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This work is dedicated to Professor Leon Mikołajczyk
on the occasion of his 85th birthday.

SUFFICIENT OPTIMALITY CRITERIA AND DUALITY FOR MULTIOBJECTIVE VARIATIONAL CONTROL PROBLEMS WITH B-(p,r)-INVEX FUNCTIONS

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Abstract. In this paper, we generalize the notion of B-(p,r)-invexity introduced by Antczak in [*A class of B-$(p;r)$-invex functions and mathematical programming*, J. Math. Anal. Appl. 286 (2003), 187–206] for scalar optimization problems to the case of a multiobjective variational programming control problem. For such nonconvex vector optimization problems, we prove sufficient optimality conditions under the assumptions that the functions constituting them are B-(p,r)-invex. Further, for the considered multiobjective variational control problem, its dual multiobjective variational control problem in the sense of Mond-Weir is given and several duality results are established under B-(p,r)-invexity.

Keywords: multiobjective variational control problems, efficient solution, B-(p,r)-invex functions, optimality conditions, duality.

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1. INTRODUCTION

Multiobjective variational control models are very prominent amongst constrained vector optimization models because of their occurrences in a variety of popular contexts, notably, industrial process control, impulsive control problems, production and inventory, epidemic, control of a rocket, control of space structures, and other diverse fields.

In recent years, there has been significant growth in the application of invexity theory in multiobjective programming which was originated by Hanson [15] for scalar optimization problems. Since that time, it has been shown that many results in multiobjective programming, previously established for convex functions, actually hold for the wider class of invex functions. After Hanson's work, other types of differentiable functions have appeared with the intent of generalizing invex functions from different points of view. One such generalization of invexity is the concept of (p, r)-invexity introduced by Antczak [1] for scalar optimization problems and extended to the vectorial case in [2]. In [3], Antczak generalized the definition of (p, r)-invexity and he introduced the definition of B-(p, r)-invexity for scalar constrained optimization problems.

The relationship between mathematical programming and classical calculus of variation was explored and extended by Hanson [14]. Thereafter variational control programming problems have attracted some attention in the literature. Optimality conditions and duality for multiobjective variational control problems have been of much interest in recent years, and several contributions have been made to its development (see, for example, [5–9, 13, 16–18, 21, 24, 26], and references here). Bhatia and Kumar [8] derived duality theorems for multiobjective control problems under generalized ρ-invexity assumptions. Nahak and Nanda [22] discussed the efficiency and duality for multiobjective variational control problems with (F, ρ)-convexity. In [9], Bhatia and Mehra extended the concepts of B-type I and generalized B-type I functions to the continuous case and they used these concepts to establish sufficient optimality conditions and duality results for multiobjective variational programming problems. Xiuhong [24] proved duality relations through a parametric approach to relate properly efficient solutions of multiobjective control problems under invexity assumptions. In [12], Gulati *et al.* established optimality conditions and duality results for multiobjective control problems involving generalized invex functions. Hachimi and Aghezzaf [13] obtained several mixed type duality results for multiobjective variational programming problems under the introduced concept of generalized type I functions. Nahak and Nanda [23] obtained sufficient optimality criteria and duality results for multiobjective variational control problems under V-invexity assumptions. In [16], Khazafi *et al.* introduced the classes of (B, ρ)-type I functions and generalized (B, ρ)-type I functions and derived a series of sufficient optimality conditions and mixed type duality results for multiobjective control problems. Arana-Jiménez *et al.* [5] provided new pseudoinvexity conditions on the involved functionals of a multiobjective variational problem such that all vector Kuhn-Tucker or Fritz John points are weakly efficient solutions if and only if these conditions are fulfilled. In [6], Arana-Jiménez *et al.* defined the V-KT-pseudoinvex multiobjective control problem and proved that a V-KT-pseudoinvex multiobjective control problem is characterized so that a Kuhn-Tucker point is an efficient solution. Recently, Arana-Jiménez *et al.* [7] introduced new classes of pseudoinvex functions and established a necessary and sufficient condition for duality results in the considered multiobjective control problem.

In this paper, we extend the definition of B-(p, r)-invexity introduced by Antczak [3] for differentiable scalar optimization problems to the continuous vectorial case. We prove sufficient optimality conditions for weakly efficient, efficient and properly efficient optimal solutions in the considered multiobjective variational control problem involved B-(p, r)-invex functions with respect to the same function η and, not necessarily, with respect to the same function b. Further, for the considered multiobjective variational control problem, we define its vector variational control dual problem in the sense of Mond-Weir. Then, we prove various duality results between the considered multiobjective variational control programming problem and its vector variational control dual problem under assumption that the functions constituting these vector optimization problems are B-(p,r)-invex.

2. MULTIOBJECTIVE VARIATIONAL CONTROL PROBLEM AND B-(p,r)-INVEXITY

In this section, we provide some definitions and some results that we shall use in the sequel. The following convention for equalities and inequalities will be used throughout the paper.

For any $x = (x_1, x_2, \ldots, x_n)^T$, $y = (y_1, y_2, \ldots, y_n)^T$, we define:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, \ldots, n$,
(ii) $x < y$ if and only if $x_i < y_i$ for all $i = 1, 2, \ldots, n$,
(iii) $x \leqq y$ if and only if $x_i \leqq y_i$ for all $i = 1, 2, \ldots, n$,
(iv) $x \leq y$ if and only if $x \leqq y$ and $x \neq y$.

Let $I = [a, b]$ be a real interval and let $K = \{1, 2, \ldots, k\}$, $J = \{1, 2, \ldots, m\}$.

In this paper, we assume $x(t)$ is an n-dimensional piecewise smooth function of t, and $\dot{x}(t)$ is the derivative of $x(t)$ with respect to t in $[a, b]$.

Denote by $C(I, R^n)$ the space of piecewise smooth functions $x : I \to R^n$ with norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where $\|\cdot\|_\infty$ is the uniform norm and the differentiation operator D is given by

$$z = Dx \Longleftrightarrow x(t) = x(a) + \int_a^t z(s)\, ds,$$

where $x(a)$ is a given boundary value. Therefore, $\frac{d}{dt} \equiv D$ except at discontinuities.

