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WEAK HETEROCLINIC SOLUTIONS AND COMPETITION PHENOMENA TO ANISOTROPIC DIFFERENCE EQUATIONS WITH VARIABLE EXPONENTS

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Abstract. In this paper, we prove the existence of weak heteroclinic solutions for a family of anisotropic difference equations under competition phenomena between parameters.

Keywords: anisotropic difference equations, heteroclinic solutions, discrete Hölder type inequality, competition phenomena.

Mathematics Subject Classification: 47A75, 35B38, 35P30, 34L05, 34L30.

1. INTRODUCTION

In this paper, we study the following nonlinear anisotropic discrete problem with heteroclinic condition at the boundary

$$\begin{cases} -\Delta(a(k-1,\Delta u(k-1))) + \alpha(k)g(k,u(k)) = \delta(k)f(k,u(k)), \quad k\in\mathbb{Z}^*, \\ u(0)=0, \quad \lim\limits_{k\to-\infty} u(k) = -1, \quad \lim\limits_{k\to+\infty} u(k) = 1, \end{cases} \tag{1.1}$$

where $\Delta u(k) = u(k+1)-u(k)$ is the forward difference operator, $\mathbb{Z}^* := \{k\in\mathbb{Z} : k\neq 0\}$ and a,α,δ,f,g are functions to be defined later.

Difference equations can be seen as a discrete counterpart of PDEs and are usually studied in connection with numerical analysis. In this way, the main operator in Problem (1.1)

$$-\Delta(a(k-1,\Delta u(k-1)))$$

can be seen as a discrete counterpart of the anisotropic operator

$$-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a\Big(x,\frac{\partial}{\partial x_i}u\Big).$$

Note that anisotropic PDEs with as main operators, the operator above was studied by many authors under Leray-Lions type conditions (see [6]) in the context of variable exponents (see [3, 5, 7, 9–11]). Therefore, the problem (1.1) can be seen as a discrete counterpart of such PDEs under nonhomogeneous Dirichlet boundary conditions.

We adapt in this paper the classical minimization methods used for the study of anisotropic PDEs to prove the existence of solution of problem (1.1). Note that we examine anisotropic difference equations on unbounded discrete interval, typically, on the whole set $\mathbb{Z}$, with asymptotic conditions of heteroclinic type. The first study in that direction for constant exponents was done by Cabada *et al.* [2] and for variable exponent by Mihailescu *et al.* [8] (see also [4]). In [4], the authors studied the following problem:

$$\begin{cases} -\Delta(a(k-1, \Delta u(k-1))) + |u(k)|^{p(k)-2}u(k) = f(k), \quad k \in \mathbb{Z}, \\ \lim\limits_{|k|\to\infty} u(k) = 0. \end{cases} \tag{1.2}$$

They proved an existence result of weak homoclinic solution of (1.2).

In this paper, we prove an existence result of (1.1) and for that, we define other new spaces and new associated norms compared to that of [4]. Some of the norms defined may be equivalent in order to prove the main result of this paper. Note also that in our study, we show some competition phenomena between $\alpha(\cdot)$ and $\delta(\cdot)$. Such competition phenomena are also necessary for the proof of the existence of weak heteroclinic solution of (1.1).

The study of heteroclinic connections for boundary value problems has had a certain impulse in recent years, motivated by applications in various biological, physical and chemical models, such has phase-transition, physical processes in which the variable transits from an unstable equilibrium to a stable one, or front-propagation in reaction-diffusion equations. Indeed, heteroclinic solutions are often called transitional solutions.

The paper is organized as follows. Section 2 is devoted to the mathematical preliminary. In Section 3, we study problem (1.1), therefore, we prove the existence of weak heteroclinic solutions of (1.1).

2. AUXILIARY RESULTS

We set $\mathbb{Z}_*^- := \{k \in \mathbb{Z} : k < 0\}$, $\mathbb{Z}_*^+ := \{k \in \mathbb{Z} : k > 0\}$, $\mathbb{Z}^- := \{k \in \mathbb{Z} : k \leq 0\}$ and $\mathbb{Z}^+ := \{k \in \mathbb{Z} : k \geq 0\}$. For the data f, α and a, we assume the following.

- (H_1) $a(k,\cdot) : \mathbb{R} \to \mathbb{R}$, $k \in \mathbb{Z}$, and there exists a mapping $A : \mathbb{Z}\times\mathbb{R} \to \mathbb{R}$ which satisfies $a(k,\xi) = \frac{\partial}{\partial\xi}A(k,\xi)$ for all $k \in \mathbb{Z}$ and $A(k,0) = 0$ for all $k \in \mathbb{Z}$.
- (H_2) $p : \mathbb{Z} \to (1,+\infty)$ with $1 < p^- \leq p^+ < +\infty$, where $p^+ := \sup_{k\in\mathbb{Z}} p(k)$ and $p^- := \inf_{k\in\mathbb{Z}} p(k)$.
- (H_3) $|\xi|^{p(k)} \leq a(k,\xi)\xi \leq p(k)A(k,\xi)$ for all $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}$.
- (H_4) there exists $C_1 > 0$ such that for all $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}$ we have $|a(k,\xi)| \leq C_1(j(k) + |\xi|^{p(k)-1})$ with $j \in l^{p'(\cdot)}$, where $\frac{1}{p(k)} + \frac{1}{p'(k)} = 1$.

