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EXISTENCE OF THREE SOLUTIONS FOR PERTURBED NONLINEAR DIFFERENCE EQUATIONS

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Abstract. Using critical point theory, we study the existence of at least three solutions for perturbed nonlinear difference equations with discrete boundary-value condition depending on two positive parameters.

Keywords: nonlinear difference equations, discrete boundary value problem, three solutions, critical point theory, variational methods.

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1. INTRODUCTION

In this paper we study the following discrete boundary-value problem

$$\begin{cases} -\Delta(\phi_p(\Delta u(k-1))) + q_k\phi_p(u(k)) = \lambda f(k,u(k)) + \mu g(k,u(k)), & k \in [1,T], \\ u(0) = u(T+1) = 0, \end{cases} \tag{1.1}$$

where T is a fixed positive integer, $[1,T]$ is the discrete interval $\{1,\ldots,T\}$, $f,g : [1,T]\times\mathbb{R}\to\mathbb{R}$ are two continuous functions, $\lambda > 0$, $\mu \geq 0$ are two parameters, $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator and $q_k \in \mathbb{R}_0^+$ for all $k \in [0,T]$ and $\phi_p(s) = |s|^{p-2}s$, $1 < p < +\infty$.

The theory of nonlinear difference equations has been widely used to study discrete models in many fields such as computer science, economics, neural network, ecology, cybernetics, etc. In recent years, a great deal of work has been done in the study of the existence and multiplicity of solutions for discrete boundary value problems, by using classical methods such as fixed point theorems lower and upper solution methods, critical point theory, variational methods, Morse theory and the mountain-pass theorem. For background and recent results, we refer the reader to

[1–4, 6, 8–12, 14, 15, 17, 18, 20, 25–29] and the references therein. For instance, Candito and Giovannelli in [11], employing three critical point theorem established the existence of at least three solutions for the following problem

$$\begin{cases} -\Delta(\phi_p(\Delta u(k-1))) = \lambda f(k, u(k)), & k \in [1, T], \\ u(0) = u(T+1) = 0, \end{cases} \tag{1.2}$$

where T is a fixed positive integer, $[1, T]$ is the discrete interval $\{1, \dots, T\}$, $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\lambda > 0$ and $1 < p < +\infty$. Bonanno and Candito in [6], based on critical point theorems in the setting of finite dimensional Banach spaces, studied the multiplicity of solutions for the nonlinear difference equations (1.2), while the same authors in [4], using critical point theory investigated the existence of infinitely many solutions for the discrete non-linear Dirichlet problem (1.1) when $\mu = 0$, under appropriate oscillating behaviours of the non-linear term. In [9], based on three critical points theorems, the authors investigated different sets of assumptions which guarantee the existence and multiplicity of solutions for a nonlinear Neumann boundary value problem, while in [10] using critical point theory, they also studied the existence of at least three solutions for a perturbed nonlinear Dirichlet boundary value problem for difference equations depending on two positive parameters.

In the present paper, motivated by the above papers, using two kinds of three critical point theorems obtained by Bonanno and Candito in [5], and Bonanno and Marano in [7] (see Theorems 2.1 and 2.2 below) we are interested to ensure the existence of at least three solutions for the problem (1.1); see Theorems 3.1 and 3.2. We point out that in Theorems 3.1 and 3.2, precise estimates of parameters λ and μ are given.

A special case of Theorem 3.1 is the following theorem in which we have no symmetric assumption on f.

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function. Put $F(t) := \int_0^t f(\xi)d\xi$ for each $t \in \mathbb{R}$. Assume that*

$$\liminf_{\xi \to 0} \frac{F(\xi)}{\xi^p} = \limsup_{\xi \to +\infty} \frac{F(\xi)}{\xi^p} = 0.$$

Then, for each continuous function $g : [1, T] \times \mathbb{R} \to \mathbb{R}$ satisfying the asymptotical condition

$$\limsup_{|\xi| \to +\infty} \frac{\sum_{k=1}^{T} G(k, \xi)}{|\xi|^p} < +\infty,$$

there exists $\delta > 0$ such that, for each $\mu \in [0, \delta)$, the problem

$$\begin{cases} -\Delta(\phi_p(\Delta u(k-1))) = f(u(k)) + \mu g(k, u(k)), & k \in [1, T], \\ u(0) = u(T+1) = 0, \end{cases}$$

admits at least three solutions.

It is worth to mention that Galewski and Głąb in [13], using critical point theory, studied the following anisotropic (unperturbed) problem

$$\begin{cases} -\Delta(|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)) = \lambda f_k(u(k)), & k \in [1,T], \\ u(0) = u(T+1) = 0. \end{cases} \tag{1.3}$$

In fact, firstly they applied the direct method of the calculus of variations and the mountain pass technique in order to reach the existence of at least one non-trivial solution. Secondly they derived some version of a discrete three critical point theorem which they applied in order to get the existence of at least two non-trivial solutions. In particular, Molica Bisci and Repovs in [22], using related variational arguments employed in the present paper, studied the problem (1.3), and substantially improved the results obtained by Bonanno and Candito in [4].

For a through review of the subject, we also refer the reader to [19, 21, 23].

2. PRELIMINARIES

Our main tools are the following three critical point theorems. In the first one the coercivity of the functional $\Phi - \lambda\Psi$ is required, in the second one a suitable sign hypothesis is assumed.

