supremum metric.

Keywords: Darboux property, strong Świątkowski property, Baire property, $\mathcal{I}$-approximate continuity, quasi-continuity.

Mathematics Subject Classification: 26A15, 54C08.

1. INTRODUCTION

Studies concerning the "size" of some subsets of the metric space have a long tradition, also in the case when a space is a family of functions. One of the best known and interesting examples of this kind is a result of S. Banach. In 1931 S. Banach using the category method showed that the set of nowhere differentiable functions is residual in the space of continuous functions under the supremum metric.

It is natural to consider similar problems for different spaces of functions. Our paper is devoted to investigations of subclasses of the class $\mathcal{D}$ of Darboux functions in this context.

Functions with the Darboux (intermediate value) property continue to hold interest for a variety of reasons. Many papers which appeared during the last few years contain results concerning Darboux-like functions in relation to dynamical systems (see [3, 8, 20–23]).

The notion of porosity in spaces of Darboux-like functions was studied among others by J. Kucner, R. Pawlak, B. Świątek in [10] and by H. Rosen in [26].

Modifications of the Darboux property are also considered. In particular, in the case of the strong Świątkowski property we demand from a point where a function takes on the intermediate value that it be a continuity point of this function (see [14, 15]).

In [4] Z. Grande considered subclass $\mathcal{D}_{ap}$ of the family of Darboux functions replacing continuity at a point, where a functions takes on the intermediate value, with approximate continuity, i.e. continuity with respect to the density topology. In the paper mentioned above Z. Grande proved among others that the family $\mathcal{D}_{ap}\mathcal{B}_1$ is a nowhere dense set in the space $\mathcal{DB}_1$ equipped with the supremum metric, where $\mathcal{DB}_1$ is the family of Darboux functions of Baire class 1 (and throughout, as here, we omit the intersection sign when context permits: $\mathcal{DB}_1 \equiv \mathcal{D} \cap \mathcal{B}_1$).

In this note, following Z. Grande, we consider functions with the $\mathcal{I}$-ap Darboux property, replacing approximate continuity with $\mathcal{I}$-approximate continuity, i.e. continuity with respect to the $\mathcal{I}$-density topology.

We prove that the set $\mathcal{DQB}_1$ of all Darboux quasi-continuous Baire 1 functions is strongly porous (so also nowhere dense) in the space $\mathcal{DB}_1$.

2. PRELIMINARIES

A function $f : \mathbb{R} \to \mathbb{R}$ has the *intermediate value property* if on each interval $(a, b) \subset \mathbb{R}$ function f assumes every real value between $f(a)$ and $f(b)$. In 1875 J. Darboux showed that this property is not equivalent to continuity, and that every derivative has the intermediate value property.

Let us introduce a metric ρ on the space $\mathcal{D}$ as follows:

$$\rho(f, g) = \min\{1, \sup\{|f(t) - g(t)| : t \in \mathbb{R}\}\}.$$

To simplify notation, we shall write

$$\langle a, b\rangle = (\min\{a, b\}, \max\{a, b\}).$$

In 1977 T. Mańk and T. Świątkowski in [18] define a modification of the Darboux property. They considered a family of functions with the so-called Świątkowski property.

Definition 2.1 ([18])**.** A function $f : \mathbb{R} \to \mathbb{R}$ has the Świątkowski property iff for each interval $(a, b) \subset \mathbb{R}$ there exists a point $x_0 \in (a, b)$ such that $f(x_0) \in \langle f(a), f(b)\rangle$ and f is continuous at x_0.

In 1995 A. Maliszewski investigated a class of functions which possess a stronger property.

Definition 2.2 ([15])**.** A function $f : \mathbb{R} \to \mathbb{R}$ has the strong Świątkowski property (briefly $f \in \mathcal{D}_s$) iff for each interval $(a, b) \subset \mathbb{R}$ and for each $\lambda \in \langle f(a), f(b)\rangle$ there exists a point $x_0 \in (a, b)$ such that $f(x_0) = \lambda$ and f is continuous at x_0.

For more details about the strong Świątkowski property see [14–17].

Z. Grande in 2009 considered a modification of the strong Świątkowski property replacing continuity with approximate continuity.

Definition 2.3 ([4])**.** A function $f:\mathbb{R}\to\mathbb{R}$ has the ap-Darboux property (briefly $f\in\mathcal{D}_{ap}$) iff for each interval $(a,b)\subset\mathbb{R}$ and for each $\lambda\in\langle f(a),f(b)\rangle$ there exists a point $x_0\in(a,b)$ such that $f(x_0)=\lambda$ and f is approximately continuous at x_0.

