SIGNED STAR \((k,k)\)-DOMATIC NUMBER OF A GRAPH

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Abstract. Let \(G\) be a simple graph without isolated vertices with vertex set \(V(G)\) and edge set \(E(G)\) and let \(k\) be a positive integer. A function \(f : E(G) \rightarrow \{-1, 1\}\) is said to be a signed star \(k\)-dominating function on \(G\) if \(\sum_{e \in E(v)} f(e) \geq k\) for every vertex \(v\) of \(G\), where \(E(v) = \{uv \in E(G) \mid u \in N(v)\}\). A set \(\{f_1, f_2, \ldots, f_d\}\) of signed star \(k\)-dominating functions on \(G\) with the property that \(\sum_{i=1}^{d} f_i(e) \leq k\) for each \(e \in E(G)\), is called a signed star \((k, k)\)-dominating family (of functions) on \(G\). The maximum number of functions in a signed star \((k, k)\)-dominating family on \(G\) is the signed star \((k, k)\)-domatic number of \(G\), denoted by \(d_{SS}^{(k,k)}(G)\). In this paper we study properties of the signed star \((k, k)\)-domatic number. In particular, we present bounds on \(d_{SS}^{(k,k)}(G)\), and we determine the signed \((k, k)\)-domatic number of some regular graphs. Some of our results extend these given by Atapour, Sheikholeslami, Ghameslou and Volkmann [Signed star domatic number of a graph, Discrete Appl. Math. 158 (2010), 213–218] for the signed star domatic number.

Keywords: signed star \((k,k)\)-domatic number, signed star domatic number, signed star \(k\)-dominating function, signed star dominating function, signed star \(k\)-domination number, signed star domination number, regular graphs.

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1. INTRODUCTION

Let \(G\) be a graph with vertex set \(V(G)\) and edge set \(E(G)\). We use [8] for terminology and notation which are not defined here and consider simple graphs without isolated vertices only. For every nonempty subset \(E'\) of \(E(G)\), the subgraph \(G[E']\) induced by \(E'\) is the graph whose vertex set consists of those vertices of \(G\) incident with at least one edge of \(E'\) and whose edge set is \(E'\).

Two edges \(e_1, e_2\) of \(G\) are called adjacent if they are distinct and have a common vertex. The open neighborhood \(N_G(e)\) of an edge \(e \in E(G)\) is the set of all edges adjacent to \(e\). Its closed neighborhood is \(N_G[e] = N_G(e) \cup \{e\}\). For a function \(f : E(G) \rightarrow \{-1, 1\}\) and a subset \(S\) of \(E(G)\) we define \(f(S) = \sum_{e \in S} f(e)\). The
edge-neighborhood $E_G(v) = E(v)$ of a vertex $v \in V(G)$ is the set of all edges incident with the vertex $v$. For each vertex $v \in V(G)$, we also define $f(v) = \sum_{e \in E_G(v)} f(e)$.

Let $k$ be a positive integer. A function $f : E(G) \rightarrow \{-1, 1\}$ is called a signed star $k$-dominating function (SSkDF) on $G$, if $f(v) \geq k$ for every vertex $v$ of $G$. The signed star $k$-domination number of a graph $G$ is

$$\gamma_{kSS}(G) = \min \left\{ \sum_{e \in E(G)} f(e) \mid f \text{ is a SSkDF on } G \right\}.$$  

The signed star $k$-dominating function $f$ on $G$ with $f(E(G)) = \gamma_{kSS}(G)$ is called a $\gamma_{kSS}(G)$-function. As the assumption $\delta(G) \geq k$ is clearly necessary, we will always assume that when we discuss $\gamma_{kSS}(G)$ all graphs involved satisfy $\delta(G) \geq k$. The signed star $k$-domination number was introduced by Xu and Li in [11] and has been studied by several authors (see for instance [4,5]). The signed star 1-domination number is the usual signed star domination number which has been introduced by Xu in [9] and has been studied by several authors (see for instance [4,6,10]).