Theorem 2.1 ([7, Theorem 2.6])**.** *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $\Phi(0) = \Psi(0) = 0$. Assume that there exists $r > 0$ and $\overline{v} \in X$, with $r < \Phi(\overline{v})$ such that*

(a_1) $\frac{\sup_{\Phi(u)\le r}\Psi(u)}{r} < \frac{\Psi(\overline{v})}{\Phi(\overline{v})}$,

(a_2) *for each* $\lambda \in \Lambda_r := \left(\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \frac{r}{\sup_{\Phi(u)\le r}\Psi(u)}\right)$ *the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Theorem 2.2 ([5, Theorem 3.3])**.** *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose derivative is compact, such that:*

1. $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$,
2. *for each $\lambda > 0$ and for every $u_1, u_2 \in X$ which are local minima for the functional $\Phi - \lambda\Psi$ and such that $\Psi(u_1) \ge 0$ and $\Psi(u_2) \ge 0$, one has*

$$\inf_{s\in[0,1]} \Psi(su_1 + (1-s)u_2) \ge 0.$$

Assume that there are two positive constants r_1, r_2 and $\overline{v} \in X$, with $2r_1 < \Phi(\overline{v}) < \frac{r_2}{2}$, such that:

(b_1) $\frac{\sup_{u\in\Phi^{-1}((-\infty,r_1))}\Psi(u)}{r_1} < \frac{2}{3}\frac{\Psi(\overline{v})}{\Phi(\overline{v})}$,

(b_2) $\frac{\sup_{u\in\Phi^{-1}((-\infty,r_2))}\Psi(u)}{r_2} < \frac{1}{3}\frac{\Psi(\overline{v})}{\Phi(\overline{v})}$.

Then, for each

$$\lambda \in \left(\frac{3}{2}\frac{\Phi(\overline{v})}{\Psi(\overline{v})},\ \min\left\{\frac{r_1}{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)},\ \frac{\frac{r_2}{2}}{\sup_{u\in\Phi^{-1}((-\infty,r_2))}\Psi(u)}\right\}\right),$$

the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which lie in $\Phi^{-1}((-\infty, r_2))$.

For an overview on three critical point theorems we refer to [24].

For the reader's convenience we state the following consequence of the strong comparison principle [3, Lemma 2.3] (see also [4, Theorem 2.2] which we will use in the sequel in order to obtain positive solutions to the problem (1.1), i.e. $u(k) > 0$ for each $k \in [1, T]$.

Lemma 2.3. *Let*

$$\begin{gathered} -\Delta(\phi_p(\Delta u(k-1))) + q_k\phi_p(u(k)) \geq 0, \quad k \in [1,T], \\ u(0) \geq 0,\ u(k+1) \geq 0. \end{gathered}$$

Then either u is positive or $u \equiv 0$.

In order to give the variational formulation of the problem (1.1), on a T-dimensional Banach space

$$W := \{u : [0, T+1] \to \mathbb{R} : u(0) = u(T+1) = 0\},$$

equipped with the norm

$$\|u\| := \left\{\sum_{k=1}^{T+1} |\Delta u(k-1)|^p + q_k|u(k)|^p\right\}^{1/p},$$

we set

$$\Phi(u) := \frac{\|u\|^p}{p} \quad \text{and} \quad \Psi(u) := \sum_{k=1}^{T}\left(F(k,u(k)) + \frac{\mu}{\lambda}G(k,u(k))\right) \tag{2.1}$$

for every $u \in W$, where $F(k,t) := \int_0^t f(k,\xi)d\xi$ and $G(k,t) := \int_0^t g(k,\xi)d\xi$ for every $(k,t) \in [1,T] \times \mathbb{R}$. An easy computation ensures that Φ and Ψ turn out to be of class C^1 on W with

$$\begin{aligned} \Phi'(u)(v) &= \sum_{k=1}^{T+1}\left[\phi_p(\Delta u(k-1))\Delta v(k-1) + q_k|u(k)|^{p-2}u(k)v(k)\right] \\ &= -\sum_{k=1}^{T}\left[\Delta(\phi_p(\Delta u(k-1))v(k) - q_k|u(k)|^{p-2}u(k)v(k)\right] \end{aligned}$$

and

$$\Psi'(u)(v) = \sum_{k=1}^{T} \left[f(k, u(k)) + \frac{\mu}{\lambda} g(k, u(k))) \right] v(k)$$

for all $u, v \in W$. It is clear that the critical points of $\Phi - \lambda\Psi$ are exactly the solutions of the problem (1.1).

In the sequel, we will use the following inequality

$$\max_{k\in[1,T]} |u(k)| \le \frac{(T+1)^{(p-1)/p}}{2} \|u\|, \tag{2.2}$$

for every $u \in W$. It immediately follows, for instance, from Lemma 2.2 of [16].

3. MAIN RESULTS

For our convenience, set

$$G^c := \sum_{k=1}^{T} \max_{|\xi|\le c} G(k,\xi) \quad \text{for all } c > 0 \quad \text{and} \quad G_d := \sum_{k=1}^{T} G(k,d) \quad \text{for all } d > 0.$$

In order to introduce our first result, fix $c, d > 0$ such that

$$\frac{(2 + \sum_{k=1}^{T} q_k) d^p}{\sum_{k=1}^{T} F(k,d)} < \frac{(2c)^p}{(T+1)^{p-1} \sum_{k=1}^{T} \max_{|\xi|\le c} F(k,\xi)}$$

and pick

$$\lambda \in \Lambda := \left(\frac{(2 + \sum_{k=1}^{T} q_k) d^p}{p \sum_{k=1}^{T} F(k,d)}, \; \frac{(2c)^p}{p(T+1)^{p-1} \sum_{k=1}^{T} \max_{|\xi|\le c} F(k,\xi)} \right).$$

Moreover, put

$$\delta_{\lambda,g} := \min \left\{ \frac{(2c)^p - \lambda p (T+1)^{p-1} \sum_{k=1}^{T} \max_{|\xi|\le c} F(k,\xi)}{p(T+1)^{p-1} G^c}, \right.$$
$$\left. \left| \frac{(2 + \sum_{k=1}^{T} q_k) d^p - \lambda p \sum_{k=1}^{T} F(k,d)}{p \min\{0, G_d\}} \right| \right\} \tag{3.1}$$

and

$$\overline{\delta}_{\lambda,g} := \min \left\{ \delta_{\lambda,g}, \frac{1}{\max\left\{0, \frac{p(T+1)^{p-1}}{2^{p-1}} \limsup_{|\xi|\to+\infty} \frac{\sum_{k=1}^{T} G(k,\xi)}{|\xi|^p} \right\}} \right\}, \tag{3.2}$$

where we read $\frac{1}{0} := +\infty$ whenever this case occurs.