In [4] Z. Grande proved that the family $\mathcal{D}_{ap}$ is nowhere dense set in the space $\mathcal{D}$, and $\mathcal{D}_{ap}\mathcal{B}_1$ is nowhere dense in $\mathcal{D}\mathcal{B}_1$.

Let $\mathcal{I}$ be the σ-ideal of sets of the first category.

In [5–7] we introduced the analogous modification of the strong Świątkowski property replacing continuity with $\mathcal{I}$-approximate continuity, i.e. by continuity with respect to the $\mathcal{I}$-density topology in the domain (see [2, 24, 25, 28, 29]).

We say that a property holds $\mathcal{I}$-almost everywhere (briefly $\mathcal{I}$-a.e.) iff the set of all points which do not have this property belongs to $\mathcal{I}$.

Definition 2.4 ([24])**.** The sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions with the Baire property converges with respect to $\mathcal{I}$ to some real function f with the Baire property ($f_n \xrightarrow[n\to\infty]{\mathcal{I}} f$) iff every subsequence $\{f_{m_n}\}_{n\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$ contains a subsubsequence $\{f_{m_{p_n}}\}_{n\in\mathbb{N}}$ which converges to f $\mathcal{I}$-a.e.

Let A be a set with the Baire property and $n\cdot A=\{n\cdot a: a\in A\}$ for $n\in\mathbb{N}$.

Definition 2.5 ([24])**.** A point 0 is an $\mathcal{I}$-density point of A iff

$$\chi_{(n\cdot A)\cap(-1,1)} \xrightarrow[n\to\infty]{\mathcal{I}} \chi_{(-1,1)}.$$

A point 0 is an $\mathcal{I}$-dispersion point of A iff 0 is an $\mathcal{I}$-density point of $\mathbb{R}\backslash A$. We say that x is an $\mathcal{I}$-density point of A if 0 is an $\mathcal{I}$-density point of $A-x=\{a-x: a\in A\}$.

Put

$$\Phi(A)=\{x\in\mathbb{R}:\ x \text{ is an } \mathcal{I}\text{-density point of } A\}.$$

The family

$$\tau_{\mathcal{I}}=\{A\subset\mathbb{R}:\ \text{A has the Baire property and } A\subset\Phi(A)\}$$

called the $\mathcal{I}$-density topology was first studied in [24, 25, 29].

Recall that a function $f:\mathbb{R}\to\mathbb{R}$ is $\mathcal{I}$-approximately continuous at $x_0\in\mathbb{R}$ iff for every $\epsilon>0$ there exists $U\in\tau_{\mathcal{I}}$ such that $x_0\in U$ and $f(U)\subset(f(x_0)-\epsilon, f(x_0)+\epsilon)$.

Definition 2.6 ([5–7])**.** A function $f:\mathbb{R}\to\mathbb{R}$ has the $\mathcal{I}$-ap-Darboux property (briefly $f\in\mathcal{D}_{\mathcal{I}-ap}$) iff for each interval $(a,b)\subset\mathbb{R}$ and for each $\lambda\in\langle f(a),f(b)\rangle$ there exists a point $x_0\in(a,b)$ such that $f(x_0)=\lambda$ and f is $\mathcal{I}$-approximately continuous at x_0.

We have

$$D_s\subset\mathcal{D}_{ap}\cap\mathcal{D}_{\mathcal{I}-ap}\subset\mathcal{D}_{ap}\cup\mathcal{D}_{\mathcal{I}-ap}\subset\mathcal{D},$$

and in [5] it is proved that all these inclusions are proper.

Let us denote by $\overline{A}$ ($Int\,(A)$) the closure (interior) of the set A in the Euclidean topology, respectively. A set $A \subset \mathbb{R}$ is said to be *semi-open* iff there is an open set U such that $U \subset A \subset \overline{U}$ (see [13]). It is not difficult to see that A is semi-open iff $A \subset \overline{Int\,(A)}$. The family of all semi-open sets will be denoted by $\mathcal{S}$. A function $f : \mathbb{R} \to \mathbb{R}$ is semi-continuous iff for each set V open in the Euclidean topology the set $f^{-1}(V)$ is semi-open (compare [13]).