We consider the following multiobjective variational control problem in which the state vector $x(t)$ is brought from the specified initial state $x(a) = \alpha$ to some specified final state $x(b) = \beta$ in such a way to minimize a given functional. A more precise mathematical formulation is given in the following multiobjective variational control problem as follows:

$$\begin{aligned} &V\text{-Minimize} \int_a^b f\left(t, x(t), \dot{x}(t)\right) dt \\ &= \left(\int_a^b f^1\left(t, x(t), \dot{x}(t)\right) dt, \ldots, \int_a^b f^k\left(t, x(t), \dot{x}(t)\right) dt \right) \qquad \text{(MVP)} \\ &\text{subject to } g\left(t, x(t), \dot{x}(t)\right) \leqq 0, \ \ t \in I, \\ &x(a) = \alpha, \quad x(b) = \beta, \end{aligned}$$

where $f = \left(f^1, \ldots, f^k\right) : I \times R^n \times R^n \to R^k$ is a k-dimensional function and each of its components is a continuously differentiable real scalar function and $g : I \times R^n \times R^n \to R^m$ is assumed to be a continuously differentiable m-dimensional function.

For notational simplicity, we write $x(t)$ and $\dot{x}(t)$ as x and $\dot{x}$, respectively. We denote the partial derivatives of f^1 with respect to t, x and $\dot{x}$, respectively, by f_t^1, f_x^1, $f_{\dot{x}}^1$ such that $f_x^1 = \left(\frac{\partial f^1}{\partial x_1}, \ldots, \frac{\partial f^1}{\partial x_n}\right)$ and $f_{\dot{x}}^1 = \left(\frac{\partial f^1}{\partial \dot{x}_1}, \ldots, \frac{\partial f^1}{\partial \dot{x}_n}\right)$. Similarly, the partial derivatives of the vector function g can be written, using matrices with m rows instead of one, respectively.

Let Ω denote the set of all feasible points of (MVP), i.e.

$$\Omega = \{x \in C(I, R^n) : x(t) \text{ verifying the constraints of (MVP) for all } t \in I\}.$$

Definition 2.1. A solution $\overline{x} \in \Omega$ is said to be weakly efficient of (MVP) if there exists no other $x \in \Omega$ such that

$$\int_a^b f\left(t, x(t), \dot{x}(t)\right) dt < \int_a^b f\left(t, \overline{x}(t), \dot{\overline{x}}(t)\right) dt.$$

Definition 2.2. A solution $\overline{x} \in \Omega$ is said to be efficient of (MVP) if there exists no other $x \in \Omega$ such that

$$\int_a^b f\left(t, x(t), \dot{x}(t)\right) dt \leq \int_a^b f\left(t, \overline{x}(t), \dot{\overline{x}}(t)\right) dt.$$

It is known that some efficient solutions present an undesirable property with respect to the ratio between the marginal profit of an objective function and the loss of some other. The concept of proper efficiency given by Geoffrion [11] is a slightly restricted definition of efficiency which eliminates efficient points of a certain anomalous type.

Definition 2.3. A solution $\overline{x} \in \Omega$ is said to be properly efficient of (MVP) if there exists a scalar $M > 0$ such that, for each $i = 1, \ldots, p$, the following inequality

$$\int_a^b f^i\left(t, \overline{x}(t), \dot{\overline{x}}(t)\right) dt - \int_a^b f^i\left(t, x(t), \dot{x}(t)\right) dt$$
$$\leqq M\left(\int_a^b f^j\left(t, x(t), \dot{x}(t)\right) dt - \int_a^b f^j\left(t, \overline{x}(t), \dot{\overline{x}}(t)\right) dt\right)$$

holds for some j, satisfying $\int_a^b f^j(t, x(t), \dot{x}(t))dt > \int_a^b f^j(t, \overline{x}(t), \dot{\overline{x}}(t))dt$, whenever $x(t) \in \Omega$ and $\int_a^b f^i(t, x(t), \dot{x}(t))dt < \int_a^b f^i\left(t, \overline{x}(t), \dot{\overline{x}}(t)\right)dt$.

For notational convenience, we use $f^i(t,x,\dot{x})$ for $f^i(t,x(t),\dot{x}(t))$, x for $x(t)$ and $\dot{x}$ for $\dot{x}(t)$.

Now, we generalize the definition of a B-(p,r)-invex function introduced by Antczak [3] for scalar optimization problems to the continuous vectorial case.

Definition 2.4. Let $f : I\times R^n\times R^n \to R^k$ be a continuously differentiable function. If there exist real numbers p, r, a function $\eta : I\times R^n\times R^n \to R^n$ with $\eta\left(t,x(t),\overline{x}(t)\right)=0$ at t if $x(t)=\overline{x}(t)$ and a function $b_f : C(I,R^n)\times C(I,R^n)\to R^k$, where $b_{f_i} : C(I,R^n)\times C(I,R^n)\to R_+\backslash\{0\}$, $i=1,\dots,k$, such that, for all $x\in C(I,R^n)$ and $i=1,\dots,k$,

$$\begin{aligned}
&\frac{1}{r}b_{f_i}(x,\overline{x})\left(e^{r\left(\int_a^b f^i\left(t,x,\dot{x}\right)dt-\int_a^b f^i(t,\overline{x},\dot{\overline{x}})dt\right)}-1\right)\\
&\geqq \frac{1}{p}\int_a^b\left[\left(e^{p\eta(t,x,\overline{x})}-\mathbf{1}\right)\right]^T\left[f^i_x\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}f^i_{\dot{x}}\left(t,\overline{x},\dot{\overline{x}}\right)\right]dt \quad \text{if } p\neq 0, r\neq 0,\\
&\frac{1}{r}b_{f_i}(x,\overline{x})\left(e^{r\left(\int_a^b f^i(t,x,\dot{x})dt-\int_a^b f^i(t,\overline{x},\dot{\overline{x}})dt\right)}-1\right)\\
&\geqq \int_a^b\left[\eta\left(t,x,\overline{x}\right)\right]^T\left[f^i_x\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}f^i_{\dot{x}}\left(t,\overline{x},\dot{\overline{x}}\right)\right]dt \quad \text{if } p=0, r\neq 0,\\
&b_{f_i}(x,\overline{x})\left(\int_a^b f^i\left(t,x,\dot{x}\right)dt-\int_a^b f^i\left(t,\overline{x},\dot{\overline{x}}\right)dt\right)\\
&\geqq \frac{1}{p}\int_a^b\left[\left(e^{p\eta(t,x,\overline{x})}-\mathbf{1}\right)\right]^T\left[f^i_x\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}f^i_{\dot{x}}\left(t,\overline{x},\dot{\overline{x}}\right)\right]dt \quad \text{if } p\neq 0, r=0,\\
&b_{f_i}(x,\overline{x})\left(\int_a^b f^i\left(t,x,\dot{x}\right)dt-\int_a^b f^i\left(t,\overline{x},\dot{\overline{x}}\right)dt\right)\\
&\geqq \int_a^b\left[\eta\left(t,x,\overline{x}\right)\right]^T\left[f^i_x\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}f^i_{\dot{x}}\left(t,\overline{x},\dot{\overline{x}}\right)\right]dt \quad \text{if } p=0, r=0,
\end{aligned} \tag{2.1}$$