Theorem 3.1. *Assume that there exist two positive constants c and d with*

$$2c < d\Big(2+\sum_{k=1}^{T} q_k\Big)^{\frac{1}{p}}(T+1)^{\frac{p-1}{p}}$$

such that:

(A1) $\frac{\sum_{k=1}^{T}\max_{|\xi|\leq c}F(k,\xi)}{c^p} < \frac{2^p}{(T+1)^{p-1}(2+\sum_{k=1}^{T}q_k)}\frac{\sum_{k=1}^{T}F(k,d)}{d^p}$,

(A2) $\limsup_{|\xi|\to+\infty}\frac{\sum_{k=1}^{T}F(k,\xi)}{|\xi|^p} < \frac{\sum_{k=1}^{T}\max_{|\xi|\leq c}F(k,\xi)}{2c^p}$.

Then, for any $\lambda \in \Lambda$ and for every continuous function $g : [1,T]\times\mathbb{R}\to\mathbb{R}$ such that

(A2) $\limsup_{|\xi|\to+\infty}\frac{\sum_{k=1}^{T}G(k,\xi)}{|\xi|^p} < +\infty$,

there exits $\overline{\delta}_{\lambda,g} > 0$ given by (3.2) *such that, for each $\mu \in (0,\overline{\delta}_{\lambda,g})$, the problem* (1.1) *has at least three solutions.*

Proof. Our aim is to apply Theorem 2.1 to our problem. To this end, take $X = W$, Φ and Ψ as given in (2.1). Put $r = \frac{(2c)^p}{p(T+1)^{p-1}}$ and

$$\bar{v}(k) = \begin{cases} d, & k\in[1,T],\\ 0, & \text{otherwise.}\end{cases} \tag{3.3}$$

Clearly $\bar{v}\in W$ and $\Phi(\bar{v}) = \frac{d^p}{p}(2+\sum_{k=1}^{T}q_k)$. Since $2c < d(2+\sum_{k=1}^{T}q_k)^{\frac{1}{p}}(T+1)^{\frac{p-1}{p}}$, we get $r < \Phi(\bar{v})$. Taking (2.2) into account, we obtain

$$\begin{aligned}\frac{\sup_{u\in\Phi^{-1}((-\infty,r])}\Psi(u)}{r} &= \frac{\sup_{\|u\|\leq(pr)^{\frac{1}{p}}}\sum_{k=1}^{T}\big[F(k,u(k))+\frac{\mu}{\lambda}G(k,u(k))\big]}{\frac{(2c)^p}{p(T+1)^{p-1}}}\\ &\leq \frac{\sum_{k=1}^{T}\max_{|\xi|\leq c}\big[F(k,\xi)+\frac{\mu}{\lambda}G(k,\xi)\big]}{\frac{(2c)^p}{p(T+1)^{p-1}}}\\ &= \frac{\sum_{k=1}^{T}\max_{|\xi|\leq c}F(k,\xi)}{\frac{(2c)^p}{p(T+1)^{p-1}}} + \frac{\mu}{\lambda}\frac{G^c}{\frac{(2c)^p}{p(T+1)^{p-1}}}.\end{aligned}$$

From this, if $G^c = 0$, clearly we get

$$\frac{\sup_{u\in\Phi^{-1}((-\infty,r])}\Psi(u)}{r} < \frac{1}{\lambda}, \tag{3.4}$$

while, if $G^c > 0$, it turns out to be true bearing in mind that $\mu < \delta_{\lambda,g}$. Moreover, one has

$$\frac{\Psi(\bar{v})}{\Phi(\bar{v})} = \frac{\sum_{k=1}^{T}\big[F(k,d)+\frac{\mu}{\lambda}G(k,d)\big]}{\frac{d^p}{p}(2+\sum_{k=1}^{T}q_k)}.$$

Hence, if $G_d \geq 0$, it follows

$$\frac{\Psi(\bar{v})}{\Phi(\bar{v})} > \frac{1}{\lambda}, \tag{3.5}$$

and if $G_d < 0$, $\frac{\Psi(\bar{v})}{\Phi(\bar{v})} > \frac{1}{\lambda}$ again holds since $\mu < \delta_{\lambda,g}$. Thus, from (3.4) and (3.5), (a_1) of Theorem 2.1 follows. Now, we prove the coercivity of the functional $\Phi - \lambda\Psi$. First, we assume that

$$\limsup_{|\xi|\to+\infty} \frac{\sum_{k=1}^{T} F(k,\xi)}{|\xi|^p} > 0.$$

Therefore, fix

$$\limsup_{|\xi|\to+\infty} \frac{\sum_{k=1}^{T} F(k,\xi)}{|\xi|^p} < \varepsilon < \frac{\sum_{k=1}^{T} \max_{|\xi|\le c} F(k,\xi)}{2c^p},$$

from (A2) there is a positive constant h_ε such that

$$\sum_{k=1}^{T} F(k,\xi) \le \varepsilon|\xi|^p + h_\varepsilon \quad \text{for each} \quad \xi \in \mathbb{R}.$$