(H_5) $(a(k,\xi)-a(k,\eta)).(\xi-\eta)>0$ for all $k\in\mathbb{Z}$ and $\xi,\eta\in\mathbb{R}$ such that $\xi\neq\eta$.
(H_6) $f:\mathbb{Z}\times\mathbb{R}\to\mathbb{R}$ and there exists $C_2>0$ such that

$$|f(k,t)|\le \mathrm{C}_2(1+|t-1|^{p(k)-1})\chi_{\mathbb{Z}^+}+\mathrm{C}_2(1+|t+1|^{p(k)-1})\chi_{\mathbb{Z}^-_*},$$

for all $k\in\mathbb{Z}$, $t\in\mathbb{R}$, where $\chi_A(k)=1$ if $k\in A$ and $\chi_A(k)=0$ if $k\notin A$.

This assumption implies that

$$\begin{cases}|f(k,t+1)|\le \mathrm{C}_2(1+|t|^{p(k)-1}) & \text{if} \quad k\ge 0,\\ |f(k,t-1)|\le \mathrm{C}_2(1+|t|^{p(k)-1}) & \text{if} \quad k<0,\end{cases}$$

so by denoting

$$F(k,t)=\int_0^t f(k,\tau)d\tau \quad \text{for} \quad k\in\mathbb{Z}, t\in\mathbb{R},$$

we deduce that there exists a positive constant $C_2'>1$ such that

$$\begin{cases}|F(k,t+1)|\le C_2'(1+|t|^{p(k)}) & \text{if} \quad k\ge 0,\\ |F(k,t-1)|\le C_2'(1+|t|^{p(k)}) & \text{if} \quad k<0.\end{cases}$$

(H_7) $\alpha:\mathbb{Z}\to\mathbb{R}$ and $\delta:\mathbb{Z}\to\mathbb{R}$ are such that $\alpha(k)\ge\alpha_0>0$ for all $k\in\mathbb{Z}$,

$$0<\delta(k)\le\bar{\delta}=\sup_{k\in\mathbb{Z}}|\delta(k)|<+\infty \quad \text{and} \quad \delta\in l^1:=\Big\{u:\mathbb{Z}\to\mathbb{R};\ \sum_{k\in\mathbb{Z}}|u(k)|<+\infty\Big\}.$$

(H_8) $\alpha_0>\bar{\delta}p^+C_2'$.
(H_9) $g(k,t)=|t-1|^{p(k)-2}(t-1)\chi_{\mathbb{Z}^+}(k)+|t+1|^{p(k)-2}(t+1)\chi_{\mathbb{Z}^-}(k)$.

Remark 2.1. The condition $\alpha_0>\bar{\delta}p^+C_2'$ on the data means that the parameter $\alpha(\cdot)$ should be bigger than the parameter $\bar{\delta}$. This condition is called competition phenomena between $\alpha(\cdot)$ and $\delta(\cdot)$.

In order to present the main result, for each $p(\cdot):\mathbb{Z}\to(1,+\infty)$, we introduce the following spaces:

$$l^\infty=\Big\{u:\mathbb{Z}\to\mathbb{R};\sup_{k\in\mathbb{Z}}|u(k)|<\infty\Big\},$$

$$l_0^{p(\cdot)}=\Big\{u:\mathbb{Z}\to\mathbb{R};\ u(0)=0 \text{ and } \rho_{p(\cdot)}(u):=\sum_{k\in\mathbb{Z}}|u(k)|^{p(k)}<+\infty\Big\},$$

$$l_{0,+}^{p(\cdot)}=\Big\{u:\mathbb{Z}\to\mathbb{R};\ u(0)=0 \text{ and } \rho_{p_+(\cdot)}(u):=\sum_{k\in\mathbb{Z}^+}|u(k)|^{p(k)}<+\infty\Big\},$$

$$l_{0,-}^{p(\cdot)}=\Big\{u:\mathbb{Z}\to\mathbb{R};\ u(0)=0 \text{ and } \rho_{p_-(\cdot)}(u):=\sum_{k\in\mathbb{Z}^-}|u(k)|^{p(k)}<+\infty\Big\},$$

$$l^{p(\cdot)}_{0,\alpha(\cdot)} = \Big\{u : \mathbb{Z} \to \mathbb{R};\ u(0) = 0 \text{ and } \rho_{\alpha(\cdot),p(\cdot)}(u) := \sum_{k\in\mathbb{Z}} \alpha(k)|u(k)|^{p(k)} < +\infty\Big\},$$

$$l^{p(\cdot)}_{0,+,\alpha(\cdot)} = \Big\{u : \mathbb{Z} \to \mathbb{R};\ u(0) = 0 \text{ and } \rho_{\alpha(\cdot),p_+(\cdot)}(u) := \sum_{k\in\mathbb{Z}^+} \alpha(k)|u(k)|^{p(k)} < +\infty\Big\},$$

