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This work is dedicated to Professor Leon Mikołajczyk
on the occasion of his 85th birthday.

OPTIMIZATION OF A FRACTIONAL MAYER PROBLEM – EXISTENCE OF SOLUTIONS, MAXIMUM PRINCIPLE, GRADIENT METHODS

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Abstract. In the paper, we study a linear-quadratic optimal control problem of Mayer type given by a fractional control system. First, we prove a theorem on the existence of a solution to such a problem. Next, using the local implicit function theorem, we derive a formula for the gradient of a cost functional under constraints given by a control system and prove a maximum principle in the case of a control constraint set. The formula for the gradient is used to implement the gradient methods for the problem under consideration.

Keywords: fractional Riemann-Liouville derivative, Mayer problem, existence of an optimal solution, maximum principle, gradient method.

Mathematics Subject Classification: 26A33, 49J15, 49K15, 49M37.

1. INTRODUCTION

In the paper, we consider the following fractional linear control system of order $\alpha \in (0,1)$

$$\begin{cases} D^{\alpha}_{a+}x(t) = Ax(s) + Bu(t), \quad t \in [a,b] \text{ a.e.}, \\ I^{1-\alpha}_{a+}x(a) = 0 \end{cases} \tag{1.1}$$

with the performance index of Mayer type

$$J(u) = \frac{1}{2}\left|I^{1-\alpha}_{a+}x_u(b) - c\right|^2. \tag{1.2}$$

Here $D^{\alpha}_{a+}x$ denotes the left Riemann-Liouville derivative of a function $x : [a,b] \to \mathbb{R}^n$, x_u is a unique solution to system (1.1), corresponding to a control u (cf. [3]) and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $c \in \mathbb{R}^n$, n, $m \in \mathbb{N}$. We consider the above optimal control problem in the space $AC^{\alpha,2}_{a+} = AC^{\alpha,2}_{a+}([a,b],\mathbb{R}^n)$ of solutions and in a subset U of the space $L^2 = L^2([a,b],\mathbb{R}^m)$ of controls ($AC^{\alpha,2}_{a+}$ is the set of all functions $x : [a,b] \to \mathbb{R}^n$ possessing Riemann-Liouville derivative of order α belonging to L^2). Problems of such a type can be used in the study of pointwise controllability of systems (1.1).

The paper is organized as follows. In the first part, we obtain existence of a unique solution to problem (1.1)–(1.2). In the second part, assuming that $\alpha \in (\frac{1}{2}, 1)$, we derive a formula for the gradient of the functional J under constraints given by (1.1) and prove a maximum principle. In the third part, we show that the gradient is Lipschitzian and next use this fact to describe the gradient and projection of the gradient methods for (1.1)–(1.2).

Results of such a type for $\alpha = 1$ can be found in [7]. To our best knowledge the case of $\alpha < 1$ has not been studied by other authors so far.

2. BASICS OF FRACTIONAL CALCULUS

Let $\alpha > 0$, $h \in L^1 = L^1([a,b],\mathbb{R}^n)$. By the left Riemann-Liouville integral of order α of the function h on the interval $[a,b]$ we mean ([6]) a function $I^{\alpha}_{a+}h$ given by

$$I^{\alpha}_{a+}h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{h(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \quad t \in [a,b] \text{ a.e.}, \tag{2.1}$$

where Γ is the Euler function. By the right Riemann-Liouville derivative we mean the function

$$I^{\alpha}_{b-}h(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{h(\tau)}{(\tau-t)^{1-\alpha}} d\tau, \quad t \in [a,b] \text{ a.e.}$$

One can show that the above integrals exist and are finite a.e. on $[a,b]$. Moreover, if $h \in L^p = L^p([a,b],\mathbb{R}^n)$, $1 \le p < \infty$, then $I^{\alpha}_{a+}h \in L^p$.