Taking (2.2) into account and since $\lambda < \frac{(2c)^p}{p(T+1)^{p-1}\sum_{k=1}^{T}\max_{|\xi|\le c} F(k,\xi)}$, it follows that

$$\begin{aligned}\lambda\sum_{k=1}^{T} F(k,u(k)) \le \lambda\varepsilon|u(k)|^p + \lambda h_\varepsilon \le \varepsilon &\frac{c^p}{p\sum_{k=1}^{T}\max_{|\xi|\le c} F(k,\xi)}\|u\|^p \\ &+ h_\varepsilon \frac{(2c)^p}{p(T+1)^{p-1}\sum_{k=1}^{T}\max_{|\xi|\le c} F(k,\xi)}\end{aligned} \tag{3.6}$$

for each $u \in X$. Moreover, taking $\mu < \overline{\delta}_{\lambda,g}$ into account, it follows that

$$\limsup_{|\xi|\to+\infty} \frac{\sum_{k=1}^{T} G(k,\xi)}{|\xi|^p} < \frac{2^{p-1}}{\mu p(T+1)^{p-1}},$$

then, for some constant $\tau_\mu > 0$ and for every $\xi \in \mathbb{R}$, one has

$$\sum_{k=1}^{T} G(k,\xi) \le \frac{2^{p-1}}{\mu p(T+1)^{p-1}}|\xi|^p + \tau_\mu.$$

Hence, by using again (2.2) for each $u \in X$, we get

$$\sum_{k=1}^{T} G(k,u(k)) \le \frac{2^{p-1}}{\mu p(T+1)^{p-1}}|u(k)|^p + \tau_\mu \le \frac{\|u\|^p}{2p\mu} + \tau_\mu. \tag{3.7}$$

Therefore, from (3.6) and (3.7) we have

$$\begin{aligned}\Phi(u) - \lambda\Psi(u) \ge \frac{1}{p}\left(\frac{1}{2} - \varepsilon\frac{c^p}{\sum_{k=1}^{T}\max_{|\xi|\le c} F(k,\xi)}\right)&\|u\|^p \\ - h_\varepsilon \frac{(2c)^p}{p(T+1)^{p-1}\sum_{k=1}^{T}\max_{|\xi|\le c} F(k,\xi)} &- \mu\tau_\mu.\end{aligned}$$

On the other hand, if

$$\limsup_{|\xi|\to+\infty} \frac{\sum_{k=1}^{T} F(k,\xi)}{|\xi|^p} \leq 0,$$

there exists h_ε such that $\sum_{k=1}^{T} F(k,\xi) \leq h_\varepsilon$ for all $\xi \in \mathbb{R}$, and arguing as before we obtain

$$\Phi(u) - \lambda\Psi(u) \geq \frac{1}{2p}\|u\|^p - h_\varepsilon \frac{(2c)^p}{p(T+1)^{p-1}\sum_{k=1}^{T}\max_{|\xi|\leq c} F(k,\xi)} - \mu\tau_\mu.$$

Both cases lead to the coercivity of $\Phi - \lambda\Psi$. Thus, (a_2) holds. By using relations (3.4) and (3.5) one also has

$$\lambda \in \left(\frac{\Phi(\overline{v})}{\Psi(\overline{v})}, \frac{r}{\sup_{\Phi(u)\leq r}\Psi(u)}\right).$$

Finally, Theorem 2.1 ensures the conclusion. □

Now, we present a variant of Theorem 3.1 in which no asymptotic condition on the nonlinear term is requested.

For our goal, let us fix positive constants c_1, c_2 and d such that

$$\frac{3}{2}\frac{d^p(2+\sum_{k=1}^{T} q_k)}{\sum_{k=1}^{T} F(k,d)} < \frac{2^p}{(T+1)^{p-1}} \min\left\{\frac{c_1^p}{\sum_{k=1}^{T}\max_{|\xi|\leq c_1} F(k,\xi)}, \frac{c_2^p}{2\sum_{k=1}^{T}\max_{|\xi|\leq c_2} F(k,\xi)}\right\},$$

and taking

$$\lambda \in \Lambda' := \left(\frac{3}{2}\frac{\frac{d^p}{p}(2+\sum_{k=1}^{T} q_k)}{\sum_{k=1}^{T} F(k,d)}, \frac{2^p}{p(T+1)^{p-1}} \min\left\{\frac{c_1^p}{\sum_{k=1}^{T}\max_{|\xi|\leq c_1} F(k,\xi)}, \frac{c_2^p}{2\sum_{k=1}^{T}\max_{|\xi|\leq c_2} F(k,\xi)}\right\}\right).$$

Theorem 3.2. *Assume that there exist three positive constants c_1, c_2, d with*

$$2^{\frac{2}{p}} c_1 < d\Big(2 + \sum_{k=1}^{T} q_k\Big)^{\frac{1}{p}} \Big(\frac{T+1}{2}\Big)^{\frac{p-1}{p}} < c_2$$

such that:

(B1) $f(k,t) \geq 0$ *for every* $(k,t) \in [1,T] \times [0,c_2)$,

(B2)

$$\frac{\sum_{k=1}^{T}\max_{|\xi|\leq c_1} F(k,\xi)}{c_1^p} < \frac{2}{3}\frac{2^p}{(T+1)^{p-1}(2+\sum_{k=1}^{T} q_k)}\frac{\sum_{k=1}^{T} F(k,d)}{d^p},$$