A set $\{f_1, f_2, \ldots, f_d\}$ of signed star $k$-dominating functions on $G$ with $\sum_{i=1}^{d} f_i(e) \leq k$ for each $e \in E(G)$, is called a signed star $(k,k)$-dominating family (of functions) on $G$. The maximum number of functions in a signed star $(k,k)$-dominating family on $G$ is the signed star $(k,k)$-domatic number of $G$, denoted by $d_{SS}^{(k,k)}(G)$. The signed star $(k,k)$-domatic number is well-defined and

$$d_{SS}^{(k,k)}(G) \geq 1 \quad \text{(1.1)}$$

for all graphs $G$ with $\delta(G) \geq k$, since the set consisting of any one SSkD function forms a SS(k,k)D family on $G$. A $d_{SS}^{(k,k)}$-family of a graph $G$ is a SS(k,k)D family containing $d_{SS}^{(k,k)}(D)$ SSkD functions. The signed star $(1,1)$-domatic number $d_{SS}^{(1,1)}(G)$ is the usual signed star domatic number $d_{SS}(G)$ which was introduced by Atapour, Sheikholeslami, Ghameslou and Volkmann in [1].

Our purpose in this paper is to initiate the study of signed star $(k,k)$-domatic numbers in graphs. We first study basic properties and bounds for the signed star $(k,k)$-domatic number of a graph where some of them are analogous to those of the signed star domatic number $d_{SS}(G)$ in [1]. In addition, we determine the signed star $(k,k)$-domatic number of some regular graphs.

We start with a simple known observation which is important for our investigations.

**Observation 1.1** ([5]). Let $G$ be a graph of size $m$ with $\delta(G) \geq k$. Then $\gamma_{kSS}(G) = m$ if and only if each edge $e \in E(G)$ has an endpoint $u$ such that $\deg(u) = k$ or $\deg(u) = k + 1$.

2. BASIC PROPERTIES OF THE SIGNED STAR $(k,k)$-DOMATIC NUMBER

In this section we study basic properties of $d_{SS}^{(k,k)}(G)$. 
Proposition 2.1. If \( k \geq 1 \) is an integer and \( G \) is a graph of minimum degree \( \delta(G) \geq k \), then
\[
d_{SS}^{(k,k)}(G) \leq \delta(G).
\]
Moreover, if \( d_{SS}^{(k,k)}(G) = \delta(G) \), then for each function of any signed star \((k,k)\)-dominating family \( \{f_1, f_2, \ldots, f_d\} \) with \( d = d_{SS}^{(k,k)}(G) \), and for all vertices \( v \) of degree \( \delta(G) \), \( \sum_{e \in E(v)} f_i(e) = k \) and \( \sum_{i=1}^{d} f_i(e) = k \) for every \( e \in E(v) \).

Proof. Let \( \{f_1, f_2, \ldots, f_d\} \) be a signed star \((k,k)\)-dominating family on \( G \) such that \( d = d_{SS}^{(k,k)}(G) \). If \( v \in V(G) \) is a vertex of minimum degree \( \delta(G) \), then it follows that
\[
d \cdot k = \sum_{i=1}^{d} k \leq \sum_{i=1}^{d} \sum_{e \in E(v)} f_i(e) = \sum_{e \in E(v)} \sum_{i=1}^{d} f_i(e) \leq \sum_{e \in E(v)} k = k \cdot \delta(G),
\]
and this implies the desired upper bound on the signed star \((k,k)\)-domatic number.

If \( d_{SS}^{(k,k)}(G) = \delta(G) \), then the two inequalities occurring in the proof become equalities, which leads to the two properties given in the statement.

The special case \( k = 1 \) in Proposition 2.1 can be found in [1]. As an application of Proposition 2.1, we will prove the following Nordhaus-Gaddum type result.