then f is said to be a B_f-(p,r)-invex function with respect to η at $\overline{x}$ on $C(I,R^n)$. Further, every function f_i, $i=1,\dots,k$, satisfying (2.1) is said to be b_{f_i}-(p,r)-invex with respect to η at $\overline{x}$ on $C(I,R^n)$. If the inequality (2.1) is satisfied at each point $\overline{x}\in C(I,R^n)$, then f_i, $i=1,\dots,k$, is said to be b_{f_i}-(p,r)-invex with respect to η on $C(I,R^n)$.

Remark 2.5. It should be pointed out that the exponentials appearing in inequalities (2.1) are understood to be taken componentwise and $\mathbf{1} = (1, 1, \ldots, 1) \in R^n$.

Definition 2.6. If the inequalities (2.1) are strict, then f is said to be a strictly B_f-(p,r)-invex function with respect to η at $\overline{x}$ on $C(I, R^n)$ and every function f_i, $i = 1, \ldots, k$, is said to be strictly b_{f_i}-(p,r)-invex with respect to η at $\overline{x}$ on $C(I, R^n)$.

3. OPTIMALITY CONDITIONS

In this section, for the considered multiobjective continuous programming problem (MVP), we prove the sufficient optimality conditions for weak efficiency, efficiency and properly efficiency under assumptions that the functions constituting it are B-(p,r)-invex (with respect to the same function η and with respect to, not necessarily, the same function b).

Theorem 3.1. *Let $\overline{x}$ be a feasible solution in the considered multiobjective continuous programming problem* (MVP)*. Assume that there exist $\overline{\lambda} \in R^k$ and a piecewise smooth function $\overline{\xi}(\cdot) : I \to R^m$ such that the following conditions*

$$\begin{aligned} &\overline{\lambda}^T f_x\left(t, \overline{x}, \dot{\overline{x}}\right) + \overline{\xi}(t)^T g_x\left(t, \overline{x}, \dot{\overline{x}}\right) \\ &= \frac{d}{dt}\left[\overline{\lambda}^T f_{\dot{x}}\left(t, \overline{x}, \dot{\overline{x}}\right) + \overline{\xi}(t)^T g_{\dot{x}}\left(t, \overline{x}, \dot{\overline{x}}\right)\right], \quad t \in I, \end{aligned} \tag{3.1}$$

$$\overline{\xi}_j(t) \int_a^b g^j\left(t, \overline{x}, \dot{\overline{x}}\right) dt = 0, \quad t \in I, j = 1, \ldots, m, \tag{3.2}$$

$$\overline{\lambda} \geq 0, \quad \overline{\lambda}^T e = 1, \quad \overline{\xi}(t) \geqq 0 \tag{3.3}$$

hold, where $e = (1, \ldots, 1) \in R^k$. Further, assume that f is strictly B_f-(p,r)-invex at $\overline{x}$ on Ω with respect to η and $\overline{\xi}(t)^T g$ is B_g-(p,r)-invex at $\overline{x}$ on Ω with respect to η. Then $\overline{x}$ is an efficient solution in (MVP).

Proof. Suppose, contrary to the result, that $\overline{x} \in \Omega$ is not an efficient solution in (MVP). Hence, there exists $\widetilde{x} \in \Omega$ such that

$$\int_a^b f\left(t, \widetilde{x}, \dot{\widetilde{x}}\right) dt \leq \int_a^b f\left(t, \overline{x}, \dot{\overline{x}}\right) dt. \tag{3.4}$$

This means that

$$\int_a^b f^i\left(t, \widetilde{x}, \dot{\widetilde{x}}\right) dt \leqq \int_a^b f^i\left(t, \overline{x}, \dot{\overline{x}}\right) dt, \quad i = 1, \ldots, k \tag{3.5}$$

and

$$\int_a^b f^{i^*}\left(t,\widetilde{x},\dot{\widetilde{x}}\right)dt < \int_a^b f^{i^*}\left(t,\overline{x},\dot{\overline{x}}\right)dt \quad \text{for some } i^* \in K. \tag{3.6}$$

By assumption, f is strictly B_f-(p,r)-invex at $\overline{x}$ on Ω with respect to η and $\overline{\xi}(t)^T g$ is B_g-(p,r)-invex at $\overline{x}$ on Ω with respect to η. Then, by Definition 2.4, the following inequalities are satisfied

$$\begin{aligned}&\frac{1}{r}b_{f_i}(\widetilde{x},\overline{x})\left(e^{r\left(\int_a^b f^i\left(t,\widetilde{x},\dot{\widetilde{x}}\right)dt-\int_a^b f^i\left(t,\overline{x},\dot{\overline{x}}\right)dt\right)}-1\right)\\&>\frac{1}{p}\int_a^b\left[\left(e^{p\eta(t,\widetilde{x},\overline{x})}-\mathbf{1}\right)\right]^T\left[f_x^i\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}f_{\dot{x}}^i\left(t,\overline{x},\dot{\overline{x}}\right)\right]dt, \quad i=1,\ldots,k,\end{aligned} \tag{3.7}$$

$$\begin{aligned}&\frac{1}{r}b_{g_j}(\widetilde{x},\overline{x})\left(e^{r\left(\int_a^b \overline{\xi}_j(t)g^j\left(t,\widetilde{x},\dot{\widetilde{x}}\right)dt-\int_a^b \overline{\xi}_j(t)g^j\left(t,\overline{x},\dot{\overline{x}}\right)dt\right)}-1\right)\\&\geqq\frac{1}{p}\int_a^b\left[\left(e^{p\eta(t,\widetilde{x},\overline{x})}-\mathbf{1}\right)\right]^T\overline{\xi}_j(t)\left[g_x^j\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}g_{\dot{x}}^j\left(t,\overline{x},\dot{\overline{x}}\right)\right]dt, \quad j=1,\ldots,m.\end{aligned} \tag{3.8}$$