Now, let $\alpha \in (0,1)$. We say ([6]) that $x \in L^1$ possesses the left Riemann-Liouville derivative $D^{\alpha}_{a+}x$ of order α on the interval $[a,b]$ if the integral $I^{1-\alpha}_{a+}x$ is absolutely continuous on $[a,b]$ (more precisely, if $I^{1-\alpha}_{a+}x$ has an absolutely continuous representant a.e. on $[a,b]$). By this derivative we mean the classical derivative $D^1(I^{1-\alpha}_{a+}x)$, i.e.

$$(D^{\alpha}_{a+}x)(t) = \frac{1}{\Gamma(1-\alpha)} D^1\left(\int_a^t \frac{x(\tau)}{(t-\tau)^{\alpha}} d\tau \right), \quad t \in [a,b] \text{ a.e.}$$

In a similar way, by the right Riemann-Liouville derivative $D^{\alpha}_{b-}x$ of order $\alpha \in (0,1)$ of the function $x \in L^1$ on the interval $[a,b]$ we mean the function $-D^1(I^{1-\alpha}_{b-}x)$, i.e.

$$D^{\alpha}_{b-}x(t) = -\frac{1}{\Gamma(1-\alpha)} D^1\left(\int_t^b \frac{x(\tau)}{(\tau-t)^{\alpha}} d\tau \right), \quad t \in [a,b] \text{ a.e.}$$

provided that the integral $I^{1-\alpha}_{b-}x$ is absolutely continuous on $[a,b]$.

Properties of the fractional integrals and derivatives can be found in [6].

Let $\alpha \in (0,1)$, $1 \leq p < \infty$. By $AC^{\alpha,p}_{a+} = AC^{\alpha,p}_{a+}([a,b],\mathbb{R}^n)$ we denote the set of all functions $x : [a,b] \to \mathbb{R}^n$ of the form

$$x(t) = \frac{1}{\Gamma(\alpha)} \frac{c}{(t-a)^{1-\alpha}} + I^{\alpha}_{a+}\varphi(t), \quad t \in [a,b] \text{ a.e.},$$

with $c \in \mathbb{R}^n$, $\varphi \in L^p$. One can show ([1]) that $x \in AC^{\alpha,p}_{a+}$ if and only if x possesses the left Riemann-Liouville derivative $D^{\alpha}_{a+}x \in L^p$. In such a case

$$I^{1-\alpha}_{a+}x(a) = c$$

and

$$D^{\alpha}_{a+}x = \varphi.$$

It is easy to show that $AC^{\alpha,2}_{a+}$ with the scalar product

$$\langle x, y\rangle = I^{1-\alpha}_{a+}x(a) I^{1-\alpha}_{a+}y(a) + \int_a^b D^{\alpha}_{a+}x(t) D^{\alpha}_{a+}y(t)dt$$

for x, $y \in AC^{\alpha,2}_{a+}$, is complete.

Similarly, by $AC^{\alpha,p}_{b-} = AC^{\alpha,p}_{b-}([a,b],\mathbb{R}^n)$ we mean the set of all functions $x : [a,b] \to \mathbb{R}^n$ of the form

$$x(t) = \frac{1}{\Gamma(\alpha)} \frac{d}{(b-t)^{1-\alpha}} + I^{\alpha}_{a+}\psi(t), \quad t \in [a,b] \text{ a.e.},$$

with $d \in \mathbb{R}^n$, $\psi \in L^p$. As in the „left" case $x \in AC^{\alpha,p}_{b-}$ if and only if x possesses the right Riemann-Liouville derivative $D^{\alpha}_{b-}x \in L^p$ and, in such a case,

$$I^{1-\alpha}_{b-}x(a) = d,$$

$$D^{\alpha}_{b-}x = \psi.$$

Using the method applied in [3] one can obtain (cf. [5]) the following lemma.

Lemma 2.1. *If $1 \leq p < \infty$, $\frac{p-1}{p} < \alpha < 1$, $c \in \mathbb{R}^n$, $u \in L^p$, then the problem*

$$\begin{cases} D^{\alpha}_{a+}x(t) = Ax(t) + Bu(t), & t \in [a,b] \text{ a.e.}, \\ I^{1-\alpha}_{a+}x(a) = c \end{cases}$$

has a unique solution x_u in $AC^{\alpha,p}_{a+}$. It is given by

$$x_u(t) = \Phi^{A}_{\alpha,\alpha}(t-a)c + \int_a^t \Phi^{A}_{\alpha,\alpha}(t-s)Bu(s)ds, \quad t \in [a,b] \text{ a.e.},$$

where $\Phi^{A}_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma(k\alpha+\beta)}$. In particular, a unique solution x_0 corresponding to control $u(\cdot) \equiv 0$ is given by

$$x_0(t) = \Phi^{A}_{\alpha,\alpha}(t-a)c, \quad t \in [a,b] \text{ a.e.},$$

From the above lemma the following corollary can be deduced.