$$l^{p(\cdot)}_{0,-,\alpha(\cdot)} = \Big\{u : \mathbb{Z} \to \mathbb{R};\ u(0) = 0 \text{ and } \rho_{\alpha(\cdot),p_-(\cdot)}(u) := \sum_{k\in\mathbb{Z}^-} \alpha(k)|u(k)|^{p(k)} < +\infty\Big\},$$

$$\begin{aligned}\mathcal{W}^{1,p(\cdot)}_{0,\alpha(\cdot)} = \Big\{u : \mathbb{Z} \to \mathbb{R};\ u(0) = 0 \text{ and } \rho_{1,\alpha(\cdot),p(\cdot)}(u) := &\sum_{k\in\mathbb{Z}} \alpha(k)|u(k)|^{p(k)} \\ &+ \sum_{k\in\mathbb{Z}} |\Delta u(k)|^{p(k)} < +\infty\Big\}.\end{aligned}$$

$$\begin{aligned}\mathcal{W}^{1,p(\cdot)}_{0,+,\alpha(\cdot)} &= \Big\{u : \mathbb{Z} \to \mathbb{R}; u(0) = 0 \text{ and } \rho_{1,\alpha(\cdot),p_+(\cdot)}(u) := \sum_{k\in\mathbb{Z}^+} \alpha(k)|u(k)|^{p(k)} \\ &\qquad + \sum_{k\in\mathbb{Z}^+} |\Delta u(k)|^{p(k)} < +\infty\Big\} \\ &= \Big\{u : \mathbb{Z} \to \mathbb{R}; u \in l^{p(\cdot)}_{0,+,\alpha(\cdot)},\ \ \Delta u(k) \in l^{p(\cdot)}_{0,+} \text{ and } u(0) = 0\Big\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{W}^{1,p(\cdot)}_{0,-,\alpha(\cdot)} &= \Big\{u : \mathbb{Z} \to \mathbb{R}; u(0) = 0 \text{ and } \rho_{1,\alpha(\cdot),p_+(\cdot)}(u) := \sum_{k\in\mathbb{Z}^-} \alpha(k)|u(k)|^{p(k)} \\ &\qquad + \sum_{k\in\mathbb{Z}^-} |\Delta u(k)|^{p(k)} < +\infty\Big\} \\ &= \Big\{u : \mathbb{Z} \to \mathbb{R}; u \in l^{p(\cdot)}_{0,-,\alpha(\cdot)},\ \ \Delta u(k) \in l^{p(\cdot)}_{0,-} \text{ and } u(0) = 0\Big\}.\end{aligned}$$

On $l^{p(\cdot)}_{0,+}$ and $l^{p(\cdot)}_{0,+,\alpha(\cdot)}$ we introduce the Luxemburg norms

$$\|u\|_{p_+(\cdot)} := \inf\left\{\lambda > 0; \sum_{k\in\mathbb{Z}^+} \left|\frac{u(k)}{\lambda}\right|^{p(k)} \le 1\right\},$$

$$\|u\|_{\alpha(\cdot),p_+(\cdot)} := \inf\left\{\lambda > 0; \sum_{k\in\mathbb{Z}^+} \alpha(k)\left|\frac{u(k)}{\lambda}\right|^{p(k)} \le 1\right\}$$

and we deduce that

$$\|u\|_{1,\alpha(\cdot),p_+(\cdot)} = \|u\|_{\alpha(\cdot),p_+(\cdot)} + \|\Delta u\|_{p_+(\cdot)}$$

is a norm on the space $\mathcal{W}^{1,p(\cdot)}_{0,+,\alpha(\cdot)}$. We replace $\mathbb{Z}^+$ by $\mathbb{Z}^-$ to get the norms on $l^{p(\cdot)}_{0,-}$, $l^{p(\cdot)}_{0,-,\alpha(\cdot)}$ and $\mathcal{W}^{1,p(\cdot)}_{0,-,\alpha(\cdot)}$.

Remark 2.2.

1) $l^{p(\cdot)}_{0,+,\alpha(\cdot)} \supset l^{p(\cdot)}_{0,\alpha(\cdot)}$, $l^{p(\cdot)}_{0,-,\alpha(\cdot)} \supset l^{p(\cdot)}_{0,\alpha(\cdot)}$, $\mathcal{W}^{1,p(\cdot)}_{0,+,\alpha(\cdot)} \supset \mathcal{W}^{1,p(\cdot)}_{0,\alpha(\cdot)}$ and $\mathcal{W}^{1,p(\cdot)}_{0,-,\alpha(\cdot)} \supset \mathcal{W}^{1,p(\cdot)}_{0,\alpha(\cdot)}$. Indeed, $\alpha(k)|u(k)|^{p(k)}$ is nonnegative for all $k \in \mathbb{Z}$. Therefore, if $\sum_{k\in\mathbb{Z}} \alpha(k)|u(k)|^{p(k)} < +\infty$, then $\sum_{k\in\mathbb{Z}^+} \alpha(k)|u(k)|^{p(k)} < +\infty$.
2) Since for every $k \in \mathbb{Z}$, $a(k,\cdot)$ is a gradient and is monotone, then the primitive $A(k,\cdot)$ of $a(k,\cdot)$ is necessarily convex.
3) As an example of functions which satisfy the assumptions (H_1)–(H_5), we can give the following:
   a) $A(k,\xi) = \frac{1}{p(k)} |\xi|^{p(k)}$, where $a(k,\xi) = |\xi|^{p(k)-2}\,\xi$ for all $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}$.
   b) $A(k,\xi) = \frac{1}{p(k)} \left[\left(1+|\xi|^2\right)^{p(k)/2} - 1\right]$, where $a(k,\xi) = \left(1+|\xi|^2\right)^{(p(k)-2)/2}\xi$ for all $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}$.