Definition 2.7 ([9])**.** A function $f : \mathbb{R} \to \mathbb{R}$ is quasi-continuous at a point x iff for every neighbourhood U of x and for every neighbourhood V of $f(x)$ there exists a non-empty open set $G \subset U$ such that $f(G) \subset V$. A function $f : \mathbb{R} \to \mathbb{R}$ is quasi-continuous (briefly $f \in \mathcal{Q}$) if it is quasi-continuous at each point.

A. Neubrunnová proved in [19] that f is semi-continuous if and only if it is quasi-continuous.

3. MAIN RESULTS

In [6] it is proved that the family $\mathcal{D}_{\mathcal{I}-ap}$ is a strongly porous set in the space of Darboux functions having the Baire property, and that each function from $\mathcal{D}_{\mathcal{I}-ap}$ is quasi-continuous.

Observe that even the family $\mathcal{D}_s$ is not contained in the class $\mathcal{DB}_1$ of Darboux Baire 1 functions.

Lemma 3.1. *There exists a function $f : \mathbb{R} \to [0,1]$ having the strong Świątkowski property which is not in the first class of Baire.*

Proof. Let C be a Cantor set and let $\{(a_n, b_n)\}_{n\in\mathbb{N}}$ be a sequence of all the component intervals of the set $[0,1]\setminus C$. Put

$$f(x) = \begin{cases} 1 & \text{for } x \in \left[a_n, \frac{3a_n+b_n}{4}\right] \cup \left[\frac{a_n+3b_n}{4}, b_n\right],\ n \in \mathbb{N}, \\ 0 & \text{for } x \in \left(\mathbb{R}\setminus \bigcup\limits_{n=1}^{\infty} [a_n, b_n]\right) \cup \bigcup\limits_{n=1}^{\infty} \left\{\frac{a_n+b_n}{2}\right\}, \\ \text{linear} & \text{on the intervals } \left[\frac{3a_n+b_n}{4}, \frac{a_n+b_n}{2}\right] \cup \left[\frac{a_n+b_n}{2}, \frac{a_n+3b_n}{4}\right],\ n \in \mathbb{N}. \end{cases}$$

Clearly, f is not in the first class of Baire, as $f \mid C$ has no points of continuity. On the other hand, $f \in \mathcal{D}_s$ since C is a nowhere dense set. □

We have a sequence of proper inclusions

$$D_s\mathcal{B}_1 \subsetneq \mathcal{D}_{ap}\mathcal{B}_1 \cap \mathcal{D}_{\mathcal{I}-ap}\mathcal{B}_1 \subsetneq \mathcal{D}_{ap}\mathcal{B}_1 \cup \mathcal{D}_{\mathcal{I}-ap}\mathcal{B}_1 \subsetneq \mathcal{DB}_1,$$

as all the functions constructed in [5] are in the first class of Baire.

Definition 3.2 ([6])**.** A function $f : \mathbb{R} \to \mathbb{R}$ has the q-property iff for each interval $(a,b) \subset \mathbb{R}$ such that $f \mid (a,b)$ is not constant and for each interval $(C,D) \subset f((a,b))$ there exists an interval $(c,d) \subset (a,b)$ such that $f((c,d)) \subset (C,D)$.

In [6] it is proved that if $f \in \mathcal{D}_{\mathcal{I}-ap}$, then f has the q-property.

We shall prove that $\mathcal{DQB}_1$ is a strongly porous set in $(\mathcal{DB}_1, \rho)$. For this purpose we need some auxiliary lemmas.

Lemma 3.3. *A Darboux function f has the q-property iff f is quasi-continuous.*

Proof. Let f be a Darboux function having the q-property. Let V be an open set and $A = f^{-1}(V)$. Observe that

$$A \subset \overline{Int\,(A)}. \tag{3.1}$$

Let $x \in A$. There are two cases:
Case 1. f is constant on some neighbourhood of x. Then $f^{-1}(V)$ contains this neighbourhood, so $x \in \overline{Int\,(A)}$.
Case 2. f is constant on no neighbourhood of x. From the Darboux property $f((x-1, x+1))$ is a non-degenerate interval. Since f has the q-property, there exists an interval $(c_1, d_1) \subset (x-1, x+1)$ such that

$$f((c_1, d_1)) \subset V \cap Int\,(f((x-1, x+1)))\,. \tag{3.2}$$

Let x_1 be the centre of (c_1, d_1). From (3.2) we obtain $x_1 \in Int\,\big(f^{-1}(V)\big)$, so $x_1 \in Int\,(A)$. Analogously, for each $n \in \mathbb{N}$ there exists a point $x_n \in (x - 1/n, x + 1/n)$ such that $x_n \in Int\,(A)$. Consequently $x \in \overline{Int\,(A)}$, as $x_n \xrightarrow{n\to\infty} x$. Hence A is semi-open set. From [19] it follows that f is quasi-continuous.