Corollary 2.2. *If* $1 \leq p < \infty$, $\frac{p-1}{p} < \alpha < 1$, $e \in \mathbb{R}^n$, *then problem*

$$\begin{cases} D^{\alpha}_{b-}y(t) = Ay(t), & t \in [a,b] \text{ a.e.}, \\ I^{1-\alpha}_{b-}y(b) = e \end{cases} \tag{2.2}$$

has a unique solution $y \in AC^{\alpha,p}_{b-}$ *and it is given by*

$$y(t) = \Phi^{A}_{\alpha,\alpha}(b-t)e. \tag{2.3}$$

The above corollary implies what follows.

Corollary 2.3. *If* $1 \leq p < \infty$, $\frac{p-1}{p} < \alpha < 1$, *then the function*

$$[a,b] \ni t \longmapsto \Phi^{A}_{\alpha,\alpha}(b-t) \in \mathbb{R}^{n\times n}$$

belongs to L^p.

Denoting by $I^{\alpha}_{a+}(L^2)$ the range of the operator $I^{\alpha}_{a+} : L^2 \to L^2$ we see that problem (1.1) (with a fixed control $u \in L^2$) in $AC^{\alpha,2}_{a+}$ is equivalent to the problem

$$D^{\alpha}_{a+}x(t) = Ax(t) + Bu(t), \quad t \in [a,b] \text{ a.e.}, \tag{2.4}$$

in $I^{\alpha}_{a+}(L^2)$, i.e. the set of solutions to (1.1) in $AC^{\alpha,2}_{a+}$ is the same as the set of solutions to (2.4) in $I^{\alpha}_{a+}(L^2)$.

Of course, $I^{\alpha}_{a+}(L^2)$ with the scalar product

$$\langle x, y\rangle = \int_a^b D^{\alpha}_{a+}x(t)D^{\alpha}_{a+}y(t)dt$$

is complete.

3. EXISTENCE OF A SOLUTION TO (1.1)–(1.2)

Let us recall that a function $J : U \to \mathbb{R}$ where U is a convex subset of a Hilbert space H, is called strongly convex, if

$$J(\gamma u + (1-\gamma)v) \leq \gamma J(u) + (1-\gamma)J(v) - \gamma(1-\gamma)\kappa \|u - v\|^2$$

for some $\kappa > 0$ and all $\gamma \in [0,1]$, u, $v \in U$. One can show that J is strongly convex on U with a constant κ if and only if function $g(u) = J(u) - \kappa \|u\|^2$ is convex on U. So, functional $H \ni u \longmapsto \|u\|^2 \in \mathbb{R}$ is strongly convex with constant $\kappa = 1$.

We have the following corollary of a continuous dependence result obtained in [4, Theorem 2].

Lemma 3.1. *If* $1 \leq p < \infty$, $0 < \alpha < 1$ *and* $u_j \longrightarrow u_0$ *in* L^p, *then* $I_{a+}^{1-\alpha} x_j \longrightarrow I_{a+}^{1-\alpha} x_0$ *uniformly on* $[a,b]$ (*here* x_j *is the unique solution to problem* (1.1), *corresponding to control* u_j).

Now, let us fix a convex closed set $M \subset \mathbb{R}^m$ and consider problem (1.1)–(1.2) in the set $U = L_M = \{u \in L^2;\ u(t) \in M \text{ for } t \in [a,b] \text{ a.e.}\}$ of controls.

Theorem 3.2. *The set of minimum points of the functional* J *on the set* L_M *is nonempty and consists of one point.*

Proof. The set L_M is convex and closed in L^2. The above lemma implies continuity of the strongly convex functional (1.2). So, from [7, Theorem I.3.8] the assertion follows. □