As in [4], we can prove the following results.

Lemma 2.3. *Under assumption (H_2), we have:*

a) $\rho_{\alpha(\cdot),p_+(\cdot)}(u+v) \le 2^{p+}(\rho_{\alpha(\cdot),p_+(\cdot)}(u) + \rho_{\alpha(\cdot),p_+(\cdot)}(v))$ *for all* $u, v \in l^{p(\cdot)}_{0,+,\alpha(\cdot)}$,
b) *for* $u \in l^{p(\cdot)}_{0,+,\alpha(\cdot)}$, *if* $\lambda > 1$, *then*

$$\begin{aligned}\rho_{\alpha(\cdot),p_+(\cdot)}(u) \le \lambda\rho_{\alpha(\cdot),p_+(\cdot)}(u) &\le \lambda^{p^-}\rho_{\alpha(\cdot),p_+(\cdot)}(u)\\ &\le \rho_{\alpha(\cdot),p_+(\cdot)}(\lambda u) \le \lambda^{p^+}\rho_{\alpha(\cdot),p_+(\cdot)}(u)\end{aligned}$$

and if $0 < \lambda < 1$, *then*

$$\begin{aligned}\lambda^{p^+}\rho_{\alpha(\cdot),p_+(\cdot)}(u) \le \rho_{\alpha(\cdot),p(\cdot)}(\lambda u) &\le \lambda^{p^-}\rho_{\alpha(\cdot),p_+(\cdot)}(u)\\ &\le \lambda\rho_{\alpha(\cdot),p_+(\cdot)}(u) \le \rho_{\alpha(\cdot),p_+(\cdot)}(u),\end{aligned}$$

c) *for every fixed* $u \in l^{p(\cdot)}_{0,+,\alpha(\cdot)} \setminus \{0\}$, $\rho_{\alpha(\cdot),p_+(\cdot)}(\lambda u)$ *is continuous convex even function in* λ, *and it increases strictly when* $\lambda \in [0,\infty)$.

Proposition 2.4. *Let* $u \in l^{p(\cdot)}_{0,+,\alpha(\cdot)} \setminus \{0\}$. *Then*

$$\|u\|_{\alpha(\cdot),p_+(\cdot)} = \gamma \Leftrightarrow \rho_{\alpha(\cdot),p_+(\cdot)}\left(\frac{u}{\gamma}\right) = 1.$$

Proposition 2.5. *If* $u \in l^{p(\cdot)}_{0,+,\alpha(\cdot)}$ *and* $p^+ < +\infty$, *then the following properties hold:*

1) $\|u\|_{\alpha(\cdot),p_+(\cdot)} < 1 (= 1;\ > 1) \Leftrightarrow \rho_{\alpha(\cdot),p_+(\cdot)}(u) < 1 (= 1; > 1)$,
2) $\|u\|_{\alpha(\cdot),p_+(\cdot)} > 1 \Rightarrow \|u\|^{p^-}_{\alpha(\cdot),p_+(\cdot)} \le \rho_{\alpha(\cdot),p_+(\cdot)}(u) \le \|u\|^{p+}_{\alpha(\cdot),p_+(\cdot)}$,
3) $\|u\|_{\alpha(\cdot),p_+(\cdot)} < 1 \Rightarrow \|u\|^{p^+}_{\alpha(\cdot),p_+(\cdot)} \le \rho_{\alpha(\cdot),p_+(\cdot)}(u) \le \|u\|^{p-}_{\alpha(\cdot),p_+(\cdot)}$,
4) $\|u\|_{\alpha(\cdot),p_+(\cdot)} \to 0 \Leftrightarrow \rho_{\alpha(\cdot),p_+(\cdot)}(u) \to 0$.