Since $b_{f_i}(\widetilde{x},\overline{x})>0$, $i=1,\ldots,k$, combining (3.5), (3.6) and (3.7), we obtain

$$\frac{1}{p}\int_a^b\left[\left(e^{p\eta(t,\widetilde{x},\overline{x})}-\mathbf{1}\right)\right]^T\left[f_x^i\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}f_{\dot{x}}^i\left(t,\overline{x},\dot{\overline{x}}\right)\right]dt<0, \quad i\in K. \tag{3.9}$$

Multiplying each inequality (3.9) by $\overline{\lambda}_i$, where $\overline{\lambda}=\left(\overline{\lambda}_1,\ldots,\overline{\lambda}_k\right)\geq 0$, and then adding both sides of the obtained inequalities, we get

$$\frac{1}{p}\sum_{i=1}^{k}\int_a^b\left[\left(e^{p\eta(t,\widetilde{x},\overline{x})}-\mathbf{1}\right)\right]^T\overline{\lambda}_i\left[f_x^i\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}f_{\dot{x}}^i\left(t,\overline{x},\dot{\overline{x}}\right)\right]dt<0. \tag{3.10}$$

Taking into account that $\widetilde{x}\in\Omega$ and $b_{g_j}(\widetilde{x},\overline{x})>0$, $j=1,\ldots,m$, by (3.2) and (3.8), it follows that

$$\frac{1}{p}\int_a^b\left[\left(e^{p\eta(t,\widetilde{x},\overline{x})}-\mathbf{1}\right)\right]^T\overline{\xi}_j(t)\left[g_x^j\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}g_{\dot{x}}^j\left(t,\overline{x},\dot{\overline{x}}\right)\right]dt\leqq 0, \quad j=1,\ldots,m. \tag{3.11}$$

Adding both sides of the inequalities above, we get

$$\frac{1}{p}\sum_{j=1}^{m}\int_a^b\left[\left(e^{p\eta(t,\widetilde{x},\overline{x})}-\mathbf{1}\right)\right]^T\overline{\xi}_j(t)\left[g_x^j\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}g_{\dot{x}}^j\left(t,\overline{x},\dot{\overline{x}}\right)\right]dt\leqq 0. \tag{3.12}$$

By (3.10) and (3.12), it follows that

$$\frac{1}{p}\sum_{i=1}^{k}\int_a^b \left[\left(e^{p\eta(t,\widetilde{x},\overline{x})}-\mathbf{1}\right)\right]^T \overline{\lambda}_i \left[f_x^i\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}f_{\dot{x}}^i\left(t,\overline{x},\dot{\overline{x}}\right)\right]dt$$
$$+\frac{1}{p}\sum_{j=1}^{m}\int_a^b \left[\left(e^{p\eta(t,\widetilde{x},\overline{x})}-\mathbf{1}\right)\right]^T \overline{\xi}_j(t)\left[g_x^j\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}g_{\dot{x}}^j\left(t,\overline{x},\dot{\overline{x}}\right)\right]dt<0.$$

Thus, we obtain the following inequality

$$\frac{1}{p}\int_a^b \left[\left(e^{p\eta(t,\widetilde{x},\overline{x})}-\mathbf{1}\right)\right]^T \left[\overline{\lambda}^T f_x(t,\overline{x},\dot{\overline{x}})+\overline{\xi}(t)^T g_x(t,\overline{x},\dot{\overline{x}})\right.$$
$$\left.-\frac{d}{dt}\left(\overline{\lambda}^T f_{\dot{x}}(t,\overline{x},\dot{\overline{x}})-\overline{\xi}(t)^T g_{\dot{x}}(t,\overline{x},\dot{\overline{x}})\right)\right]dt<0,$$

contradicting (3.1). Thus, $\overline{x}$ is an efficient solution in (MVP) and the proof is complete. □

In order to prove that $\overline{x}\in\Omega$ is a weakly efficient solution in the multiobjective variational programming problem (MVP), weaker B-(p,r)-invexity hypotheses are needed.

Theorem 3.2. *Let $\overline{x}$ be a feasible solution in the considered multiobjective continuous programming problem* (MVP). *Assume that there exist $\overline{\lambda}\in R^k$ and a piecewise smooth function $\overline{\xi}(\cdot):I\to R^r$ such that the conditions* (3.1)–(3.3) *are satisfied. Further, assume that f is B_f-(p,r)-invex at $\overline{x}$ on Ω with respect to η and $\overline{\xi}(t)^T g$ is B_g-(p,r)-invex at $\overline{x}$ on Ω with respect to η. Then $\overline{x}$ is a weakly efficient solution in* (MVP).

Proof. The proof of this theorem is similar to the proof of Theorem 3.1. □

Theorem 3.3. *Assume that all hypotheses of Theorem 3.1 are fulfilled. If $\overline{\lambda}>0$, then $\overline{x}$ a properly efficient solution in* (MVP).

Proof. Since all hypotheses of Theorem 3.1 are fulfilled, $\overline{x}$ is efficient in problem (MVP).

Now, we prove that $\overline{x}$ is a properly efficient solution in problem (MVP). Suppose, contrary to the result, that $\overline{x}$ is not a properly efficient solution in problem (MVP). If we assume that $p\geqq 2$, then we choose

$$M=(k-1)\max\left\{\frac{\overline{\lambda}_q}{\overline{\lambda}_i}:i,q\in P,\ i\neq q\right\}. \tag{3.13}$$

Then, there exist $\widetilde{x}\in\Omega$ and $i\in P$, such that

$$\int_a^b f^i\left(t,\widetilde{x}(t),\dot{\widetilde{x}}(t)\right)dt<\int_a^b f^i\left(t,\overline{x}(t),\dot{\overline{x}}(t)\right)dt$$

and

$$\frac{\int\limits_a^b f^i\left(t,\overline{x}(t),\dot{\overline{x}}(t)\right)dt-\int\limits_a^b f^i\left(t,\widetilde{x}(t),\dot{\widetilde{x}}(t)\right)dt}{\int\limits_a^b f^q\left(t,\widetilde{x}(t),\dot{\widetilde{x}}(t)\right)dt-\int\limits_a^b f^q\left(t,\overline{x}(t),\dot{\overline{x}}(t)\right)dt}>M \tag{3.14}$$

for each $q\neq i$ such that

$$\int\limits_a^b f^q\left(t,\widetilde{x}(t),\dot{\widetilde{x}}(t)\right)dt>\int\limits_a^b f^q\left(t,\overline{x}(t),\dot{\overline{x}}(t)\right)dt.$$