4. GRADIENT OF THE COST FUNCTIONAL

Let $\alpha \in (\frac{1}{2}, 1)$. To calculate the gradient of the functional J we write equation (2.4) in the form

$$F(x,u) = 0$$

where $F : I_{a+}^{\alpha}(L^2) \times L^2 \to L^2$ is given by

$$F(x,u) = D_{a+}^{\alpha} x(\cdot) - Ax(\cdot) - Bu(\cdot).$$

In a standard way we check that F is of class C^1 and its partial differentials are given by

$$F_x(x,u)h = D_{a+}^{\alpha} h(\cdot) - Ah(\cdot),$$
$$F_u(x,u)v = -Bv(\cdot)$$

for x, $h \in I_{a+}^{\alpha}(L^2)$, u, $v \in L^2$. From Lemmas 2.1, 3.1 and the Banach inverse mapping theorem it follows that $F_x(x,u) : I_{a+}^{\alpha}(L^2) \to L^2$ is a homeomorphism for any $(x,u) \in I_{a+}^{\alpha}(L^2) \times L^2$. So, using the implicit function theorem we assert that the mapping

$$\lambda : L^2 \ni u \longmapsto x_u \in I_{a+}^{\alpha}(L^2)$$

is of class C^1 and its differential $\lambda'(u)$ at a point $u \in L^2$ has the form

$$\begin{aligned} \lambda'(u)v &= -\left([F_x(\lambda(u),u)]^{-1} \circ F_u(\lambda(u),u)\right) v \\ &= [F_x(\lambda(u),u)]^{-1}(Bv) = h_v \end{aligned}$$

for any $v \in L^2$, where $h_v \in I_{a+}^{\alpha}(L^2)$ is a unique solution of system (2.4), corresponding to control v, i.e.

$$D_{a+}^{\alpha} h_v(t) = Ah_v(t) + Bv(t), \quad t \in [a,b] \text{ a.e.} \tag{4.1}$$

Remark 4.1. Differentiability and the form of the differential of λ can be deduced from the fact that $F_x(x,u)$ is a homeomorphism. We present the method based on the local implicit function theorem, because it is more general and can also be used in a nonlinear case.

Since J is the following superposition

$$J : L^2 \ni u \longmapsto \lambda(u) \longmapsto I_{a+}^{1-\alpha}\lambda(u)(b) - c \longmapsto \frac{1}{2}\left|I_{a+}^{1-\alpha}\lambda(u)(b) - c\right|^2 \in \mathbb{R},$$

its differential $J'(u) : L^2 \to \mathbb{R}$ at a point $u \in L^2$ is given by

$$J'(u)v = \left(I_{a+}^{1-\alpha}\lambda(u)(b) - c\right) I_{a+}^{1-\alpha}h_v(b) = \left(I_{a+}^{1-\alpha}x_u(b) - c\right) I_{a+}^{1-\alpha}h_v(b)$$

for $v \in L^2$.

Now, let us observe that if $\alpha \in (\frac{1}{2}, 1)$, then

$$(I_{a+}^{1-\alpha}x_u(b) - c)I_{a+}^{1-\alpha}h_v(b) = \int_a^b \frac{d}{dt}\left(I_{b-}^{1-\alpha}g_u(t)I_{a+}^{1-\alpha}h_v(t)\right)dt$$

where $g_u \in AC_{b-}^{\alpha,2}$ is a unique solution to problem

$$\begin{cases} D_{b-}^{\alpha}y(t) = A^T y(s), \quad t \in [a,b] \text{ a.e.}, \\ I_{b-}^{1-\alpha}y(b) = I_{a+}^{1-\alpha}x_u(b) - c. \end{cases} \tag{4.2}$$

But

$$\begin{aligned}
&\int_a^b \frac{d}{dt}\left(I_{b-}^{1-\alpha}g_u(t)I_{a+}^{1-\alpha}h_v(t)\right)dt \\
&= \int_a^b -D_{b-}^{\alpha}g_u(t)I_{a+}^{1-\alpha}h_v(t)dt + \int_a^b I_{b-}^{1-\alpha}g_u(t)D_{a+}^{\alpha}h_v(t)dt \\
&= \int_a^b -A^T g_u(t)I_{a+}^{1-\alpha}h_v(t)dt + \int_a^b I_{b-}^{1-\alpha}g_u(t)Ah_v(t)dt + \int_a^b I_{b-}^{1-\alpha}g_u(t)Bv(t)dt \\
&= -\int_a^b g_u(t)I_{a+}^{1-\alpha}(Ah_v)(t)dt + \int_a^b g_u(t)I_{a+}^{1-\alpha}(Ah_v)(t)dt + \int_a^b I_{b-}^{1-\alpha}g_u(t)Bv(t)dt \\
&= \int_a^b B^T I_{b-}^{1-\alpha}g_u(t)v(t)dt.
\end{aligned}$$