(B3)
$$\frac{\sum_{k=1}^{T}\max_{|\xi|\le c_2}F(k,\xi)}{c_2^p} < \frac{1}{3}\frac{2^p}{(T+1)^{p-1}(2+\sum_{k=1}^{T}q_k)}\frac{\sum_{k=1}^{T}F(k,d)}{d^p}.$$

Then, for each $\lambda\in\Lambda'$ and for every continuous function $g:[1,T]\times\mathbb{R}\to\mathbb{R}$ such that is nonnegative in $[1,T]\times[0,c_2)$, there exists $\delta^*_{\lambda,g}>0$ given by

$$\min\left\{\frac{(2c_1)^p-\lambda p(T+1)^{p-1}\sum_{k=1}^{T}\max_{|\xi|\le c_1}F(k,\xi)}{p(T+1)^{p-1}G^{c_1}},\right.$$
$$\left.\frac{(2c_2)^p-2\lambda p(T+1)^{p-1}\sum_{k=1}^{T}\max_{|\xi|\le c_2}F(k,\xi)}{2p(T+1)^{p-1}G^{c_2}}\right\}.$$

such that, for each $\mu\in[0,\delta^*_{\lambda,g})$, the problem (1.1) *admits at least three solutions u_i for $i=1,2,3$, such that*

$$0\le u_i(k)<c_2,\quad \textit{for each}\quad k\in[1,T],\ (i=1,2,3).$$

Proof. Fix λ, g and μ as in the conclusion and take Φ and Ψ as in the proof of Theorem 3.1. We observe that the regularity assumptions of Theorem 2.2 on Φ and Ψ are satisfied. Then, our aim is to verify (b_1) and (b_2).

To this end, put $\overline{v}$ as given in (3.3), as well as $r_1=\frac{(2c_1)^p}{p(T+1)^{p-1}}$ and $r_2=\frac{(2c_2)^p}{p(T+1)^{p-1}}$. By using condition

$$2^{\frac{2}{p}}c_1<d\Big(2+\sum_{k=1}^{T}q_k\Big)^{\frac{1}{p}}\Big(\frac{T+1}{2}\Big)^{\frac{p-1}{p}}<c_2,$$

and bearing in mind that $\Phi(\bar v)=\frac{d^p}{p}(2+\sum_{k=1}^{T}q_k)$, we get $2r_1<\Phi(\bar v)<\frac{r_2}{2}$. Since $\mu<\delta^*_{\lambda,g}$ and $G_d\ge 0$, one has

$$\begin{aligned}\frac{\sup_{u\in\Phi^{-1}((-\infty,r_1))}\Psi(u)}{r_1}&=\frac{\sup_{u\in\Phi^{-1}((-\infty,r_1))}\sum_{k=1}^{T}(F(k,u(k))+\frac{\mu}{\lambda}G(k,u(k)))}{r_1}\\&\le\frac{\sum_{k=1}^{T}\max_{|\xi|\le c_1}F(k,\xi)+\frac{\mu}{\lambda}G^{c_1}}{\frac{(2c_1)^p}{p(T+1)^{p-1}}}\\&<\frac{1}{\lambda}<\frac{2}{3}\frac{\sum_{k=1}^{T}F(k,d)+\frac{\mu}{\lambda}G_d}{\frac{d^p}{p}(2+\sum_{k=1}^{T}q_k)}=\frac{2}{3}\frac{\Psi(\overline{v})}{\Phi(\overline{v})},\end{aligned}$$

and

$$\begin{aligned}\frac{2\sup_{u\in\Phi^{-1}((-\infty,r_2))}\Psi(u)}{r_2}&=\frac{2\sup_{u\in\Phi^{-1}((-\infty,r_2))}\sum_{k=1}^{T}(F(k,u(k))+\frac{\mu}{\lambda}G(k,u(k)))}{r_2}\\&\le\frac{2\sum_{k=1}^{T}\max_{|\xi|\le c_2}F(k,\xi)+2\frac{\mu}{\lambda}G^{c_2}}{\frac{(2c_2)^p}{p(T+1)^{p-1}}}\\&<\frac{1}{\lambda}<\frac{2}{3}\frac{\sum_{k=1}^{T}F(k,d)+\frac{\mu}{\lambda}G_d}{\frac{d^p}{p}(2+\sum_{k=1}^{T}q_k)}=\frac{2}{3}\frac{\Psi(\overline{v})}{\Phi(\overline{v})}.\end{aligned}$$

Therefore, (b_1) and (b_2) of Theorem 2.2 are verified. Finally, we verify that $\Phi - \lambda\Psi$ satisfies the assumption 2 of Theorem 2.2. Let u_1 and u_2 be two local minima for $\Phi - \lambda\Psi$. Then u_1 and u_2 are critical points for $\Phi - \lambda\Psi$, and so, they are solutions for problem (1.1). Then, since Φ is convex, by Lemma 2.3, we deduce u_1 and u_2 are positive. Thus, it follows that $su_1 + (1-s)u_2 \geq 0$ for all $s \in [0,1]$, and that

$$(\lambda f + \mu g)(k, su_1 + (1-s)u_2) \geq 0,$$

and consequently, $\Psi(su_1 + (1-s)u_2) \geq 0$, for every $s \in [0,1]$.