Hence, for each $q\neq i$, (3.13) and (3.14) yield

$$\begin{aligned}&\int\limits_a^b f^i\left(t,\overline{x}(t),\dot{\overline{x}}(t)\right)dt-\int\limits_a^b f^i\left(t,\widetilde{x}(t),\dot{\widetilde{x}}(t)\right)dt\\&>(k-1)\max_{i,q\in P,i\neq q}\frac{\overline{\lambda}_q}{\overline{\lambda}_i}\left(\int\limits_a^b f^q\left(t,\widetilde{x}(t),\dot{\widetilde{x}}(t)\right)dt-\int\limits_a^b f^q\left(t,\overline{x}(t),\dot{\overline{x}}(t)\right)dt\right).\end{aligned}\tag{3.15}$$

Since $\int_a^b f^q\left(t,\widetilde{x}(t),\dot{\widetilde{x}}(t)\right)dt>\int_a^b f^q\left(t,\overline{x}(t),\dot{\overline{x}}(t)\right)dt$, the inequality (3.15) gives

$$\begin{aligned}&(k-1)\max_{i,q\in P,i\neq q}\frac{\overline{\lambda}_q}{\overline{\lambda}_i}\left(\int\limits_a^b f^q\left(t,\widetilde{x}(t),\dot{\widetilde{x}}(t)\right)dt-\int\limits_a^b f^q\left(t,\overline{x}(t),\dot{\overline{x}}(t)\right)dt\right)\\&\geqq(k-1)\frac{\overline{\lambda}_q}{\overline{\lambda}_i}\left(\int\limits_a^b f^q\left(t,\widetilde{x}(t),\dot{\widetilde{x}}(t)\right)dt-\int\limits_a^b f^q\left(t,\overline{x}(t),\dot{\overline{x}}(t)\right)dt\right).\end{aligned}\tag{3.16}$$

Combining (3.15) and (3.16), we get

$$\begin{aligned}&\overline{\lambda}_i\left(\int\limits_a^b f^i\left(t,\overline{x}(t),\dot{\overline{x}}(t)\right)dt-\int\limits_a^b f^i\left(t,\widetilde{x}(t),\dot{\widetilde{x}}(t)\right)dt\right)\\&>(k-1)\,\overline{\lambda}_q\left(\int\limits_a^b f^q\left(t,\widetilde{x}(t),\dot{\widetilde{x}}(t)\right)dt-\int\limits_a^b f^q\left(t,\overline{x}(t),\dot{\overline{x}}(t)\right)dt\right).\end{aligned}$$

Summing over $q \neq i$ both sides of the inequalities above, we obtain

$$\overline{\lambda}_i \left(\int_a^b f^i \left(t, \overline{x}(t), \dot{\overline{x}}(t) \right) dt - \int_a^b f^i \left(t, \widetilde{x}(t), \dot{\widetilde{x}}(t) \right) dt \right)$$

$$> \sum_{q \neq i} \overline{\lambda}_q \left(\int_a^b f^q \left(t, \widetilde{x}(t), \dot{\widetilde{x}}(t) \right) dt - \int_a^b f^q \left(t, \overline{x}(t), \dot{\overline{x}}(t) \right) dt \right).$$

Thus, the following inequality

$$\overline{\lambda}_i \int_a^b f^i \left(t, \overline{x}(t), \dot{\overline{x}}(t) \right) dt + \sum_{q \neq i} \overline{\lambda}_q \int_a^b f^q \left(t, \overline{x}(t), \dot{\overline{x}}(t) \right) dt$$

$$> \overline{\lambda}_i \int_a^b f^i \left(t, \widetilde{x}(t), \dot{\widetilde{x}}(t) \right) dt + \sum_{q \neq i} \overline{\lambda}_q \int_a^b f^q \left(t, \widetilde{x}(t), \dot{\widetilde{x}}(t) \right) dt$$

holds, contradicting the efficiency of $\overline{x}$ in (MVP). Thus, $\overline{x}$ is a properly efficient solution in the considered multiobjective continuous programming problem (MVP) and the proof is complete. $\square$

4. DUALITY

In this section, for the considered multiobjective variational control problem (MVP), we define its vector variational control dual problem in the sense of Mond-Weir (MWDP). We prove various duality results between (MVP) and (MWDP) under suitable B-(p, r)-invex hypotheses.

Consider the following vector variational control dual problem in the sense of Mond-Weir:

$$\begin{aligned}
&V\text{-Minimize} \int_a^b f \left(t, y(t), \dot{y}(t) \right) dt \\
&= \left(\int_a^b f^1 \left(t, y(t), \dot{y}(t) \right) dt, \ldots, \int_a^b f^k \left(t, y(t), \dot{y}(t) \right) dt \right), \\
&\sum_{i=1}^p \lambda_i f_y^i \left(t, y(t), \dot{y}(t) \right) + \sum_{j=1}^m \xi_j(t) g_y^j \left(y(t), \dot{y}(t) \right) \\
&= \frac{d}{dt} \left[\sum_{i=1}^p \lambda_i f_{\dot{y}}^i \left(t, y(t), \dot{y}(t) \right) + \sum_{j=1}^m \xi_j(t) g_{\dot{y}}^j \left(y(t), \dot{y}(t) \right) \right], \\
&\text{subject to} \int_a^b \xi_j(t) g^j \left(t, y(t), \dot{y}(t) \right) dt \geqq 0, \quad t \in I, j \in J,
\end{aligned} \tag{MWDP}$$

$$\lambda \in R^k, \quad \lambda \geq 0, \quad \xi(t) \in R^m, \quad \xi(t) \geqq 0, \quad y(a) = \alpha \ , \ y(b) = \beta,$$

where $f=\left(f^1,\ldots,f^k\right): I\times R^n\times R^n\to R^k$ is a k-dimensional function and each of its components is a continuously differentiable real scalar function and $g=\left(g^1,\ldots,g^m\right):$ $I\times R^n\times R^n\to R^m$ is assumed to be a continuously differentiable m-dimensional function.