Now let $f \in \mathcal{DQ}$. Let (a, b) and (C, D) be two intervals such that $f \mid (a, b)$ is not constant and $(C, D) \subset f((a, b))$. Let $y \in (C, D)$. Then there exists a point $x \in (a, b)$ such that $f(x) = y$ and f is quasi-continuous at x. Hence there exists a semi-open set A such that $x \in A$ and $f(A) \subset (C, D)$.

Let δ be a positive number such that $(x - \delta, x + \delta) \subset (a, b)$. As $x \in A$ and $A \subset \overline{Int\,(A)}$, there exists a point $x_\delta \in (x - \delta, x + \delta) \cap Int\,(A)$. So there exists $\delta_1 > 0$ such that

$$(x_\delta - \delta_1, x_\delta + \delta_1) \subset (x - \delta, x + \delta) \cap Int\,(A)\,.$$

Put $(c, d) = (x_\delta - \delta_1, x_\delta + \delta_1)$. Obviously, $(c, d) \subset (a, b)$ and $f((c, d)) \subset f(A) \subset (C, D)$. □

Lemma 3.4. *Let f be a Darboux function. If there exist two intervals (a, b) and (A, B) such that $f^{-1}((A, B)) \cap (a, b)$ is a non-empty set with an empty interior, then f does not have the q-property.*

Proof. Suppose, on the contrary, that f does have the q-property. Obviously, f is not constant on (a, b), so from the Darboux property $f((a, b)) \cap (A, B)$ is a non-degenerate interval. Put

$$(C, D) = Int\,(f((a, b))) \cap (A, B)\,.$$

From the q-property there exists an interval $(c, d) \subset (a, b)$ such that $f((c, d)) \subset (C, D) \subset (A, B)$. Consequently,

$$(c, d) \subset f^{-1}((A, B)) \cap (a, b)\,,$$

a contradiction. □

Let $B(f,r)$ denote the open ball with centre f and with radius r in the space $(\mathcal{D},\rho)$.

Lemma 3.5. *For an arbitrary interval (a,b) there exists a function $f:\mathbb{R}\xrightarrow{onto}[0,1]$ vanishing $\mathcal{I}$-a.e. on (a,b) and everywhere on $\mathbb{R}\setminus(a,b)$ such that*

(i) $f\in\mathcal{DB}_1$,
(ii) $B\left(f,\frac{1}{2}\right)\cap\mathcal{DQ}=\emptyset$,
(iii) *there exists a function $h\in\mathcal{DQ}$ such that $\rho(f,h)=\frac{1}{2}$.*

Proof. Let $(a,b)\subset\mathbb{R}$ and let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint closed nowhere dense subsets of $[a,b]$ of cardinality continuum such that for each interval $(c,d)\subset(a,b)$ there exists a natural number n with $C_n\subset(c,d)$ (see [6]). Put $C=\bigcup_{n\in\mathbb{N}}C_n$. Obviously, C is a set of type F_σ which is bilaterally $\mathfrak{c}$-dense-in-itself, so using Theorem 2.4 of [1, Chapter II], there exists a function $f\in\mathcal{DB}_1$ such that $f(x)=0$ if $x\notin C$ and $0<f(x)\le 1$ for $x\in C$.

As the set $\{x\in\mathbb{R}:f(x)\neq 0\}$ is of the first category, f is a function having the Baire property. Since $f^{-1}((0,1))\cap(a,b)$ is a non-empty set of the first category, using Lemma 3.4, we obtain that f does not have the q-property and, consequently, by Lemma 3.3, f is not quasi-continuous.