By using Theorem 2.2, for every

$$\lambda \in \left(\frac{3}{2}\frac{\Phi(\overline{v})}{\Psi(\overline{v})},\ \min\left\{ \frac{r_1}{\sup_{u\in\Phi^{-1}((-\infty,r_1))}\Psi(u)},\ \frac{r_2/2}{\sup_{u\in\Phi^{-1}((-\infty,r_2))}\Psi(u)} \right\} \right),$$

the functional $\Phi - \lambda\Psi$ has at least three critical points which are the solutions of the problem (1.1) and the desired conclusion is achieved. $\square$

As a special case of problem (1.1), we consider the following problem

$$\begin{cases} -\Delta(\phi_p(\Delta u(k-1))) + q_k\phi_p(u(k)) = \lambda h_1(k)h_2(u(k)) + \mu g(k,u(k)), & k \in [1,T], \\ u(0) = u(T+1) = 0 \end{cases} \tag{3.8}$$

where $h_1 : [1,T] \to \mathbb{R}$ is continuous and $h_2 \in C(\mathbb{R},\mathbb{R})$. Put

$$H_2(t) = \int_0^t h_2(\xi)d\xi \quad \text{for all } t \in \mathbb{R}.$$

Note that

$$\sum_{k=1}^{T} \max_{|\xi|\leq c} F(k,\xi) = \max_{|\xi|\leq c} H_2(\xi) \sum_{k=1}^{T} h_1(k) \quad \text{for } c > 0,$$

Theorems 3.1 and 3.2 take the following simple forms, respectively.

Fix $c, d > 0$ such that

$$\frac{(2 + \sum_{k=1}^T q_k)d^p}{H_2(d)\sum_{k=1}^T h_1(k)} < \frac{(2c)^p}{(T+1)^{p-1}\max_{|\xi|\leq c} H_2(\xi)\sum_{k=1}^T h_1(k)}$$

and pick

$$\lambda \in \overline{\Lambda} := \left(\frac{(2 + \sum_{k=1}^T q_k)d^p}{pH_2(d)\sum_{k=1}^T h_1(k)},\ \frac{(2c)^p}{p(T+1)^{p-1}\max_{|\xi|\leq c} H_2(\xi)\sum_{k=1}^T h_1(k)} \right).$$

Moreover, put

$$\begin{aligned} \delta'_{\lambda,g} := \min\Bigg\{ & \frac{(2c)^p - \lambda p(T+1)^{p-1}\max_{|\xi|\leq c} H_2(\xi)\sum_{k=1}^T h_1(k)}{p(T+1)^{p-1}G^c}, \\ & \left| \frac{(2 + \sum_{k=1}^T q_k)d^p - \lambda pH_2(d)\sum_{k=1}^T h_1(k)}{p\min\{0, G_d\}} \right| \Bigg\} \end{aligned} \tag{3.9}$$

and

$$\overline{\delta'}_{\lambda,g} := \min\left\{\delta'_{\lambda,g}, \frac{1}{\max\left\{0, \frac{p(T+1)^{p-1}}{2^{p-1}} \limsup_{|\xi|\to+\infty} \frac{\sum_{k=1}^{T} G(k,\xi)}{|\xi|^p}\right\}}\right\}, \tag{3.10}$$

where we read $\frac{1}{0} := +\infty$ whenever this case occurs.

Theorem 3.3. *Assume that there exist two positive constants c and d with $2c < d(2+\sum_{k=1}^{T} q_k)^{\frac{1}{p}}(T+1)^{\frac{p-1}{p}}$ such that*

(A4) $\frac{\max_{|\xi|\le c} H_2(\xi)\sum_{k=1}^{T} h_1(k)}{c^p} < \frac{2^p}{(T+1)^{p-1}(2+\sum_{k=1}^{T} q_k)} \frac{H_2(d)\sum_{k=1}^{T} h_1(k)}{d^p}$,

(A5) $(\sum_{k=1}^{T} h_1(k)) \limsup_{|\xi|\to+\infty} \frac{H_2(\xi)}{|\xi|^p} < \frac{H_2(c)\sum_{k=1}^{T} h_1(k)}{2c^p}$.

Then, for any $\lambda \in \overline{\Lambda}$ and for every continuous function $g : [1,T]\times\mathbb{R} \to \mathbb{R}$ satisfying (A3)*, there exits $\overline{\delta'}_{\lambda,g} > 0$ given by* (3.10) *such that, for each $\mu \in (0, \overline{\delta'}_{\lambda,g})$, problem* (3.8) *has at least three solutions.*

Let $h_1 : [1,T] \to \mathbb{R}$ be a nonnegative continuous function and $h_2 \in C(\mathbb{R},\mathbb{R})$. Fix positive constants c_1, c_2 and d such that h_2 is nonnegative in $[0, c_2)$ and

$$\frac{3}{2}\frac{d^p(2+\sum_{k=1}^{T} q_k)}{H_2(d)} < \frac{2^p}{(T+1)^{p-1}} \min\left\{\frac{c_1^p}{\max_{|\xi|\le c_1} H_2(\xi)}, \frac{c_2^p}{2\max_{|\xi|\le c_2} H_2(\xi)}\right\},$$

and taking

$$\lambda \in \overline{\Lambda'} := \Bigg(\frac{3}{2}\frac{\frac{d^p}{p}(2+\sum_{k=1}^{T} q_k)}{H_2(d)\sum_{k=1}^{T} h_1(k)}, \frac{2^p}{p(T+1)^{p-1}} \min\left\{\frac{c_1^p}{\max_{|\xi|\le c_1} H_2(\xi)\sum_{k=1}^{T} h_1(k)}, \frac{c_2^p}{2\max_{|\xi|\le c_2} H_2(\xi)\sum_{k=1}^{T} h_1(k)}\right\}\Bigg).$$

Theorem 3.4. *Assume that there exist three positive constants c_1, c_2 and d with*

$$2^{\frac{2}{p}} c_1 < d\Big(2+\sum_{k=1}^{T} q_k\Big)^{\frac{1}{p}} \Big(\frac{T+1}{2}\Big)^{\frac{p-1}{p}} < c_2$$

such that

(B4)

$$\frac{\max_{|\xi|\le c_1} H_2(\xi)}{c_1^p} < \frac{2}{3}\frac{2^p}{(T+1)^{p-1}(2+\sum_{k=1}^{T} q_k)}\frac{H_2(d)}{d^p},$$