Corollary 2.2. If \( k \geq 1 \) is an integer and \( G \) is a graph of order \( n \) such that \( \delta(G) \geq k \) and \( \delta(G) \geq k \), then
\[
d_{SS}^{(k,k)}(G) + d_{SS}^{(k,k)}(\overline{G}) \leq n - 1.
\]
If \( d_{SS}^{(k,k)}(G) + d_{SS}^{(k,k)}(\overline{G}) = n - 1 \), then \( G \) is regular.

Proof. Since \( \delta(G) \geq k \) and \( \delta(\overline{G}) \geq k \), it follows from Proposition 2.1 that
\[
d_{SS}^{(k,k)}(G) + d_{SS}^{(k,k)}(\overline{G}) \leq \delta(G) + \delta(\overline{G}) = \delta(G) + (n - \Delta(G) - 1) \leq n - 1,
\]
and this is the desired Nordhaus-Gaddum inequality. If \( G \) is not regular, then \( \Delta(G) \geq 1 \), and the above inequality chain leads to the better bound \( d_{SS}^{(k,k)}(G) + d_{SS}^{(k,k)}(\overline{G}) \leq n - 2 \). This completes the proof.

Theorem 2.3. If \( v \) is a vertex of a graph \( G \) such that \( d(v) \) is odd and \( k \) is even or \( d(v) \) is even and \( k \) is odd, then
\[
d_{SS}^{(k,k)}(G) \leq \frac{k}{k+1} \cdot d(v).
\]
Proof. Let \( \{f_1, f_2, \ldots, f_d\} \) be a signed star \((k,k)\)-dominating family on \( G \) such that \( d = d_S^{(k,k)}(G) \). Assume first that \( d(v) \) is odd and \( k \) is even. The definition yields \( \sum_{e \in E(v)} f_i(e) \geq k \) for each \( i \in \{1, 2, \ldots, d\} \). On the left-hand side of this inequality a sum of an odd number of odd summands occurs. Therefore it is an odd number, and as \( k \) is even, we obtain \( \sum_{e \in E(v)} f_i(e) \geq k + 1 \) for each \( i \in \{1, 2, \ldots, d\} \). It follows that

\[
k \cdot d(v) = \sum_{e \in E(v)} k \geq \sum_{e \in E(v)} \sum_{i=1}^{d} f_i(e) = \sum_{i=1}^{d} \sum_{e \in E(v)} f_i(e) \geq \sum_{i=1}^{d} (k + 1) = d(k + 1),
\]

and this leads to the desired bound. Assume next that \( d(v) \) is even and \( k \) is odd. Note that \( \sum_{e \in E(v)} f_i(e) \geq k \) for each \( i \in \{1, 2, \ldots, d\} \). On the left-hand side of this inequality a sum of an even number of odd summands occurs. Therefore it is an even number, and as \( k \) is odd, we obtain \( \sum_{e \in E(v)} f_i(e) \geq k + 1 \) for each \( i \in \{1, 2, \ldots, d\} \). Now the desired bound follows as above, and the proof is complete. \( \square \)

The next result is an immediate consequence of Theorem 2.3.

**Corollary 2.4.** If \( G \) is a graph such that \( \delta(G) \) is odd and \( k \) is even or \( \delta(G) \) is even and \( k \) is odd, then

\[ d_S^{(k,k)}(G) \leq \frac{k}{k+1} \cdot \delta(G). \]

The bound is sharp for cycles when \( k = 1 \).

As an application of Corollary 2.4, we will improve the Nordhaus-Gaddum bound in Corollary 2.2 for many cases.

**Theorem 2.5.** Let \( k \geq 1 \) be an integer, and let \( G \) be a graph of order \( n \) such that \( \delta(G) \geq k \) and \( \delta(G) \geq k \). If \( \Delta(G) - \delta(G) \geq 1 \) or \( k \) is odd or \( k \) is even and \( \delta(G) \) is odd or \( k \), \( \delta(G) \) and \( n \) are even, then

\[ d_S^{(k,k)}(G) + d_S^{(k,k)}(\overline{G}) \leq n - 2. \]

**Proof.** If \( \Delta(G) - \delta(G) \geq 1 \), then Corollary 2.2 implies the desired bound. Thus assume now that \( G \) is \( \delta(G) \)-regular.