Now we shall prove that $B\left(f,\frac{1}{2}\right)\cap\mathcal{DQ}=\emptyset$. For this purpose we shall prove that for each $n>2$

$$B\left(f,\frac{1}{2}-\frac{1}{n}\right)\cap\mathcal{DQ}=\emptyset. \tag{3.3}$$

Let $n>2$ and $g\in B\left(f,\frac{1}{2}-\frac{1}{n}\right)$. Let $a',b'\in(a,b)$ be such that $f(a')=0$ and $f(b')=1$. Then $g(a')<\frac{1}{2}-\frac{1}{n}$ and $g(b')>\frac{1}{2}+\frac{1}{n}$. From the Darboux property there exists a point $x\in\langle a',b'\rangle$ such that $g(x)=\frac{1}{2}$. Obviously, $x\in g^{-1}\left(\left(\frac{1}{2}-\frac{1}{n},\frac{1}{2}+\frac{1}{n}\right)\right)$. At the same time

$$g^{-1}\left(\left(\frac{1}{2}-\frac{1}{n},\frac{1}{2}+\frac{1}{n}\right)\right)\cap(a,b)\subset f^{-1}((0,1])\subset C,$$

so from Lemma 3.4 g does not have the q-property and, consequently, by Lemma 3.3 g is not quasi-continuous. As

$$B\left(f,\frac{1}{2}\right)=\bigcup_{n=3}^{\infty}B\left(f,\frac{1}{2}-\frac{1}{n}\right),$$

using (3.3) we obtain (ii).

Now let

$$h(x)=\begin{cases}0 & \text{for } x\in\mathbb{R}\setminus\left(a-\frac{1}{2},b+\frac{1}{2}\right),\\ \frac{1}{2} & \text{for } x\in[a,b],\\ \text{linear} & \text{on the intervals } \left[a-\frac{1}{2},a\right] \text{ and } \left[b,b+\frac{1}{2}\right].\end{cases}$$

Clearly, $h\in\mathcal{DQ}$ and $\rho(f,h)=\frac{1}{2}$. □

Lemma 3.6. *Let $g \in \mathcal{DB}_1$ and $r \in (0,1)$. For each $\epsilon \in (0, r/4)$ there exists a function $h \in \mathcal{DB}_1$ such that*

$$B\left(h, \frac{r}{2} - \epsilon\right) \subset B(g, r) \setminus \mathcal{DQB}_1.$$

Proof. Let $g \in \mathcal{DB}_1$, $r \in (0,1)$ and $\epsilon \in (0, r/4)$. Since $g \in \mathcal{B}_1$, we can find a point x_0 such that g is continuous at x_0, consequently there exists an interval $(a,b) \subset \mathbb{R}$ with

$$\operatorname{diam}(g([a,b])) < 2\epsilon. \tag{3.4}$$

Put

$$B = \frac{\inf_{x\in[a,b]} \{g(x)\} + \sup_{x\in[a,b]} \{g(x)\}}{2}, \quad J = \left[B - \frac{r}{2} + \epsilon, B + \frac{r}{2} - \epsilon\right]$$

and

$$A = B - \frac{r}{2} + \epsilon.$$

Then $J = [A, A + r - 2\epsilon]$. As $\epsilon < \frac{r}{2} - \epsilon$, from (3.4) we obtain

$$g([a,b]) \subset [B - \epsilon, B + \epsilon] \subset J.$$

Let $[a_0, b_0] \subset (a,b)$ and let f be a function constructed as in Lemma 3.5 for the interval (a_0, b_0). Put

$$h(x) = \begin{cases} g(x) & \text{for } x \in \mathbb{R} \setminus (a,b), \\ A + (r - 2\epsilon) f(x) & \text{for } x \in [a_0, b_0], \\ \text{linear} & \text{on the intervals } [a, a_0],\ [b_0, b]. \end{cases}$$

Observe, that $\rho(g,h) < \frac{r}{2}$. If $x \in \mathbb{R} \setminus (a,b)$, $|h(x) - g(x)| = 0$. Let $x \in (a,b)$. As $h([a,b]) \subset J$ and $g([a,b]) \subset [B - \epsilon, B + \epsilon]$, we obtain

$$|h(x) - g(x)| \le |h(x) - B| + |B - g(x)| < \frac{r}{2} - \epsilon + \epsilon = \frac{r}{2}.$$

Consequently, $\rho(g,h) < \frac{r}{2}$.

From the definition of h it follows that $h \in \mathcal{DB}_1$.