Let W denote the set of all feasible solutions in (MWDP), that is, the set

$$W=\Big\{(y,\lambda,\xi): y(t)\in C(I,R^n),\ \lambda\in R^k,\ \xi(t)\in R^m \\ \text{verifying the constraints of (MWDP) for all } t\in I\Big\}$$

and $Y=\Omega\cup pr_{C(I,R^n)}W$.

Theorem 4.1 (Weak duality)**.** *Let x and (y,λ,ξ) be any arbitrary feasible solutions in* (MVP) *and* (MWDP)*, respectively. Further, assume that f is strictly B_f-(p,r)-invex at y on Y with respect to η and $\xi(t)^Tg$ is B_g-(p,r)-invex at y on Y with respect to η. Then the following inequality cannot hold:*

$$\int_a^b f\left(t,x(t),\dot{x}(t)\right)dt\le\int_a^b f\left(t,y(t),\dot{y}(t)\right)dt. \tag{4.1}$$

Proof. From the feasibility of x in (MVP) and the feasibility of (y,λ,ξ) in problem (MWDP), it follows that

$$\int_a^b \xi_j(t)g\left(t,y(t),\dot{y}(t)\right)dt\geqq 0,\quad t\in I,\ j=1,\ldots,m, \tag{4.2}$$

$$\int_a^b \xi_j(t)g\left(t,x(t),\dot{x}(t)\right)dt\leqq 0,\quad t\in I,\ j=1,\ldots,m. \tag{4.3}$$

By assumption, $\overline{\xi}(t)^Tg$ is B_g-(p,r)-invex at y on Y with respect to η. As it follows from Definition 2.4, $b_{g_j}(x,y)>0$, $j=1,\ldots,m$. Hence, (4.2) and (4.3) yield

$$\frac{1}{r}b_{g_j}(x,y)\left(e^{r\left(\int_a^b \xi_j(t)g^j\left(t,x,\dot{x}\right)dt-\int_a^b \xi_j(t)g^j\left(t,y,\dot{y}\right)dt\right)}-1\right)\leqq 0. \tag{4.4}$$

Hence, by Definition 2.4, the inequality (4.4) implies

$$\frac{1}{p}\int_a^b\left[\left(e^{p\eta(t,x,y)}-\mathbf{1}\right)\right]^T\sum_{j=1}^m\overline{\xi}_j(t)\left[g_y^j\left(t,y,\dot{y}\right)-\frac{d}{dt}g_{\dot{y}}^j\left(t,y,\dot{y}\right)\right]dt\leqq 0. \tag{4.5}$$

By assumption, f is B_f-(p,r)-invex at y on Y with respect to η. Hence, by Definition 2.4, we have

$$\frac{1}{r}b_{f_i}(x,y)\left(e^{r\left(\int_a^b f^i\left(t,x,\dot{x}\right)dt-\int_a^b f^i\left(t,y,\dot{y}\right)dt\right)}-1\right)>\frac{1}{p}\int_a^b\left[\left(e^{p\eta(t,x,y)}-\mathbf{1}\right)\right]^T \tag{4.6}$$
$$\cdot\left[f^i_y\left(t,y,\dot{y}\right)-\tfrac{d}{dt}f^i_{\dot{y}}\left(t,y,\dot{y}\right)\right]dt,\quad i=1,\ldots,k.$$

Suppose, contrary to the result, that (4.1) holds. Since $b_{f_i}(x,y)>0$, $i=1,\ldots,k$, the inequalities (4.1) and (4.6) yield

$$\frac{1}{p}\int_a^b\left[\left(e^{p\eta(t,x,y)}-\mathbf{1}\right)\right]^T\left[f^i_y\left(t,y,\dot{y}\right)-\frac{d}{dt}f^i_{\dot{y}}\left(t,y,\dot{y}\right)\right]dt<0,\quad i=1,\ldots,k. \tag{4.7}$$

Multiplying each inequality (4.7) by λ_i, where $\lambda=(\lambda_1,\ldots,\lambda_k)\geq 0$, and then adding both sides of the obtained inequalities, we get

$$\frac{1}{p}\int_a^b\left[\left(e^{p\eta(t,x,y)}-\mathbf{1}\right)\right]^T\sum_{i=1}^k\lambda_i\left[f^i_y\left(t,y,\dot{y}\right)-\frac{d}{dt}f^i_{\dot{y}}\left(t,y,\dot{y}\right)\right]dt<0. \tag{4.8}$$

Adding both sides of inequalities (4.5) and (4.8), we obtain

$$\frac{1}{p}\int_a^b\left[\left(e^{p\eta(t,x,y)}-\mathbf{1}\right)\right]^T\sum_{i=1}^k\lambda_i\left[f^i_y\left(t,y,\dot{y}\right)-\frac{d}{dt}f^i_{\dot{y}}\left(t,y,\dot{y}\right)\right]dt$$
$$+\frac{1}{p}\int_a^b\left[\left(e^{p\eta(t,x,y)}-\mathbf{1}\right)\right]^T\sum_{j=1}^m\xi_j(t)\left[g^j_y\left(t,y,\dot{y}\right)-\frac{d}{dt}g^j_{\dot{y}}\left(t,y,\dot{y}\right)\right]dt<0.$$

Hence, the following inequality

$$\frac{1}{p}\int_a^b\left[\left(e^{p\eta(t,x,y)}-\mathbf{1}\right)\right]^T\left(\sum_{i=1}^k\lambda_i\Big[f^i_y(t,y,\dot{y})-\frac{d}{dt}f^i_{\dot{y}}(t,y,\dot{y})\Big]\right.$$
$$\left.+\sum_{j=1}^m\xi_j(t)\Big[g^j_y(t,y,\dot{y})-\frac{d}{dt}g^j_{\dot{y}}(t,y,\dot{y})\Big]\right)dt<0,$$

contradicting the feasibility of (y,λ,ξ) in (MWDP). Thus, the proof is completed. $\square$

If we impose weaker hypotheses of B_f-(p,r)-invexity on the objective function, then the following weaker result is true.

Theorem 4.2 (Weak duality). *Let x and (y,λ,ξ) be any arbitrary feasible solutions in* (MVP) *and* (MWDP), *respectively. Further, assume that f is B_f-(p,r)-invex at y on Y with respect to η and $\xi(t)^T g$ is B_g-(p,r)-invex at y on Y with respect to η. Then the following inequality cannot hold:*

$$\int_a^b f\left(t,x(t),\dot{x}(t)\right)dt < \int_a^b f\left(t,y(t),\dot{y}(t)\right)dt. \tag{4.9}$$

In order to prove the strong duality theorem we will invoke the following lemma due to Chankong and Haimes [10].