Proposition 2.6. *Let* $u \in W^{1,p(\cdot)}_{0,+,\alpha(\cdot)} \setminus \{0\}$. *Then*

$$\|u\|_{1,\alpha(\cdot),p_+(\cdot)} = a \Leftrightarrow \rho_{1,\alpha(\cdot),p_+(\cdot)}\left(\frac{u}{a}\right) = 1.$$

Proposition 2.7. *If* $u \in W^{1,p(\cdot)}_{0,+,\alpha(\cdot)}$ *and* $p^+ < +\infty$, *then the following properties hold:*

1) $\|u\|_{1,\alpha(\cdot),p_+(\cdot)} < 1 (= 1;\ > 1) \Leftrightarrow \rho_{1,\alpha(\cdot),p_+(\cdot)}(u) < 1 (= 1; > 1)$,
2) $\|u\|_{1,\alpha(\cdot),p_+(\cdot)} > 1 \Rightarrow \|u\|^{p^-}_{1,\alpha(\cdot),p_+(\cdot)} \leq \rho_{1,\alpha(\cdot),p_+(\cdot)}(u) \leq \|u\|^{p+}_{1,\alpha(\cdot),p_+(\cdot)}$,
3) $\|u\|_{1,\alpha(\cdot),p_+(\cdot)} < 1 \Rightarrow \|u\|^{p^+}_{1,\alpha(\cdot),p_+(\cdot)} \leq \rho_{1,\alpha(\cdot),p_+(\cdot)}(u) \leq \|u\|^{p-}_{1,\alpha(\cdot),p_+(\cdot)}$,
4) $\|u\|_{1,\alpha(\cdot),p_+(\cdot)} \to 0 \Leftrightarrow \rho_{1,\alpha(\cdot),p_+(\cdot)}(u) \to 0$.

We also have the following lemma (see [4]).

Lemma 2.8 (Hölder type inequality)**.** *Let* $u \in l^{p(\cdot)}_{0,+,\alpha(\cdot)}$ *and* $v \in l^{q(\cdot)}_{0,+,\alpha(\cdot)}$ *such that* $\frac{1}{p(k)} + \frac{1}{q(k)} = 1$ *for all* $k \in \mathbb{Z}$. *Then*

$$\sum_{k\in\mathbb{Z}^+} |uv| \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right)\|u\|_{\alpha(\cdot),p_+(\cdot)}\|v\|_{\alpha(\cdot),q_+(\cdot)}.$$

Remark 2.9. The properties above also hold for the spaces $l^{p(\cdot)}_{0,\alpha(\cdot)}$, $l^{p(\cdot)}_{0,-,\alpha(\cdot)}$ and $\mathcal{W}^{1,p(\cdot)}_{0,-,\alpha(\cdot)}$.

3. EXISTENCE OF WEAK HETEROCLINIC SOLUTIONS

In this section, we study the existence of weak heteroclinic solutions of (1.1) where δ is a positive function.

Definition 3.1. A weak heteroclinic solution of (1.1) is a function $u \in \mathcal{W}^{1,p(\cdot)}_{0,\alpha(\cdot)}$ such that

$$\begin{aligned}&\sum_{k\in\mathbb{Z}} a(k-1, \Delta u(k-1))\Delta v(k-1) + \sum_{k\in\mathbb{Z}} \alpha(k)g(k, u(k))v(k)\\ &= \sum_{k\in\mathbb{Z}} \delta(k)f(k, u(k))v(k)\end{aligned} \tag{3.1}$$

for any $v : \mathbb{Z} \to \mathbb{R}$, with $u(0) = 0$, $\lim_{k\to-\infty} u(k) = -1$ and $\lim_{k\to+\infty} u(k) = 1$.

We have the following result.

Theorem 3.2. *Assume that* (H_1)*–*(H_9) *hold. Then, there exists at least one weak heteroclinic solution of* (1.1).

To prove Theorem 3.2, we first consider that the following problem:

$$\begin{cases} -\Delta(a(k-1, \Delta u(k-1))) + \alpha(k)\,|u(k)|^{p(k)-2}\,u(k) = \delta(k)f(k, u(k)+1), \quad k \in \mathbb{Z}^+_*,\\ u(0) = 0,\ \ \lim_{k\to+\infty} u(k) = 0,\end{cases} \tag{3.2}$$

admits at least a weak solution in the following sense.

Definition 3.3. A weak solution of (3.2) is a function $u \in \mathcal{W}^{1,p(\cdot)}_{0,+,\alpha(\cdot)}$ satisfying

$$\begin{aligned}\sum_{k=1}^{+\infty} a(k-1, \Delta u(k-1))\Delta v(k-1) + \sum_{k=1}^{+\infty} \alpha(k)\,|u(k)|^{p(k)-2}\,u(k)v(k)\\ = \sum_{k=1}^{+\infty} \delta(k) f(k, u(k)+1)v(k),\end{aligned} \tag{3.3}$$

for any $v \in \mathcal{W}^{1,p(\cdot)}_{0,+,\alpha(\cdot)}$.

Theorem 3.4. *Assume that* (H_1)–(H_9) *hold. Then, there exists at least one weak solution of* (3.2).

To prove Theorem 3.4, we first consider some auxiliary results.