Now we shall prove that

$$B\left(h, \frac{r}{2} - \epsilon\right) \subset B(g, r) \setminus \mathcal{DQB}_1.$$

Let $s \in B\left(h, \frac{r}{2} - \epsilon\right)$ and $\epsilon_s \in (\rho(s,h), r/2 - \epsilon)$. Put

$$E = s^{-1}((\sup J - \epsilon_s, \sup J + \epsilon_s)) \cap (a_0, b_0).$$

Observe that $E \neq \emptyset$. From the construction of f there exists a point $x_0 \in (a_0, b_0)$ such that $f(x_0) = 1$, i.e. $h(x_0) = A + (r - 2\epsilon) = \sup J$. Then

$$s(x_0) \in [h(x_0) - \rho(s,h), h(x_0) + \rho(s,h)] \subset (\sup J - \epsilon_s, \sup J + \epsilon_s),$$

so $x_0 \in E$.

Now we shall prove that

$$E \subset f^{-1}\left((0,1]\right).$$

Let $x \in E$. Then $s\left(x\right) \in \left(\sup J - \epsilon_s, \sup J + \epsilon_s\right)$, so

$$h\left(x\right) \in \left(\sup J - \epsilon_s - \rho\left(s,h\right), \sup J + \epsilon_s + \rho\left(s,h\right)\right).$$

Hence

$$x \in h^{-1}\left(\left(\sup J - \epsilon_s - \rho\left(s,h\right), \sup J + \epsilon_s + \rho\left(s,h\right)\right)\right) \cap \left(a_0, b_0\right). \tag{3.5}$$

At the same time, as $\rho\left(s,h\right) < \epsilon_s < \frac{r}{2} - \epsilon$ and $\sup J = A + r - 2\epsilon$,

$$\left(\sup J - \epsilon_s - \rho\left(s,h\right), \sup J + \epsilon_s + \rho\left(s,h\right)\right) \subset \left(A, A + 2r - 4\epsilon\right). \tag{3.6}$$

From the definition of h

$$h^{-1}\left(\left(A, A + 2r - 4\epsilon\right)\right) \cap \left(a_0, b_0\right) \subset f^{-1}\left((0,1]\right). \tag{3.7}$$

Using (3.5), (3.6) and (3.7) we obtain $x \in f^{-1}\left((0,1]\right)$.

As f is a function vanishing $\mathcal{I}$-a.e., E is a non-empty set of the first category. From Lemma 3.4 and Lemma 3.3 it follows that s is not quasi-continuous.

Consequently, $B\left(h, \frac{r}{2} - \epsilon\right) \cap \mathcal{DQ} = \emptyset$.

As $\rho(g,h) < \frac{r}{2}$, $B\left(h, \frac{r}{2} - \epsilon\right) \subset B\left(g,r\right)$. Finally,

$$B\left(h, \frac{r}{2} - \epsilon\right) \subset B\left(g,r\right) \setminus \mathcal{DQB}_1.$$

□

Let X be an arbitrary metric space. Assume that $B\left(x,0\right) = \emptyset$. Fix $M \subset X$, $x \in X$ and $r > 0$. Let

$$\gamma\left(x,r,M\right) = \sup\{t \geq 0 : \exists_{z \in X} B\left(z,t\right) \subset B\left(x,r\right) \setminus M\}.$$

Define the *porosity* of M at x as

$$p\left(M,x\right) = 2 \limsup_{r \to 0^+} \frac{\gamma\left(x,r,M\right)}{r}.$$

Definition 3.7 ([30])**.** The set $M \subset X$ is porous (strongly porous) iff $p\left(M,x\right) > 0$ ($p\left(M,x\right) = 1$) for each $x \in M$.

Theorem 3.8. *The set $\mathcal{DQB}_1$ is strongly porous in $\left(\mathcal{DB}_1, \rho\right)$.*

Proof. Let $g \in \mathcal{DB}_1$, $r \in (0,1)$ and $\epsilon \in (0, r/4)$. From Lemma 3.6 it follows that there exists a function $h \in \mathcal{DB}_1$ such that $B\left(h, \frac{r}{2} - \epsilon\right) \subset B\left(g,r\right) \setminus \mathcal{DQB}_1$. Then

$$\gamma\left(g,r,\mathcal{DQB}_1\right) = \frac{r}{2}$$

and

$$p\left(\mathcal{DQB}_1, g\right) = 2 \limsup_{r \to 0^+} \frac{\gamma\left(g,r,\mathcal{DQB}_1\right)}{r} = 1.$$

□

From the last theorem and from the inclusion $\mathcal{D}_{\mathcal{I}-ap} \subset \mathcal{DQ}$ it follows that $\mathcal{D}_{\mathcal{I}-ap}\mathcal{B}_1$ is strongly porous, so also nowhere dense in $\mathcal{DB}_1$. The last conclusion is analogous to the result obtained by Z. Grande, that the set $\mathcal{D}_{ap}\mathcal{B}_1$ is nowhere dense in $\mathcal{DB}_1$ (see [4, Theorem 2]).