Proposition 4.3. *A point $\overline{x}(t)\in\Omega$ is an efficient solution (a weakly efficient solution) for* (MVP) *if and only if, for every $i=1,\ldots,k$, $\overline{x}(t)$ solves*

$$\begin{gathered} \text{Minimize } \int_a^b f^i\left(t,x(t),\dot{x}(t)\right)dt \\ \text{subject to } g\left(t,x(t),\dot{x}(t)\right) \leqq 0,\ t\in I, \\ x(a)=\alpha,\quad x(b)=\beta, \\ \int_a^b f^j\left(t,x(t),\dot{x}(t)\right)dt \leqq \int_a^b f^j\left(t,\overline{x}(t),\dot{\overline{x}}(t)\right)dt,\quad j=1,\ldots,k,\ j\neq i. \end{gathered} \tag{$P_i(\overline{x})$}$$

Theorem 4.4 (Strong duality). *Let $\overline{x}\in\Omega$ be an efficient solution in* (MVP) *and, moreover, a suitable constraint qualification for* $(\mathrm{P}_i(\overline{x}))$ *be satisfied. Then there exist $\overline{\lambda}\in R^k$ and a piecewise smooth function $\overline{\xi}(\cdot): I\to R^m$ such that the conditions* (3.1)–(3.3) *are satisfied. Further, if all hypotheses of the weak duality theorem are fulfilled, then $\left(\overline{x},\overline{\lambda},\overline{\xi}\right)$ is an efficient solution (a weakly efficient solution) in* (MWDP).

Proof. Since $\overline{x}\in\Omega$ is an efficient solution in (MVP), by Proposition 4.3, $\overline{x}$ solves $(\mathrm{P}_i(\overline{x}))$ for every $i=1,\ldots,k$. Thus, by the necessary optimality conditions for each problem $(\mathrm{P}_i(\overline{x}))$, we get that $\lambda_j^i \geqq 0$ for all $j\neq i$, and $\xi^i(\cdot)\in R^m$, $\xi^i(\cdot)\geqq 0$ such that, for $i=1,\ldots,k$,

$$\begin{gathered} f_x^i\left(t,\overline{x},\dot{\overline{x}}\right) - \tfrac{d}{dt} f_{\dot{x}}^i\left(t,\overline{x},\dot{\overline{x}}\right) + \textstyle\sum_{j\neq i}\overline{\lambda}_j^i\left[f_x^j\left(t,\overline{x},\dot{\overline{x}}\right) - \tfrac{d}{dt} f_{\dot{x}}^j\left(t,\overline{x},\dot{\overline{x}}\right)\right] \\ + \textstyle\sum_{j=1}^m \xi_j^i(t)\left[g_x^j\left(t,\overline{x},\dot{\overline{x}}\right) - \tfrac{d}{dt} g_{\dot{x}}^j\left(t,\overline{x},\dot{\overline{x}}\right)\right] = 0,\quad t\in I, \end{gathered} \tag{4.10}$$

$$\int_a^b \overline{\xi}_j(t) g^j\left(t,\overline{x},\dot{\overline{x}}\right)dt = 0,\ t\in I,\quad j=1,\ldots,m. \tag{4.11}$$

Adding both sides of the inequalities (4.10), we get

$$\begin{aligned}
&\left(1+\lambda_2^1+\ldots+\lambda_k^1\right)\left[f_x^1\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}f_{\dot{x}}^1\left(t,\overline{x},\dot{\overline{x}}\right)\right]\\
&+\sum_{j=1}^m\xi_j^1(t)\left[g_x^j\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}g_{\dot{x}}^j\left(t,\overline{x},\dot{\overline{x}}\right)\right]\\
&+\left(\lambda_1^2+1+\ldots+\lambda_k^2\right)\left[f_x^2\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}f_{\dot{x}}^2\left(t,\overline{x},\dot{\overline{x}}\right)\right]\\
&+\sum_{j=1}^m\xi_j^2(t)\left[g_x^j\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}g_{\dot{x}}^j\left(t,\overline{x},\dot{\overline{x}}\right)\right]+\ldots\\
&+\left(\lambda_1^k+\lambda_2^k+\ldots+1\right)\left[f_x^k\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}f_{\dot{x}}^k\left(t,\overline{x},\dot{\overline{x}}\right)\right]\\
&+\sum_{j=1}^m\xi_j^k(t)\left[g_x^j\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}g_{\dot{x}}^j\left(t,\overline{x},\dot{\overline{x}}\right)\right]=0.
\end{aligned}$$

We introduce the following notations: $\overline{\lambda}_1=1+\lambda_2^1+\ldots+\lambda_k^1$, $\overline{\lambda}_2=\lambda_1^2+1+\ldots+\lambda_k^2$, $\ldots$, $\overline{\lambda}_k=\lambda_1^k+\lambda_2^k+\ldots+1$, $\sum_{k=1}^m\xi_j^k(t)=\overline{\xi}_j(t)$, $j\in J$. Thus, we have

$$\begin{aligned}
&\sum_{i=1}^k\overline{\lambda}_i\left[f_x^i\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}f_{\dot{x}}^i\left(t,\overline{x},\dot{\overline{x}}\right)\right]\\
&+\sum_{j=1}^m\overline{\xi}_j(t)\left[g_x\left(t,\overline{x},\dot{\overline{x}}\right)-\frac{d}{dt}g_{\dot{x}}^j\left(t,\overline{x},\dot{\overline{x}}\right)\right]=0,\quad t\in I.
\end{aligned}\tag{4.12}$$

By (4.11) and (4.12), it follows that $\left(\overline{x},\overline{\lambda},\overline{\xi}\right)$ is feasible in the vector variational control dual problem (MWDP). Hence, the efficiency of $\left(\overline{x},\overline{\lambda},\overline{\xi}\right)$ follows from weak duality – Theorem 4.1 (the weak efficiency follows from Theorem 4.2). $\square$

Theorem 4.5 (Strict converse duality)**.** *Let $\overline{x}$ and $\left(\overline{y},\overline{\lambda},\overline{\xi}\right)$ be feasible solutions in the vector variational control problems* (MVP) *and* (MWDP)*, respectively, such that*

$$\int_a^b\overline{\lambda}_if^i\left(t,\overline{x},\dot{\overline{x}}\right)dt=\int_a^b\overline{\lambda}_if^i\left(t,\overline{y},\dot{\overline{y}}\right)dt,\quad i=1,\ldots,k.\tag{4.13}$$

Further, assume that $\overline{\lambda}^Tf$ is strictly B_f-(p,r)-invex at $\overline{y}$ on Y with respect to η and $\overline{\xi}(t)^Tg$ is B_g-(p,r)-invex at $\overline{x}$ on Y with respect to η. Then $\overline{x}=\overline{y}$.