**Case 1.** Assume that \( k \) is odd. If \( \delta(G) \) is even, then it follows from Proposition 2.1 and Corollary 2.4 that

\[
d_S^{(k,k)}(G) + d_S^{(k,k)}(\overline{G}) \leq \frac{k}{k+1} \delta(G) + \delta(G) = \frac{k}{k+1} \delta(G) + \delta(G) - 1 < n - 1,
\]

and this leads to the desired bound. Assume next that \( d(v) \) is even and \( k \) is odd. Note that \( \sum_{e \in E(v)} f_i(e) \geq k \) for each \( i \in \{1, 2, \ldots, d\} \). On the left-hand side of this inequality a sum of an even number of odd summands occurs. Therefore it is an even number, and as \( k \) is odd, we obtain \( \sum_{e \in E(v)} f_i(e) \geq k + 1 \) for each \( i \in \{1, 2, \ldots, d\} \). Now the desired bound follows as above, and the proof is complete. \( \square \)
and we obtain the desired bound. If $\delta(G)$ is odd, then $n$ is even and thus $\delta(\overline{G}) = n - \delta(G) - 1$ is even. Combining Proposition 2.1 and Corollary 2.4, we find that
\[
d_{SS}^{(k,k)}(G) + d_{SS}^{(k,k)}(\overline{G}) \leq \delta(G) + \frac{k}{k+1} \delta(\overline{G}) = (n - \delta(\overline{G}) - 1) + \frac{k}{k+1} \delta(\overline{G}) < n - 1,
\]
and this completes the proof of Case 1.

Case 2. Assume that $k$ is even. If $\delta(G)$ is odd, then it follows from Proposition 2.1 and Corollary 2.4 that
\[
d_{SS}^{(k,k)}(G) + d_{SS}^{(k,k)}(\overline{G}) \leq \frac{k}{k+1} \delta(G) + (n - \delta(G) - 1) < n - 1.
\]
If $\delta(G)$ is even and $n$ is even, then $\delta(\overline{G}) = n - \delta(G) - 1$ is odd, and we obtain the desired bound as above.

**Theorem 2.6.** If $G$ is a graph such that $k$ is odd and $d_{SS}^{(k,k)}(G)$ is even or $k$ is even and $d_{SS}^{(k,k)}(G)$ is odd, then
\[
d_{SS}^{(k,k)}(G) \leq \frac{k-1}{k} \delta(G).
\]

**Proof.** Let $\{f_1, f_2, \ldots, f_d\}$ be a signed star $(k,k)$-dominating family on $G$ such that $d = d_{SS}^{(k,k)}(G)$. Assume first that $k$ is odd and $d$ is even. If $e \in E(G)$ is an arbitrary edge, then $\sum_{i=1}^{d} f_i(e) \leq k$. On the left-hand side of this inequality a sum of an even number of odd summands occurs. Therefore it is an even number, and as $k$ is odd, we obtain $\sum_{i=1}^{d} f_i(e) \leq k - 1$ for each $e \in E(G)$. If $v$ is a vertex of minimum degree, then it follows that
\[
d \cdot k = \sum_{i=1}^{d} f_i(e) = \sum_{e \in E(v)} \sum_{i=1}^{d} f_i(e) \leq \sum_{e \in E(v)} (k - 1) = \delta(G)(k - 1),
\]
and this yields to the desired bound. Assume second that $k$ is even and $d$ is odd. If $e \in E(G)$ is an arbitrary edge, then $\sum_{i=1}^{d} f_i(e) \leq k$. On the left-hand side of this inequality a sum of an odd number of odd summands occurs. Therefore it is an odd number, and as $k$ is even, we obtain $\sum_{i=1}^{d} f_i(e) \leq k - 1$ for each $e \in E(G)$. Now the desired bound follows as above, and the proof is complete. 

\[\Box\]
According to (1.1), \( d_{SS}^{(k,k)}(G) \) is a positive integer. If we suppose in the case \( k = 1 \) that \( d_{SS}(G) = d_{SS}^{(1,1)}(G) \) is an even integer, then Theorem 2.6 leads to the contradiction \( d_{SS}(G) \leq 0 \). Consequently, we obtain the next known result.