(B5)

$$\frac{\max_{|\xi|\le c_2} H_2(\xi)}{c_2^p} < \frac{1}{3}\frac{2^p}{(T+1)^{p-1}(2+\sum_{k=1}^{T} q_k)}\frac{H_2(d)}{d^p}.$$

Then, for each $\lambda \in \overline{\Lambda'}$ *and for every continuous function* $g : [1,T] \times \mathbb{R} \to \mathbb{R}$ *such that it is nonnegative in* $[1,T] \times [0,c_2)$, *there exists* $\overline{\delta^*}_{\lambda,g} > 0$ *given by*

$$\min\left\{\frac{(2c_1)^p - \lambda p(T+1)^{p-1}\max_{|\xi|\le c_1} H_2(\xi)\sum_{k=1}^T h_1(k)}{p(T+1)^{p-1}G^{c_1}},\right.$$
$$\left.\frac{(2c_2)^p - 2\lambda p(T+1)^{p-1}\max_{|\xi|\le c_2} H_2(\xi)\sum_{k=1}^T h_1(k)}{2p(T+1)^{p-1}G^{c_2}}\right\}.$$

such that, for each $\mu \in [0, \overline{\delta^*}_{\lambda,g})$, *the problem* (3.8) *admits at least three solutions* u_i *for* $i = 1,2,3$, *such that*

$$0 \le u_i(k) < c_2 \quad \textit{for all} \quad k \in [1,T],\ (i = 1,2,3).$$

Now, we present two examples to illustrate the results of Theorems 3.3 and 3.4.

Example 3.5. Choose $c = \frac{1}{7500}$, $d = \frac{1}{2}$, $p = 4$, $T = 10$, $\sum_{k=1}^T q_k = 2$ and

$$h_1(k) = 1, \quad h_2(\xi) = \xi^4(5 - 6\xi), \quad g(k,\xi) = \frac{1}{(k^2+k)(1+\xi^2)}$$

for all $k \in [1,10]$ and $\xi \in \mathbb{R}$. Therefore, since in this case,

$$\frac{H_2(c)}{c^p} = 0.0001,\ \frac{2^p}{(T+1)^{p-1}(2+\sum_{k=1}^T q_k)}\frac{H_2(d)}{d^p} = 0.0007 \text{ and } \limsup_{|\xi|\to+\infty}\frac{H_2(\xi)}{|\xi|^p} = -\infty,$$

we see that all assumptions of Theorem 3.3 are satisfied, and hence Theorem 3.3 follows that for any $\lambda \in (0.4, 2.2)$ and $\mu \in (0, \frac{10^{-15}(5-2.2\lambda)}{0.532})$, the following problem

$$\begin{cases} -\Delta(\phi_4(\Delta u(k-1))) + q_k\phi_4(u(k)) = \lambda u(k)^4(5 - 6u(k)) + \mu\frac{1}{(k^2+k)(1+u(k)^2)}, \\ \hfill k \in [1,10], \\ u(0) = u(11) = 0, \end{cases}$$

has at least three solutions.

Example 3.6. The following problem

$$\begin{cases} -\Delta^2 u(k-1) + q_k u(k) = \lambda\frac{2}{5}\ln(\frac{k+1}{k})\frac{(\sinh u(k))^3\cosh u(k)}{1+(\sinh u(k))^8} + \mu k e^{u(k)}, \quad k \in [1,10], \\ u(0) = u(11) = 0, \end{cases}$$

has at least three positive solutions with $\|u\|_\infty \le 20$, for any $\lambda \in (46, 94)$ and for each $\mu \in (0, \frac{1600-16.6\lambda}{1174099770471.7})$. Indeed, it is enough to apply Theorem 3.4 by choosing

$$c_1 = \frac{1}{20}, \quad c_2 = 20, \quad d = 1, \quad T = 10, \quad p = 2, \quad \sum_{k=1}^T q_k = 14$$

and

$$h_1(k) = \frac{1}{10}\ln\left(\frac{k+1}{k}\right), \quad h_2(\xi) = \frac{4\,(\sinh\xi)^3\cosh\xi}{1+(\sinh\xi)^8} \quad \text{and} \quad g(k,\xi) = ke^{\xi}$$

for all $k \in [1,10]$ and $\xi \in \mathbb{R}$, taking into account that in this case, $\frac{H_2(c_1)}{c_1^2} = 0.002$, $\frac{H_2(c_2)}{c_2^2} = 0.004$ and

$$\frac{1}{3}\frac{2^p}{(T+1)^{p-1}(2+\sum_{k=1}^{T} q_k)}\frac{H_2(d)}{d^p} = 0.008.$$

Finally, we prove Theorem 1.1 given in Introduction.

Proof of Theorem 1.1. From the condition

$$\liminf_{\xi\to 0}\frac{F(\xi)}{\xi^p} = 0,$$

there is a sequence $\{c_n\} \subset (0,+\infty)$ such that $\lim_{n\to\infty} c_n = 0$ and

$$\lim_{n\to\infty}\frac{\max_{|\xi|\le c_n} F(\xi)}{c_n^p} = 0.$$

Indeed, one has

$$\lim_{n\to\infty}\frac{\max_{|\xi|\le c_n} F(\xi)}{c_n^p} = \lim_{n\to\infty}\frac{F(\xi_{c_n})}{\xi_{c_n}^p}\frac{\xi_{c_n}^p}{c_n^p} = 0,$$

where $F(\xi_{c_n}) = \max_{|\xi|\le c_n} F(\xi)$. Hence, there exists $\bar{c} > 0$ such that

$$\frac{\max_{|\xi|\le\bar{c}} F(\xi)}{\bar{c}^p} < \min\left\{\frac{2^p}{(T+1)^{p-1}(2+\sum_{k=1}^{T} q_k)}\frac{F(d)}{d^p};\ \frac{2^p}{p\lambda T(T+1)^{p-1}}\right\}$$

and $\bar{c} < d$. The conclusion follows by using Theorem 3.1. □

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REFERENCES

[1] R.P. Agarwal, *Difference Equations and Inequalities: Theory, Methods and Applications,* Marcel Dekker, New York, Basel, 2000.