So,

$$J'(u)v = \int_a^b B^T I_{b-}^{1-\alpha}g_u(t)v(t)dt \tag{4.3}$$

for $v \in L^2$. This means that the gradient $\nabla J(u) \in L^2$ of the functional J at a point $u \in L^2$ is given by

$$\nabla J(u) = B^T I_{b-}^{1-\alpha}g_u$$

where g_u is a solution to (4.2), i.e.

$$g_u(t) = \Phi_{\alpha,\alpha}^{A^T}(b-t)(I_{a+}^{1-\alpha}x_u(b)-c) \tag{4.4}$$

for $t \in [a,b]$ a.e.

Now, we shall show that the gradient ∇J is Lipschitzian. In the proof of this fact we shall use the following

Lemma 4.2. *The function*

$$[a,b] \ni t \longmapsto \left(\int_a^t \left| \Phi_{\alpha,\alpha}^{A^T}(t-s) \right|^2 ds \right)^{\frac{1}{2}} \in \mathbb{R}_0^+ \tag{4.5}$$

belongs to L^2.

Proof. From Corollary 2.3 it follows that the function $\Phi_{\alpha,\alpha}^{A^T}(t-\cdot) : (a,t) \to \mathbb{R}^{n\times n}$ belongs to $L^2([a,t],\mathbb{R}^n)$ for any $t \in (a,b)$. So, the function under consideration is well defined. The mentioned corollary also implies the summability of the function

$$\varkappa : (0, b-a) \ni t \longmapsto \left| \Phi_{\alpha,\alpha}^{A^T}(t) \right|^2 \in \mathbb{R}_0^+.$$

Let us consider the function

$$\delta : P_\nabla \ni (t,s) \longmapsto \varkappa(b-s) \in \mathbb{R}_0^+$$

where

$$P_\nabla = \{(t,s) \in (a,b) \times (a,b);\ a+b-t < s\}.$$

Of course, δ being the restriction to the set P_∇ of a summable function

$$(a,b) \times (a,b) \ni (t,s) \longmapsto \varkappa(b-s) \in \mathbb{R}_0^+,$$

is summable. Now, let us consider a function

$$\mu : P_\Delta \ni (t,s) \longmapsto \varkappa(t-s) \in \mathbb{R}_0^+,$$

where

$$P_\Delta = \{(t,s) \in (a,b) \times (a,b);\ s < t\}.$$

This function is measurable because its superposition $\mu \circ F$ with a diffeomorphism F given below is equal to the measurable function δ. Moreover,

$$\begin{aligned} \int_{P_\Delta} \varkappa(t-s) ds dt &= \int_{P_\Delta} \mu(t,s) ds dt \\ &= \int_{P_\nabla} (\mu \circ F)(t,s) \left| \det F'(t,s) \right| ds dt = \int_{P_\nabla} \delta(t,s) \cdot 1 ds dt \in \mathbb{R}_0^+ \end{aligned}$$

where $F : P_\nabla \ni (t,s) \longmapsto (t, s-(b-t)) \in P_\Delta$ is a diffeomorphism with

$$F'(t,s) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

So, the function

$$P_\Delta \ni (t,s) \longmapsto \left|\Phi^{A^T}_{\alpha,\alpha}(t-s)\right|^2 \in \mathbb{R}^+_0$$

is summable. Consequently, the Fubini theorem implies summability of the function (4.5). □

Now, we can prove the following result.

Theorem 4.3. *If* $\alpha \in (\frac{1}{2}, 1)$*, then*

$$\|\nabla J(u) - \nabla J(w)\|_{L^2} \le L \|u - w\|_{L^2}$$

for u*,* $w \in L^2$*, where*

$$L = \left(2|A|^2 \left(\int_a^b \left(\int_a^t \left|\Phi^{A}_{\alpha,\alpha}(t-s)\right|^2 ds\right)^{\frac{1}{2}} dt\right)^2 + 2|B|^2 (b-a)\right)^{\frac{1}{2}} |B|^2 \left\| I^{1-\alpha}_{b-} \left|\Phi^{A}_{\alpha,\alpha}(b-\cdot)\right| \right\|_{L^2}. \tag{4.6}$$