**Corollary 2.7** ([1]). The signed star domatic number \( d_{SS}(G) \) is an odd integer.

**Proposition 2.8.** Let \( k \geq 2 \) be an integer, and let \( G \) be a graph with minimum degree \( \delta(G) \geq k \). Then \( d_{SS}^{(k,k)}(G) = 1 \) if and only if each edge \( e \in E(G) \) has an endpoint \( u \) such that \( \deg(u) = k \) or \( \deg(u) = k + 1 \).

**Proof.** Assume that each edge \( e \in E(G) \) has an endpoint \( u \) such that \( \deg(u) = k \) or \( \deg(u) = k + 1 \). It follows from Observation 1.1 that \( \gamma_{kSS}(G) = m \) and thus \( d_{SS}^{(k,k)}(G) = 1 \).

Conversely, assume that \( d_{SS}^{(k,k)}(G) = 1 \). If \( G \) contains an edge \( e = uv \) such that \( d(u) \geq k + 2 \) and \( d(v) \geq k + 2 \), then the functions \( f_1, f_2 : E(G) \to \{-1, 1\} \) such that \( f_1(x) = 1 \) for each \( x \in E(G) \) and \( f_2(e) = -1 \) and \( f_2(x) = 1 \) for each edge \( x \in E(G) \setminus \{e\} \) are signed star \( k \)-dominating functions on \( G \) such that \( f_1(x) + f_2(x) \leq 2 \) for each edge \( x \in E(G) \). Thus \( \{f_1, f_2\} \) is a signed star \((k, k)\)-dominating family on \( G \), a contradiction to \( d_{SS}^{(k,k)}(G) = 1 \).

The next result is an immediate consequence of Observation 1.1 and Proposition 2.8.

**Corollary 2.9.** Let \( k \geq 2 \) be an integer, and let \( G \) be a graph with minimum degree \( \delta(G) \geq k \). Then \( d_{SS}^{(k,k)}(G) = 1 \) if and only if \( \gamma_{kSS}(G) = m \).

Next we present a lower bound on the signed star \((k, k)\)-dominant number.

**Proposition 2.10.** Let \( k \geq 1 \) be an integer, and let \( G \) be a graph with minimum degree \( \delta(G) \geq k \). If \( G \) contains a vertex \( v \in V(G) \) such that all vertices of \( N[N[v]] \) have degree at least \( k + 2 \), then \( d_{SS}^{(k,k)}(G) \geq k \).

**Proof.** Let \( \{u_1, u_2, \ldots, u_k\} \subset N(v) \). The hypothesis that all vertices of \( N[N[v]] \) have degree at least \( k + 2 \) implies that the functions \( f_i : E(G) \to \{-1, 1\} \) such that \( f_i(u_i) = -1 \) and \( f_i(x) = 1 \) for each edge \( x \in E(G) \setminus \{u_i\} \) are signed star \( k \)-dominating functions on \( G \) for \( i \in \{1, 2, \ldots, k\} \). Since \( f_1(x) + f_2(x) + \ldots + f_k(x) \leq k \) for each edge \( x \in E(G) \), we observe that \( \{f_1, f_2, \ldots, f_k\} \) is a signed star \((k, k)\)-dominating family on \( G \), and Proposition 2.10 is proved.

**Corollary 2.11.** If \( G \) is a graph of minimum degree \( \delta(G) \geq k + 2 \), then \( d_{SS}^{(k,k)}(G) \geq k \).