[2] R.P. Agarwal, D. O'Regan, P.J.Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations,* Kluwer Academic, Dordrecht, 1999.

[3] R.P. Agarwal, K. Perera, D. O'Regan, *Multiple positive solutions of singular and nonsingular discrete problems via variational methods*, Nonlinear Anal. TMA **58** (2004), 69–73.

[4] G. Bonanno, P. Candito, *Infinitely many solutions for a class of discrete nonlinear boundary value problems,* Appl. Anal. **884** (2009), 605–616.

[5] G. Bonanno, P. Candito, *Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities,* J. Differ. Eqs **244** (2008), 3031–3059.

[6] G. Bonanno, P. Candito, *Nonlinear difference equations investigated via critical point methods,* Nonlinear Anal. TMA **70** (2009), 3180–3186.

[7] G. Bonanno, S.A. Marano, *On the structure of the critical set of non-differentiable functions with a weak compactness condition,* Appl. Anal. **89** (2010), 1–10.

[8] A. Cabada, A. Iannizzotto, S. Tersian, *Multiple solutions for discrete boundary value problem,* J. Math. Anal. Appl. **356** (2009), 418–428.

[9] P. Candito, G. D'Aguì, *Three solutions for a discrete nonlinear Neumann problem involving the p-Laplacian,* Adv. Difference Equ. 2010, Art. ID 862016, 11 pp.

[10] P. Candito, G. D'Aguì, *Three solutions to a perturbed nonlinear discrete Dirichlet problem,* J. Math. Anal. Appl. **375** (2011), 594–601.

[11] P. Candito, N. Giovannelli, *Multiple solutions for a discrete boundary value problem*, Comput. Math. Appl. **56** (2008), 959–964.

[12] J. Chu, D. Jiang, *Eigenvalues and discrete boundary value problems for the one-dimensional p-Laplacian*, J. Math. Anal. Appl. **305** (2005), 452–465.

[13] M. Galewski, S. Głąb, *On the discrete boundary value problem for anisotropic equation,* J. Math. Anal. Appl. **386** (2012), 956–965.

[14] J. Henderson, H.B. Thompson, *Existence of multiple solutions for second order discrete boundary value problems,* Comput. Math. Appl. **43** (2002), 1239–1248.

[15] D. Jiang, J. Chu, D. O'Regan, R.P. Agarwal, *Positive solutions for continuous and discrete boundary value problems to the one-dimensional p-Laplacian,* Math. Inequal. Appl. **7** (2004), 523–534.

[16] L. Jiang, Z. Zhou, *Three solutions to Dirichlet boundary value problems for p-Laplacian difference equations,* Adv. Diff. Equ. **2008** (2008), 1–10.

[17] W.G. Kelly, A.C. Peterson, *Difference Equations: An Introduction with Applications,* Academic Press, San Diego, New York, Basel, 1991.

[18] A. Kristály, M. Mihailescu, V. Rădulescu, *Discrete boundary value problems involving oscillatory nonlinearities: small and large solutions,* J. Difference Equ. Appl., doi:10.1080/10236190903555245 (to appear).

[19] N. Marcu, G. Molica Bisci, *Existence and multiplicity results for nonlinear discrete inclusions,* Electron. J. Differential Equations **2012** (2012), 1–13.

[20] M. Mihailescu, V. Rădulescu, S. Tersian, *Eigenvalue problems for anisotropic discrete boundary value problems,* J. Difference Equ. Appl. **15** (2009), 557–567.

[21] G. Molica Bisci, D. Repovs, *Nonlinear algebraic systems with discontinuous terms*, J. Math. Anal. Appl. **398** (2013), 846–856.

[22] G. Molica Bisci, D. Repovs, *On sequences of solutions for discrete anisotropic equations*, Expositiones Mathematicae (to appear).

[23] G. Molica Bisci, D. Repovs, *On some variational algebraic problems,* Adv. Nonlinear Analysis **2** (2013), 127–146.

[24] B. Ricceri, *A three critical points theorem revisited,* Nonlinear Anal. TMA **70** (2009), 3084–3089.

[25] D.B. Wang, W. Guan, *Three positive solutions of boundary value problems for p-Laplacian difference equations,* Comput. Math. Appl. **55** (2008), 1943–1949.

[26] P.J.Y. Wong, L. Xie, *Three symmetric solutions of Lidstone boundary value problems for difference and partial difference equations,* Comput. Math. Appl. **45** (2003), 1445–1460.

[27] J.S. Yu, Z.M. Guo, *On boundary value problems for a discrete generalized Emden-Fowler equation,* J. Differential Equations **231** (2006), 18–31.

[28] G. Zhang, Z.L. Yang, *Positive solutions of a general discrete boundary value problem*, J. Math. Anal. Appl. **339** (2008), 469–481.

[29] G. Zhang, W. Zhang, S. Liu, *Multiplicity result for a discrete eigenvalue problem with discontinuous nonlinearities*, J. Math. Anal. Appl. **328** (2007), 1068–1074.

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