The energy functional corresponding to problem (3.2) is defined by $J : \mathcal{W}^{1,p(\cdot)}_{0,+,\alpha(\cdot)} \to \mathbb{R}$ such that

$$J(u) = \sum_{k=1}^{+\infty} A(k-1, \Delta u(k-1)) + \sum_{k=1}^{+\infty} \frac{\alpha(k)}{p(k)}|u(k)|^{p(k)} - \sum_{k=1}^{+\infty} \delta(k) F(k, u(k)+1). \tag{3.4}$$

We first present some basic properties of J.

Proposition 3.5. *The functional J is well-defined on $\mathcal{W}^{1,p(\cdot)}_{0,+,\alpha(\cdot)}$ and is of class $C^1(\mathcal{W}^{1,p(\cdot)}_{0,+,\alpha(\cdot)}, \mathbb{R})$ with the derivative given by*

$$\begin{aligned}\langle J'(u), v\rangle = &\sum_{k=1}^{+\infty} a(k-1, \Delta u(k-1))\Delta v(k-1)\\ &+ \sum_{k=1}^{+\infty} \alpha(k)|u(k)|^{p(k)-2}u(k)v(k) - \sum_{k=1}^{+\infty} \delta(k) f(k, u(k)+1)v(k)\end{aligned} \tag{3.5}$$

for all $u, v \in \mathcal{W}^{1,p(\cdot)}_{0,+,\alpha(\cdot)}$.

Proof. Let $J(u) = I(u) + L(u) - \Lambda(u)$ with

$$I(u) = \sum_{k=1}^{+\infty} A(k-1, \Delta u(k-1)), \quad L(u) = \sum_{k=1}^{+\infty} \frac{\alpha(k)}{p(k)}|u(k)|^{p(k)}$$

and

$$\Lambda(u) = \sum_{k=1}^{+\infty} \delta(k) F(k, u(k)+1).$$

Then, by (H_4), we get

$$\begin{aligned}|I(u)| &= \left|\sum_{k=1}^{+\infty} A(k-1,\Delta u(k-1))\right| \leq \sum_{k=1}^{+\infty}|A(k-1,\Delta u(k-1))|\\ &\leq \sum_{k=1}^{+\infty} C_1\left(j(k-1)+\frac{1}{p(k-1)}|\Delta u(k-1)|^{p(k-1)-1}\right)|\Delta u(k-1)|\\ &\leq \sum_{k=1}^{+\infty} C_1 j(k-1)|\Delta u(k-1)| + \sum_{k=1}^{+\infty}\frac{C_1}{p(k-1)}|\Delta u(k-1)|^{p(k-1)} < +\infty.\end{aligned}$$

By (H_2) and (H_7), we obtain

$$|L(u)| = \left|\sum_{k=1}^{+\infty}\frac{\alpha(k)}{p(k)}|u(k)|^{p(k)}\right| \leq \frac{1}{p^-}\sum_{k=1}^{+\infty}\alpha(k)|u(k)|^{p(k)} < +\infty.$$

Owing to (H_6), we deduce that

$$\begin{aligned}|\Lambda(u)| &= \left|\sum_{k=1}^{+\infty}\delta(k)F(k,u(k)+1)\right| \leq \sum_{k=1}^{+\infty}|\delta(k)||F(k,u(k)+1)|\\ &\leq \sum_{k=1}^{+\infty}C_2'|\delta(k)|\left(1+|u(k)|^{p(k)}\right) \leq C_2'\sum_{k=1}^{+\infty}|\delta(k)| + C_2'\sum_{k=0}^{+\infty}|\delta(k)||u(k)|^{p(k)} < +\infty.\end{aligned}$$

Therefore, J is well-defined. Clearly I, L and Λ are in $C^1(\mathcal{W}^{1,p(\cdot)}_{0,+,\alpha(\cdot)},\mathbb{R})$. In what follows, we prove (3.5). Let us choose $u,v\in\mathcal{W}^{1,p(\cdot)}_{0,+,\alpha(\cdot)}$. We have

$$\langle I'(u),v\rangle = \lim_{\eta\to 0^+}\frac{I(u+\eta v)-I(u)}{\eta},\quad \langle L'(u),v\rangle = \lim_{\eta\to 0^+}\frac{L(u+\eta v)-L(u)}{\eta}$$

and

$$\langle \Lambda'(u),v\rangle = \lim_{\eta\to 0^+}\frac{\Lambda(u+\eta v)-\Lambda(u)}{\eta}.$$

Since

$$\begin{aligned}&\lim_{\eta\to 0^+}\frac{I(u+\eta v)-I(u)}{\eta}\\ &= \lim_{\eta\to 0^+}\sum_{k=1}^{+\infty}\frac{A(k-1,\Delta u(k-1)+\eta\Delta v(k-1))-A(k-1,\Delta u(k-1))}{\eta}\\ &= \sum_{k=1}^{+\infty}\lim_{\eta\to 0^+}\frac{A(k-1,\Delta u(k-1)+\eta\Delta v(k-1))-A(k-1,\Delta u(k-1))}{\eta}\\ &= \sum_{k=1}^{+\infty}a(k-1,\Delta u(k-1))\Delta v(k-1),\end{aligned}$$