**Theorem 2.12.** Let \( G \) be a graph of size \( m \) with \( \delta(G) \geq k \), signed star \( k \)-domination number \( \gamma_{kSS}(G) \) and signed star \((k, k)\)-dominant number \( d_{SS}^{(k,k)}(G) \). Then

\[
\gamma_{kSS}(G) \cdot d_{SS}^{(k,k)}(G) \leq mk.
\]

Moreover, if \( \gamma_{kSS}(G) \cdot d_{SS}^{(k,k)}(G) = mk \), then for each \( d_{SS}^{(k,k)} \)-family \( \{f_1, f_2, \ldots, f_d\} \) of \( G \), each function \( f_i \) is a \( \gamma_{kSS} \)-function and \( \sum_{i=1}^{d} f_i(e) = k \) for all \( e \in E(G) \).
Signed star \((k, k)\)-domatic number of a graph

Proof. If \(\{f_1, f_2, \ldots, f_d\}\) is a signed star \((k, k)\)-dominating family on \(G\) such that \(d = d_{SS}^{(k,k)}(G)\), then the definitions imply

\[
d \cdot \gamma_{SS}(G) = \sum_{i=1}^{d} \gamma_{SS}(G) \leq \sum_{i=1}^{d} \sum_{e \in E(G)} f_i(e) = \sum_{e \in E(G)} d \leq \sum_{e \in E(G)} k = mk
\]
as desired.

If \(\gamma_{SS}(G) \cdot d_{SS}^{(k,k)}(G) = mk\), then the two inequalities occurring in the proof become equalities. Hence for the \(d_{SS}^{(k,k)}\)-family \(\{f_1, f_2, \ldots, f_d\}\) of \(G\) and for each \(i\), \(\sum_{e \in E(G)} f_i(e) = \gamma_{SS}(G)\), thus each function \(f_i\) is a \(\gamma_{SS}\)-function, and \(\sum_{i=1}^{d} f_i(e) = k\) for all \(e \in E(G)\).

The upper bound on the product \(\gamma_{SS}(G) \cdot d_{SS}^{(k,k)}(G)\) leads to an upper bound on the sum of these two parameters.

**Theorem 2.13.** If \(k \geq 1\) is an integer and \(G\) is a graph of size \(m\) and minimum degree \(\delta(G) \geq k\), then

\[
d_{SS}^{(k,k)}(G) + \gamma_{SS}(G) \leq m + k.
\]

Proof. If \(\delta(G) = k\), then it follows from Proposition 2.1 that

\[
d_{SS}^{(k,k)}(G) + \gamma_{SS}(G) \leq \delta(G) + m = m + k.
\]

Assume next that \(\delta(G) = k + 1\). If \(\gamma_{SS}(G) = m\), then \(d_{SS}^{(k,k)}(G) = 1\) and so

\[
d_{SS}^{(k,k)}(G) + \gamma_{SS}(G) = m + 1 < m + k.
\]

In the case that \(\gamma_{SS}(G) \leq m - 1\), Proposition 2.1 implies that

\[
d_{SS}^{(k,k)}(G) + \gamma_{SS}(G) \leq \delta(G) + m - 1 = m + k.
\]

Assume now that \(\delta(G) \geq k + 2\). According to Theorem 2.12, we have

\[
d_{SS}^{(k,k)}(G) + \gamma_{SS}(G) \leq d_{SS}^{(k,k)}(G) + \frac{km}{d_{SS}^{(k,k)}(G)}.
\]

In view of Corollary 2.11, \(d_{SS}^{(k,k)}(G) \geq k\), and Proposition 2.1 implies that \(d_{SS}^{(k,k)}(G) \leq n - 1 \leq m\). Using these inequalities, and the fact that the function \(g(x) = x + (km)/x\) is decreasing for \(k \leq x \leq \sqrt{km}\) and increasing for \(\sqrt{km} \leq x \leq n\), we obtain

\[
d_{SS}^{(k,k)}(G) + \gamma_{SS}(G) \leq \max \left\{ k + \frac{km}{k}, m + \frac{km}{m} \right\} = m + k.
\]