Proof. Let us fix u, $w \in L^2$. We have

$$\|\nabla J(u) - \nabla J(w)\|^2_{L^2} \le |B|^2 \int_a^b \left|I^{1-\alpha}_{b-} g_u(t) - I^{1-\alpha}_{b-} g_w(t)\right|^2 dt.$$

But

$$\begin{aligned} \left|I^{1-\alpha}_{b-} g_u(t) - I^{1-\alpha}_{b-} g_w(t)\right| &\le I^{1-\alpha}_{b-} |g_u - g_w|(t) \\ &= I^{1-\alpha}_{b-} \left|\Phi^{A^T}_{\alpha,\alpha}(b-\cdot)(I^{1-\alpha}_{a+} x_u(b) - I^{1-\alpha}_{a+} x_w(b))\right|(t) \\ &\le \left|I^{1-\alpha}_{a+} x_u(b) - I^{1-\alpha}_{a+} x_w(b)\right| I^{1-\alpha}_{b-} \left|\Phi^{A^T}_{\alpha,\alpha}(b-\cdot)\right|(t). \end{aligned}$$

From Corollary 2.3 it follows that the function $\Phi^{A^T}_{\alpha,\alpha}(b-\cdot) : [a,b] \to \mathbb{R}^{n\times n}$ belongs to L^2. Consequently, $I^{1-\alpha}_{b-} \left|\Phi^{A}_{\alpha,\alpha}(b-\cdot)\right| \in L^2$. Thus

$$\begin{aligned} &\|\nabla J(u) - \nabla J(w)\|^2_{L^2} \\ &\le |B|^2 \int_a^b (I^{1-\alpha}_{b-} \left|\Phi^{A^T}_{\alpha,\alpha}(b-\cdot)\right|(t))^2 dt \left|I^{1-\alpha}_{a+} x_u(b) - I^{1-\alpha}_{a+} x_w(b)\right|^2 \\ &= L_1 \left|I^{1-\alpha}_{a+} x_u(b) - I^{1-\alpha}_{a+} x_w(b)\right|^2, \end{aligned}$$

where

$$L_1 = |B|^2 \int_a^b (I_{b-}^{1-\alpha} \left|\Phi_{\alpha,\alpha}^{A^T}(b-\cdot)\right|(t))^2 dt.$$

Now, let us observe that

$$\begin{aligned}
&\left|I_{a+}^{1-\alpha}x_u(b) - I_{a+}^{1-\alpha}x_w(b)\right|^2 \\
&= \left|\int_a^b \frac{d}{dt} I_{a+}^{1-\alpha}x_u(t)dt - \int_a^b \frac{d}{dt} I_{a+}^{1-\alpha}x_w(t)dt\right|^2 \\
&= \left|\int_a^b (Ax_u(t) + Bu(t))dt - \int_a^b (Ax_w(t) + Bw(t))dt\right|^2 \\
&\leq \left(|A| \int_a^b |x_u(t) - x_w(t)|\, dt + |B| \int_a^b |u(t) - w(t)|\, dt\right)^2 \\
&\leq 2\,|A|^2 \left(\int_a^b |x_u(t) - x_w(t)|\, dt\right)^2 + 2\,|B|^2 \left(\int_a^b |u(t) - w(t)|\, dt\right)^2 \\
&\leq 2|A|^2 \left(\int_a^b \left|\int_a^t \Phi_{\alpha,\alpha}^A(t-s)B(u(s) - w(s))ds\right| dt\right)^2 + 2\,|B|^2\,(b-a)\,\|u-w\|_{L^2}^2 \\
&\leq 2|A|^2|B|^2 \left(\int_a^b \left(\int_a^t \left|\Phi_{\alpha,\alpha}^A(t-s)\right|^2 ds\right)^{\frac{1}{2}} \left(\int_a^t |u(s) - w(s)|^2 ds\right)^{\frac{1}{2}} dt\right)^2 \\
&\quad + 2|B|^2(b-a)\,\|u-w\|_{L^2}^2 \\
&\leq \left(2|A|^2|B|^2 \left(\int_a^b \left(\int_a^t \left|\Phi_{\alpha,\alpha}^A(t-s)\right|^2 ds\right)^{\frac{1}{2}} dt\right)^2 + 2\,|B|^2\,(b-a)\right) \|u-w\|_{L^2}^2 .
\end{aligned}$$

From Lemma 4.2 it follows that the integral $\int_a^b (\int_a^t \left|\Phi_{\alpha,\alpha}^A(t-s)\right|^2 ds)^{\frac{1}{2}} dt$ is finite. □

5. MAXIMUM PRINCIPLE

Let us start with the following classical result ([7, Theorem I.2.5]).