according to (H_1),

$$\begin{aligned}\lim_{\eta\to 0^+}\frac{L(u+\eta v)-L(u)}{\eta} &= \lim_{\eta\to 0^+}\sum_{k=1}^{+\infty}\frac{\alpha(k)(|u(k)+\eta v(k)|^{p(k)}-|u(k)|^{p(k)})}{p(k)\eta}\\ &= \sum_{k=1}^{+\infty}\lim_{\eta\to 0^+}\frac{\alpha(k)(|u(k)+\eta v(k)|^{p(k)}-|u(k)|^{p(k)})}{p(k)\eta}\\ &= \sum_{k=1}^{+\infty}\alpha(k)|u(k)|^{p(k)-2}u(k)v(k)\end{aligned}$$

and

$$\begin{aligned}\lim_{\eta\to 0^+}\frac{\Lambda(u+\eta v)-\Lambda(u)}{\eta} &= \lim_{\eta\to 0^+}\sum_{k=1}^{+\infty}\frac{\delta(k)(F(k,u(k)+\eta v(k)+1)-F(k,u(k)+1))}{\eta}\\ &= \sum_{k=1}^{+\infty}\lim_{\eta\to 0^+}\frac{\delta(k)(F(k,u(k)+\eta v(k)+1)-F(k,u(k)+1))}{\eta}\\ &= \sum_{k=1}^{+\infty}\delta(k)f(k,u(k)+1)v(k),\end{aligned}$$

we obtain (3.5). □

Lemma 3.6. *The functional I is weakly lower semi-continuous.*

Proof. From (H_1) and (H_5), I is convex with respect to the second variable. Thus, it is enough to show that I is lower semi-continuous (see Corollary III.8 in [1]). For this, we fix $u\in\mathcal{W}^{1,p(\cdot)}_{0,+,\alpha(\cdot)}$ and $\epsilon>0$. Since I is convex, we deduce that

$$\begin{aligned}I(v) &\geq I(u)+\langle I'(u),v-u\rangle\\ &\geq I(u)+\sum_{k=1}^{+\infty}a(k-1,\Delta u(k-1))\,(\Delta v(k-1)-\Delta u(k-1))\\ &\geq I(u)-C\Big(\frac{1}{p^-}+\frac{1}{p'^-}\Big)\|\bar g\|_{p'_+(\cdot)}\|\Delta(u-v)\|_{p_+(\cdot)},\\ &\qquad \text{where } \bar g(k)=j(k-1)+|\Delta u(k-1)|^{p(k-1)-1}\\ &\geq I(u)-K\bigg(\|u-v\|_{\alpha(\cdot),p_+(\cdot)}+\|\Delta(u-v)\|_{p_+(\cdot)}\bigg)\\ &\geq I(u)-K\|u-v\|_{1,\alpha(\cdot),p_+(\cdot)}\\ &\geq I(u)-\epsilon\end{aligned}$$

for all $v\in\mathcal{W}^{1,p(\cdot)}_{0,+,\alpha(\cdot)}$ with $\|u-v\|_{1,\alpha(\cdot),p_+(\cdot)}<\xi=\frac{\epsilon}{K}$. Hence, we conclude that I is weakly lower semi-continuous. □

Proposition 3.7. *The functional J is bounded from below, coercive and weakly lower semi-continuous.*

Proof. By Lemma 3.6, since I is weakly lower semi-continuous, J is weakly lower semi-continuous. We will only prove the coerciveness of the energy functional since the boundedness from below of J arises from its coercivity. To prove the coerciveness of J, we may assume that $\|u\|_{1,\alpha(\cdot),p_+(\cdot)} > 1$. According to (H_2), (H_3), (H_6) and (H_7), we have

$$\begin{aligned}
J(u) &= \sum_{k=1}^{+\infty} A(k-1, \Delta u(k-1)) + \sum_{k=1}^{+\infty} \frac{\alpha(k)}{p(k)} |u(k)|^{p(k)} - \sum_{k=1}^{+\infty} \delta(k) F(k, u(k)+1) \\
&\geq \sum_{k=1}^{+\infty} \frac{1}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} + \sum_{k=1}^{+\infty} \frac{\alpha(k)}{p(k)} |u(k)|^{p(k)} - \sum_{k=1}^{+\infty} |\delta(k) F(k, u(k)+1)| \\
&\geq \frac{1}{p^+} \sum_{k=1}^{+\infty} |\Delta u(k-1)|^{p(k-1)} + \frac{1}{p^+} \sum_{k=1}^{+\infty} \alpha(k) |u(k)|^{p(k)} - \sum_{k=1}^{+\infty} C_2' |\delta(k)| \\
&\quad - \sum_{k=1}^{+\infty} C_2' |\delta(k)| |u(k)|^{p(k)} \\
&\geq \frac{1}{p^+} (\rho_{p_+(\cdot)}(\Delta u) + \rho_{\alpha(\cdot),p_+(\cdot)}(u)) - \frac{\bar{\delta} C_2'}{\alpha_0} \sum_{k=1}^{+\infty} \alpha(k) |u(k)|^{p(k)} - M \\
&\geq \frac{1}{p^+} \rho_{1,\alpha(\cdot),p_+(\cdot)}(u) - \frac{\bar{\delta} C_2'}{\alpha_0} \rho_{\alpha(\cdot),p_+(\cdot)}(u) - M \\
&\geq \left(\frac{1}{p^+} - \frac{\bar{\delta} C_2'}{\alpha_0} \right) \rho_{1,\alpha(\cdot),p_+(\cdot)}(u) - M \\
&\geq \left(\frac{1}{p^+} - \frac{\bar{\delta} C_2'}{\alpha_0} \right) \|u\|_{1,\alpha(\cdot),p_+(\cdot)}^{p^-} - M.
\end{aligned}$$