Since we have discussed all possible cases for the minimum degree \(\delta(G)\), the proof of Theorem 2.13 is complete.
3. REGULAR GRAPHS

**Theorem 3.1.** Let $k \geq 1$ be an integer, and let $G$ be an $r$-regular graph with $r \geq k$.

1. If $k \leq r \leq k + 1$, then $d^{(k,k)}_{SS}(G) = 1$.
2. If $r = k + 2p + 1$ with $p \geq 1$, then $k \leq d^{(k,k)}_{SS}(G) \leq r - 3$.
3. If $r = k + 2p$ with $p \geq 1$, then $d^{(k,k)}_{SS}(G) \neq r - 1$, and if $d^{(k,k)}_{SS}(G) = r$, then $G$ contains a $p$-regular factor.

**Proof.** (1) Assume that $k \leq r \leq k + 1$. According to Observation 1.1, we have $\gamma_{SS}(G) = m$ and thus $d^{(k,k)}_{SS}(G) = 1$.

(2) Assume that $r = k + 2p + 1$ with $p \geq 1$. In view of Proposition 2.1 and Corollary 2.11, we obtain $k \leq d^{(k,k)}_{SS}(G) \leq r$.

If we suppose that $d^{(k,k)}_{SS}(G) = r$, then Theorem 2.6 yields to the contradiction $r \leq (k - 1)r/k$.

Next, we suppose that $d^{(k,k)}_{SS}(G) = r - 1 = k + 2p$. In that case Theorem 2.3 leads to the contradiction $r - 1 \leq kr/(k + 1)$.

Now suppose that $d^{(k,k)}_{SS}(G) = r - 2 = k + 2p - 1$, and let $\{f_1, f_2, \ldots, f_{k+2p-1}\}$ be a signed star $(k,k)$-dominating family of $G$. If $e \in E(G)$ is an arbitrary edge, then $\sum_{i=1}^{k+2p-1} f_i(e) \leq 2e$. If $k$ is odd, then on the left-hand side of this inequality a sum of an even number of odd summands occurs. Therefore it is an even number, and as $k$ is odd, it follows that $\sum_{i=1}^{k+2p-1} f_i(e) \leq k - 1$. If $k$ is even, then we obtain analogously the same bound $\sum_{i=1}^{k+2p-1} f_i(e) \leq k - 1$. If $k \in V(G)$ is an arbitrary vertex, then $\sum_{e \in E(v)} f_i(e) \geq k$ for each $1 \leq i \leq k + 2p - 1$. Therefore $f_i(e) = -1$ for at most $p$ edges $e \in E(v)$ and thus $\sum_{e \in E(v)} f_i(e) \geq k + 1$ for each $1 \leq i \leq k + 2p - 1$. Using the identity $2|E(G)| = |V(G)|(k + 2p + 1)$, we deduce that

$$|V(G)|(k + 2p + 1)(k - 1) = 2|E(G)|(k - 1) \geq 2 \sum_{v \in V(G)} \sum_{i=1}^{r-2} f_i(e) = \sum_{v \in V(G)} \sum_{i=1}^{r-2} f_i(e) \geq \sum_{v \in V(G)} \sum_{i=1}^{r-2} (k + 1) = |V(G)|(k + 2p - 1)(k + 1).$$

It follows that $(k + 2p + 1)(k - 1) \geq (k + 2p - 1)(k + 1)$, and we obtain the contradiction $-2p \geq 2p$. Altogether, we have shown that $k \leq d^{(k,k)}_{SS}(G) \leq r - 3$ in that case.

(3) Assume that $r = k + 2p$ with $p \geq 1$. Proposition 2.1 and Corollary 2.11 imply $k \leq d^{(k,k)}_{SS}(G) \leq r$. If we suppose that $d^{(k,k)}_{SS}(G) = r - 1 = k + 2p - 1$, then it follows from Theorem 2.6 that

$$d^{(k,k)}_{SS}(G) = k + 2p - 1 \leq \frac{k - 1}{k}(k + 2p),$$

and we obtain the contradiction $2p \leq 0$. Hence $d^{(k,k)}_{SS}(G) \neq r - 1$. 