Lemma 5.1. *Let U be a convex subset of a Banach space X, J – a functional of class C^1 on U. If u_* is a minimum point of J on U, then*

$$J'(u_*)u \geq J'(u_*)u_* \tag{5.1}$$

for any $u \in U$. If, additionally, J is convex on U, then condition (5.1) *is sufficient for u_* to be the minimum point of J on U.*

Now, let us consider problem (1.1)–(1.2) in the set $U := L_M$ with a convex set $M \subset \mathbb{R}^m$, in the case of $\alpha \in (\frac{1}{2}, 1)$.

Theorem 5.2. *Control u_* is a solution to problem* (1.1)–(1.2) *in the set L_M if and only if*

$$\begin{aligned}&\min_{u\in M}(B^T I_{b-}^{1-\alpha}(\Phi_{\alpha,\alpha}^{A^T}(b-\cdot))(t)(I_{a+}^{1-\alpha}x_{u_*}(b)-c))u \\ &= (B^T I_{b-}^{1-\alpha}(\Phi_{\alpha,\alpha}^{A^T}(b-\cdot))(t)(I_{a+}^{1-\alpha}x_{u_*}(b)-c))u_*(t)\end{aligned} \tag{5.2}$$

for $t \in [a,b]$ a.e.

Proof. Let u_* be a solution to problem (1.1)–(1.2) in the set L_M. From Lemma 5.1, formula (4.3) and the convexity of functional (1.2) it follows that optimality of u_* is equivalent to the functional condition of the form

$$\min_{u(\cdot)\in L_M}\int_a^b B^T I_{b-}^{1-\alpha}g_{u_*}(t)u(t)dt = \int_a^b B^T I_{b-}^{1-\alpha}g_{u_*}(t)u_*(t)dt$$

where $g_{u_*} \in AC_{b-}^{\alpha,2}$ is a unique solution to problem

$$\begin{cases} D_{b-}^{\alpha}y(t) = A^T y(s), \quad t\in[a,b] \text{ a.e.},\\ I_{b-}^{1-\alpha}y(b) = I_{a+}^{1-\alpha}x_{u_*}(b)-c\end{cases}$$

given by (4.4). Lemma 6 of [2] gives the pointwise minimum condition

$$\min_{u\in M} B^T I_{b-}^{1-\alpha}g_{u_*}(t)u = B^T I_{b-}^{1-\alpha}g_{u_*}(t)u_*(t)$$

for $t \in [a,b]$ a.e. Formula (4.4) implies (5.2). □

6. GRADIENT METHOD

Let us recall the following classical result concerning the convergence of the gradient method for a functional J defined on a real Hilbert space H ([7, Theorem I.4.1]).

Lemma 6.1. *Let a functional $J : H \to \mathbb{R}$ be of class C^1, bounded below and with the gradient satisfying the Lipschitz condition. If $(u_k) \subset H$ is a sequence described by the formula*

$$u_{k+1} = u_k - \beta_k \nabla J(u_k), \quad k = 0,1,\ldots, \tag{6.1}$$

with any fixed $u_0 \in H$, where parameter β_k is a minimum point of the functional

$$f_k : [0,\infty) \ni \beta \longmapsto J(u_k - \beta\nabla J(u_k)) \in \mathbb{R} \tag{6.2}$$

for $k = 0, 1, \ldots$, then the sequence $(J(u_k))$ is nonincreasing and

$$\lim_{k\to\infty} \|\nabla J(u_k)\| = 0. \tag{6.3}$$

If, additionally, J is strongly convex (with a constant $\kappa > 0$), then (u_k) converges with respect to the norm to u_* - a unique minimum point of J and

$$0 \le J(u_k) - J_* \le (J(u_0) - J_*)q^k, \tag{6.4}$$

$$\|u_k - u_*\|^2 \le \frac{1}{\kappa}(J(u_0) - J_*)q^k, \tag{6.5}$$

for $k = 0, 1, \ldots$, where $J_* = \inf_{u\in H} J(u)$, $q = 1 - \frac{2\kappa}{L} \in [0, 1)$ ($L > 0$ is a Lipschitz constant for ∇J).