Therefore, by assumption (H_8), as $\|u\|_{1,\alpha(\cdot),p_+(\cdot)} \to +\infty$, then $J(u) \to +\infty$, i.e. J is coercive and so, there exists $c \in \mathbb{R}$ such that $J(u) \geq c$.

If $\|u\|_{1,\alpha(\cdot),p_+(\cdot)} \leq 1$, then

$$\begin{aligned}
J(u) &\geq \frac{1}{p^+} \rho_{1,\alpha(\cdot),p_+(\cdot)}(u) - \frac{\bar{\delta} C_2'}{\alpha_0} \rho_{\alpha(\cdot),p_+(\cdot)}(u) - M \\
&\geq -\frac{\bar{\delta} C_2'}{\alpha_0} \rho_{\alpha(\cdot),p(\cdot)}(u) - M \geq -M > -\infty.
\end{aligned}$$

Thus, J is bounded from below. □

We can now give the proof of Theorem 3.4.

Proof of Theorem 3.4. By Proposition 3.7, J has a minimizer which is a weak solution of (3.2). In order to complete the proof of Theorem 3.4, we will show that every weak solution u is homoclinic, i.e $u(k) \to 0$ as $k \to +\infty$. Let u be a weak solution of problem (3.2). Then, as $u \in W^{1,p(\cdot)}_{0,+,\alpha(\cdot)}$, we get

$$\sum_{k=1}^{+\infty} \alpha_0 |u(k)|^{p(k)} \le \sum_{k=1}^{+\infty} \alpha(k)|u(k)|^{p(k)} < +\infty.$$

Let $S_1 = \{k \in \mathbb{Z} : |u(k)| < 1\}$ and $S_2 = \{k \in \mathbb{Z} : |u(k)| \ge 1\}$. S_2 is a finite set, then

$$\sum_{k=1}^{+\infty} |u(k)|^{p(k)} = \sum_{k\in S_1} |u(k)|^{p(k)} + \sum_{k\in S_2} |u(k)|^{p(k)} \le \sum_{k\in S_1} |u(k)|^{p(k)} + R < +\infty.$$

As a consequence,

$$\sum_{k\in S_1} |u(k)|^{p^+} + R \le \sum_{k\in S_1} |u(k)|^{p(k)} + R.$$

Therefore, as S_2 is a finite set, we get

$$\sum_{k=1}^{+\infty} |u(k)|^{p^+} < +\infty.$$

Thus, $\lim_{k\to+\infty} |u(k)| = 0$, which completes the proof. □

Now, we consider the following problem:

$$\begin{cases} -\Delta(a(k-1, \Delta u(k-1))) + \alpha(k)\,|u(k)|^{p(k)-2}\,u(k) = \delta(k) f(k, u(k) - 1), & k \in \mathbb{Z}^-_*, \\ u(0) = 0, \ \ \lim_{k\to-\infty} u(k) = 0. \end{cases} \tag{3.6}$$

A weak solution of problem (3.6) is defined as follows.

Definition 3.8. A weak solution of (3.6) is a function $u \in \mathcal{W}^{1,p(\cdot)}_{0,-,\alpha(\cdot)}$ such that

$$\begin{aligned} &\sum_{k=-\infty}^{0} a(k-1, \Delta u(k-1))\Delta v(k-1) + \sum_{k=-\infty}^{0} \alpha(k)\,|u(k)|^{p(k)-2}\,u(k)v(k) \\ &= \sum_{k=-\infty}^{0} \delta(k) f(k, u(k) - 1) v(k) \end{aligned} \tag{3.7}$$

for any $v \in \mathcal{W}^{1,p(\cdot)}_{0,-,\alpha(\cdot)}$.

By mimicking the proof of Theorem 3.4, we prove the following result.

Theorem 3.9. *Assume that* (H_1)*–*(H_9) *hold. Then, there exists at least one weak solution of* (3.6).

Now, let us show the existence of weak heteroclinic solutions of problem (1.1).

Proof of Theorem 3.2. We define $v_1 = u_1 + 1$, where u_1 is a weak solution of problem (3.2) and $v_2 = u_2 - 1$, where u_2 is a weak solution of problem (3.6). Therefore, according to (H_6) and (H_9), we deduce that

$$u = v_1 \chi_{\mathbb{Z}^+} + v_2 \chi_{\mathbb{Z}^-}$$

is an heteroclinic solution of problem (1.1). □

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