Proof. Suppose, contrary to the result, that $\overline{x} \neq \overline{y}$. By assumption, f is strictly B_f-(p,r)-invex at $\overline{y}$ on Y with respect to η and $\overline{\xi}(t)^T g$ is B_g-(p,r)-invex at $\overline{y}$ on Y with respect to η. Then, by Definition 2.4, the following inequalities

$$\begin{aligned} &\frac{1}{r} b_{f_i}(\overline{x},\overline{y}) \left(e^{r\left(\int_a^b \overline{\lambda}_i f^i\left(t,\overline{x},\dot{\overline{x}}\right) dt - \int_a^b \overline{\lambda}_i f^i\left(t,\overline{y},\dot{\overline{y}}\right) dt \right)} - 1 \right) \\ &> \frac{1}{p} \int_a^b \left[\left(e^{p\eta(t,\overline{x},\overline{y})} - \mathbf{1} \right) \right]^T \overline{\lambda}_i \left[f^i_y \left(t,\overline{y},\dot{\overline{y}}\right) - \frac{d}{dt} f^i_{\dot{y}} \left(t,\overline{y},\dot{\overline{y}}\right) \right] dt, \end{aligned} \tag{4.14}$$

$$\begin{aligned} &\frac{1}{r} b_{g_j}(\overline{x},\overline{y}) \left(e^{r\left(\int_a^b \overline{\xi}_j(t) g^j(t,\overline{x},\dot{\overline{x}}) dt - \int_a^b \overline{\xi}_j(t) g^j\left(t,\overline{y},\dot{\overline{y}}\right) dt \right)} - 1 \right) \\ &\geqq \frac{1}{p} \int_a^b \left[\left(e^{p\eta(t,\overline{x},\overline{y})} - \mathbf{1} \right) \right]^T \sum_{j=1}^m \overline{\xi}_j(t) \left[g^j_y \left(t,\overline{y},\dot{\overline{y}}\right) - \frac{d}{dt} g^j_{\dot{y}} \left(t,\overline{y},\dot{\overline{y}}\right) \right] dt \end{aligned} \tag{4.15}$$

hold. Combining (4.13) and (4.14) and then adding both sides of the obtained inequalities, we get

$$\frac{1}{p} \int_a^b \left[\left(e^{p\eta(t,\overline{x},\overline{y})} - \mathbf{1} \right) \right]^T \sum_{i=1}^k \overline{\lambda}_i \left[f^i_y \left(t,\overline{y},\dot{\overline{y}}\right) - \frac{d}{dt} f^i_{\dot{y}} \left(t,\overline{y},\dot{\overline{y}}\right) \right] dt < 0. \tag{4.16}$$

By the feasibility of $\overline{x}$ and $\left(\overline{y},\overline{\lambda},\overline{\xi}\right)$ in the vector variational control problems (MVP) and (MWDP), respectively, it follows that

$$\frac{1}{r} b_{g_j}(\overline{x},\overline{y}) \left(e^{r\left(\int_a^b \overline{\xi}_j(t) g^j\left(t,\overline{x},\dot{\overline{x}}\right) dt - \int_a^b \overline{\xi}_j(t) g^j\left(t,\overline{y},\dot{\overline{y}}\right) dt \right)} - 1 \right) \leqq 0. \tag{4.17}$$

Combining (4.13) and (4.14) and then adding both sides of the obtained inequalities, we get

$$\frac{1}{p} \int_a^b \left[\left(e^{p\eta(t,\overline{x},\overline{y})} - \mathbf{1} \right) \right]^T \sum_{j=1}^m \overline{\xi}_j(t) \left[g^j_y \left(t,\overline{y},\dot{\overline{y}}\right) - \frac{d}{dt} g^j_{\dot{y}} \left(t,\overline{y},\dot{\overline{y}}\right) \right] dt \leqq 0. \tag{4.18}$$

By (4.16) and (4.18), it follows that the inequality

$$\begin{aligned} \tfrac{1}{p} \int_a^b \left[\left(e^{p\eta(t,\overline{x},\overline{y})} - \mathbf{1} \right) \right]^T &\left(\textstyle\sum_{i=1}^k \overline{\lambda}_i \left[f^i_y \left(t,\overline{y},\dot{\overline{y}}\right) - \tfrac{d}{dt} f^i_{\dot{y}} \left(t,\overline{y},\dot{\overline{y}}\right) \right] \right. \\ &\left. + \textstyle\sum_{j=1}^m \overline{\xi}_j(t) \left[g^j_y \left(t,\overline{y},\dot{\overline{y}}\right) - \tfrac{d}{dt} g^j_{\dot{y}} \left(t,\overline{y},\dot{\overline{y}}\right) \right] \right) dt < 0, \end{aligned}$$

contradicting the feasibility of $\left(\overline{y},\overline{\lambda},\overline{\xi}\right)$ in (MWDP). Thus, the proof completes. $\square$

5. CONCLUSION

In this paper, we have introduced the classes of B-(p,r)-invex functions for a multiobjective variational control problem. Then, the concept of B-(p,r)-invexity has been used to derive the sufficient optimality conditions and Mond-Weir duality results for the considered multiobjective variational control problems. Thus, the optimality conditions and duality results have been proved for a new class of nonconvex multiobjective variational control problems.

Some interesting topics for further research remain. It would be of interest to investigate whether these results are true also for a larger class of nonconvex multiobjective variational control problems, for instance, nonconvex nondifferentiable multiobjective variational control problems and nonconvex multiobjective fractional variational problems. It seems that the techniques employed in this paper can be used in proving similarly results for the classes of nonconvex multiobjective variational problems mentioned above. We shall investigate these questions in subsequent papers.

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