In [4] Z. Grande proved that the family $\mathcal{D}_{ap}\mathcal{B}_1$ is not closed under uniform convergence. Indeed there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of Baire 1 functions in $\mathcal{D}_{ap}$ uniformly convergent to a function f which does not have the ap-Darboux property ([4, Theorem 7]).

By a slightly modification of the proof of Grande we obtain its analogue for $\mathcal{I}$.

Theorem 3.9. *There exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions in $\mathcal{D}_{\mathcal{I}-ap}\mathcal{B}_1$, which is uniformly convergent to a function $f \notin \mathcal{D}_{\mathcal{I}-ap}$.*

Proof. Let $A = \bigcup_{n\in\mathbb{N}} [a_n, b_n]$ be a right-hand interval-set at zero, such that 0 is not an $\mathcal{I}$-dispersion point of A (compare [5]).

For each $n \in \mathbb{N}$ choose an interval $I_n = [c_n, d_n] \subset (b_{n+1}, a_n)$, such that $\frac{b_{n+1}+a_n}{2} \in (c_n, d_n)$, and a continuous function $g_n : [b_{n+1}, a_n] \to [c_n, 1]$ for which $g(a_n) = g(b_{n+1}) = 1$ and $g(x) = x$ for $x \in I_n$.

For each $k \in \mathbb{N}$ put

$$f_k(x) = \begin{cases} 1 & \text{for } x \in [b_1, \infty), \\ 1 & \text{for } x \in [a_n, b_n],\ n \in \mathbb{N}, \\ x & \text{for } x \in (-\infty, 0], \\ g_n(x) & \text{for } x \in [b_{n+1}, a_n],\ n \in \mathbb{N},\ n < k, \\ 0 & \text{for } x = \frac{b_{n+1}+a_n}{2},\ n \in \mathbb{N},\ n \geq k, \\ g_n(x) & \text{for } x \in [b_{n+1}, c_n] \cup [d_n, a_n],\ n \in \mathbb{N},\ n \geq k, \\ \text{linear} & \text{on the intervals } \left[c_n, \frac{b_{n+1}+a_n}{2}\right] \text{ and } \left[\frac{b_{n+1}+a_n}{2}, d_n\right],\ n \in \mathbb{N},\ n \geq k, \end{cases}$$

and

$$f(x) = \begin{cases} 1 & \text{for } x \in [b_1, \infty), \\ 1 & \text{for } x \in [a_n, b_n],\ n \in \mathbb{N}, \\ x & \text{for } x \in (-\infty, 0], \\ g_n(x) & \text{for } x \in [b_{n+1}, a_n],\ n \in \mathbb{N}. \end{cases}$$

Clearly, f and f_k, $k \in \mathbb{N}$, are continuous at each point $x \neq 0$, so they are Baire 1 functions and have the Darboux property. Observe that they are not $\mathcal{I}$-approximately continuous at zero. Let $\epsilon \in (0, 1)$ and $k \in \mathbb{N}$. The sets $f^{-1}((-\epsilon, \epsilon))$ and $f_k^{-1}((-\epsilon, \epsilon))$ are contained in $\mathbb{R}\backslash A$, as $f(A) = f_k(A) = \{1\}$, and 0 is not an $\mathcal{I}$-dispersion point of A, so 0 is not an $\mathcal{I}$-density point of either $f^{-1}((-\epsilon, \epsilon))$ or $f_k^{-1}((-\epsilon, \epsilon))$.

At the same time, for each $k \in \mathbb{N}$ and each open interval containing 0 there exists a point $x \neq 0$ such that $f_k(x) = 0$ and f_k is continuous at x. Consequently, $f_k \in \mathcal{D}_{\mathcal{I}-ap}\mathcal{B}_1$ for each $k \in \mathbb{N}$.

On the other hand, $f^{-1}(\{0\}) = \{0\}$, so $f \notin \mathcal{D}_{\mathcal{I}-ap}$.