Remark 6.2. One shows that

$$(\nabla J(u) - \nabla J(v))(u - v) \ge 2\kappa \|u - v\|^2$$

for u, $v \in U$ (cf. [7, Theorem I.2.2] and [8, Theorem IV.3.3] for the method of the proof). Consequently, $2\kappa \le L$ because $(\nabla J(u) - \nabla J(v))(u - v) \le L \|u - v\|^2$.

Now, let us consider problem (1.1)–(1.2) in the set $U = L^2$. Using the above lemma and strong convexity of J we obtain the following theorem.

Theorem 6.3. *If $\alpha \in (\frac{1}{2}, 1)$, then there exists a unique minimum point u_* of J on L^2, the sequence (u_k) given by (6.1) converges to u_* with respect to the norm and conditions (6.3), (6.4), (6.5) are satisfied (with L given by (4.6)).*

Remark 6.4. In an analogous way as in [7] we check that if

$$I_{a+}^{1-\alpha} x_{u_k - \nabla J(u_k)}(b) - I_{a+}^{1-\alpha} x_{u_k}(b) = 0$$

for some k, then $\nabla J(u_k) = 0$. This means (cf. Lemma 5.1 and Theorem 3.2) that u_k is a unique minimum point of J. If

$$I_{a+}^{1-\alpha} x_{u_k - \nabla J(u_k)}(b) - I_{a+}^{1-\alpha} x_{u_k}(b) \ne 0$$

then

$$\beta_k = \frac{1}{2} \frac{\int_a^b \left| B^T I_{b-}^{1-\alpha} g_{u_k}(t) \right|^2 dt}{\left| I_{a+}^{1-\alpha} x_{u_k - \nabla J(u_k)}(b) - I_{a+}^{1-\alpha} x_{u_k}(b) \right|^2}$$

is the unique minimum point of f_k (g_{u_k} is given by (4.4)).

7. PROJECTION OF THE GRADIENT METHOD

Let H be a real Hilbert space. By $P_U(u)$ we denote the projection of a point $u \in H$ on a convex closed set $U \subset H$. We shall use the following result on the convergence of the projection of the gradient method ([7, Theorem I.4.4]).

Lemma 7.1. *Let U be a convex closed subset of H and $J : U \to \mathbb{R}$ - a functional of class C^1, bounded below and with the gradient satisfying the Lipschitz condition (with a constant L). If $(u_k) \subset H$ is a sequence described by the formula*

$$u_{k+1} = P_U(u_k - \beta_k \nabla J(u_k)), \quad k = 0, 1, \ldots, \tag{7.1}$$

with any fixed $u_0 \in H$, where parameter β_k, $k = 0, 1, \ldots$, is such that

$$\varepsilon_0 \le \beta_k \le \frac{2}{L + 2\varepsilon} \tag{7.2}$$

(here ε_0, ε are fixed positive parameters such that $\varepsilon_0 < \frac{2}{L+2\varepsilon}$), then the sequence $(J(u_k))$ is nonincreasing and

$$\lim_{k\to\infty} \|u_k - u_{k+1}\| = 0. \tag{7.3}$$

If, additionally, J is strongly convex on U, then (u_k) converges with respect to the norm to u_* - a unique minimum point of J on U and there exists a constant $c \ge 0$ such that

$$\|u_k - u_*\|^2 \le \frac{c}{k}, \tag{7.4}$$

for $k = 1, 2, \ldots$.

Remark 7.2. The constant c can be calculated (cf. [7, Theorem I.4.4] and [8, Theorem V.2.2] for the details).

Now, let us consider problem (1.1)–(1.2) in a set $U = L_M$ where M is a convex closed subset of $\mathbb{R}^m$. From the above lemma we obtain the following theorem.

Theorem 7.3. *If $\alpha \in (\frac{1}{2}, 1)$ and ε_0, $\varepsilon > 0$ are such that $\varepsilon_0 < \frac{2}{L+2\varepsilon}$, then there exists a unique minimum point u_* of J on L_M, the sequence (u_k) given by (7.1)–(7.2) converges to u_* with respect to the norm and condition (7.4) is satisfied.*

Remark 7.4. Form the projection P_{L_M} in the case of the set

$$M = [n_1, N_1] \times \ldots \times [n_m, N_m] \subset \mathbb{R}^m$$

can be found in [7, Part I.4].

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