Clearly, $|f_k(x) - f(x)| \leq a_k$ for each $x \in \mathbb{R}$ and $k \in \mathbb{N}$, and $\lim_{k\to\infty} a_k = 0$, so the sequence $\{f_k\}_{k\in\mathbb{N}}$ converges uniformly to f. □

Let $\mathcal{A} \subset \mathcal{P}(\mathbb{R})$, where $\mathcal{P}(\mathbb{R})$ is the power set of $\mathbb{R}$.

Definition 3.10 ([7])**.** We will say that $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{A}$-continuous at the point $x \in \mathbb{R}$ iff for each open set $V \subset \mathbb{R}$ with $f(x) \in V$ there exists a set $A \in \mathcal{A}$ such that $x \in A$ and $f(A) \subset V$. We will say that $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{A}$-continuous if f is $\mathcal{A}$-continuous at each point $x \in \mathbb{R}$.

It is not difficult to see that if $\mathcal{A}$ is the Euclidean topology τ_e, then the notion of $\mathcal{A}$-continuity is equivalent to continuity in the classical sense. If $\mathcal{A}$ is the density topology τ_d, then we have approximate continuity. If $\mathcal{A}$ is the $\mathcal{I}$-density topology $\tau_{\mathcal{I}}$, then we obtain $\mathcal{I}$-approximate continuity. If $\mathcal{A}$ is an arbitrary topology τ on $\mathbb{R}$, then $\mathcal{A}$-continuity is a form of continuity between $(\mathbb{R}, \tau)$ and $(\mathbb{R}, \tau_e)$. If $\mathcal{A}$ is the family of semi-open sets $\mathcal{S}$, then $\mathcal{A}$-continuity is equivalent to quasi-continuity.

Definition 3.11. We will say that $f : \mathbb{R} \to \mathbb{R}$ has the $\mathcal{A}$-Darboux property iff for each interval $(a, b) \subset \mathbb{R}$ and each $\lambda \in \langle f(a), f(b)\rangle$ there exists a point $x \in (a, b)$ such that $f(x) = \lambda$ and f is $\mathcal{A}$-continuous at x.

Denote the family of all functions having the $\mathcal{A}$-Darboux property by $\mathcal{D}_{\mathcal{A}}$. It is easy to see that: if $\mathcal{A}$ is the Euclidean topology τ_e, then $\mathcal{D}_{\mathcal{A}} = \mathcal{D}_{\tau_e} = \mathcal{D}_s$; if $\mathcal{A}$ is the density topology τ_d, then $\mathcal{D}_{\mathcal{A}} = \mathcal{D}_{\tau_d} = \mathcal{D}_{ap}$; if $\mathcal{A}$ is the $\mathcal{I}$-density topology, then $\mathcal{D}_{\mathcal{A}} = \mathcal{D}_{\tau_{\mathcal{I}}} = \mathcal{D}_{\mathcal{I}-ap}$.

The set A is of the first category at the point x iff there exists an open neighbourhood G of x such that $A \cap G$ is of the first category (see [11]). We will denote by $D(A)$ the set of all points x such that A is not of the first category at x.

Let $\mathcal{B}a$ be the family of all sets having the Baire property.

Definition 3.12. We will say that the family $\mathcal{A} \subset \mathcal{P}(\mathbb{R})$ has the $(*)$-property iff

1. $\tau_e \subset \mathcal{A} \subset \mathcal{B}a$,
2. $A \subset D(A)$ for each $A \in \mathcal{A}$.

It is not difficult to see that a wide class of topologies has the $(*)$-property. For example, the Euclidean topology, $\mathcal{I}$-density topology, topologies constructed in [12] by E. Łazarow, R. A. Johnson, W. Wilczyński or the topology constructed by Wiertelak in [27]. Certain families of sets which are not topologies have the $(*)$-property: the family of semi-open sets being an example, but the density topology does not have this property.

In [7] we proved that if the family $\mathcal{A}$ has the $(*)$-property, then $\mathcal{D}_s \subset \mathcal{D}_{\mathcal{A}} \subset \mathcal{DQ}$. So, if $\mathcal{A}$ has the $(*)$-property, then we have:

$$\mathcal{D}_s\mathcal{B}_1 \subset \mathcal{D}_{\mathcal{A}}\mathcal{B}_1 \subset \mathcal{DQB}_1.$$

Hence, using Theorem 3.8, we obtain the following result.

Theorem 3.13. *If $\mathcal{A}$ has the $(*)$-property, then the set $\mathcal{D}_{\mathcal{A}}\mathcal{B}_1$ is strongly porous in $(\mathcal{DB}_1, \rho)$.*

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