Now assume that \( d^{(k,k)}_{SS}(G) = r = k + 2p \), and let \( \{f_1, f_2, \ldots, f_{k+2p}\} \) be a signed star \((k,k)\)-dominating family of \( G \). Applying Proposition 2.1, we deduce that 
\[
\sum_{e \in E(v)} f_i(e) = k \text{ for each } v \in V(G) \text{ and each } 1 \leq i \leq k + 2p.
\]
Then for each \( 1 \leq i \leq k + 2p \), each vertex \( v \in V(G) \) is adjacent to exactly \( p \) edges \( e_1, e_2, \ldots, e_p \) such that \( f_i(e_1) = f_i(e_2) = \ldots = f_i(e_p) = -1 \). However, this is only possible if \( G \) contains a \( p \)-regular factor, and the proof is complete. \( \square \)

Theorem 3.1 (2) implies the next result immediately.

Corollary 3.2. If \( k \geq 1 \) is an integer and \( G \) is a \((k+3)\)-regular graph, then 
\[ d^{(k,k)}_{SS}(G) = k. \]

Corollary 3.3. If \( k \geq 1 \) is an integer and \( G \) is a \((k+2p)\)-regular graph of odd order \( n \) with \( p \geq 1 \) odd, then 
\[ k \leq d^{(k,k)}_{SS}(G) = k + 2p - 2. \]

Proof. Using Theorem 3.1 (3), we see that \( d^{(k,k)}_{SS}(G) = k + 2p \) or \( d^{(k,k)}_{SS}(G) \leq k + 2p - 2 \).
If \( d^{(k,k)}_{SS}(G) = k + 2p \), then Theorem 3.1 (3) implies that \( G \) contains a \( p \)-regular factor. Since \( n \) and \( p \) are odd, this is impossible, and thus Theorem 3.1 (3) yields to 
\[ k \leq d^{(k,k)}_{SS}(G) \leq k + 2p - 2. \] \( \square \)

Corollary 3.3 leads to the following supplement to Theorem 2.5.

Corollary 3.4. Let \( k \geq 2 \) be an even integer, and let \( G \) be a \( \delta(G) \)-regular graph of odd order \( n \) such that \( \delta(G) \geq k \) and \( \delta(G) \geq k \). If \( \delta(G) = k + 2p \) with an odd integer 
\( p \geq 1 \), then 
\[ d^{(k,k)}_{SS}(G) + d^{(k,k)}_{SS}(\overline{G}) \leq n - 2. \]

Proof. In view of Corollary 2.2, we see that 
\[ d^{(k,k)}_{SS}(G) + d^{(k,k)}_{SS}(\overline{G}) \leq n - 1. \] Suppose 
the contrary that 
\[ d^{(k,k)}_{SS}(G) + d^{(k,k)}_{SS}(\overline{G}) = n - 1. \]
Then Proposition 2.1 implies that 
\[ d^{(k,k)}_{SS}(G) = \delta(G) = k + 2p. \]
However, Corollary 3.3 leads to the contradiction 
\[ d^{(k,k)}_{SS}(G) \leq k + 2p - 2, \]
and the proof is complete. \( \square \)

Corollary 3.5. If \( k \geq 1 \) is an integer and \( G \) a \((k+2)\)-regular graph of odd order \( n \), 
then \( d^{(k,k)}_{SS}(G) = k \).

Let \( H \) be a \((k+2)\)-regular bipartite graph. By a well-known result of König [3], there exists a decomposition of \( E(H) \) in perfect matchings \( M_1, M_2, \ldots, M_{k+2} \). Now define 
\[ f_i : E(H) \rightarrow \{-1,1\} \] 
by 
\[ f_i(e) = -1 \text{ when } e \in M_i \text{ and } f_i(e) = 1 \text{ when } e \in E(H) - M_i \text{ for } 1 \leq i \leq k + 2. \]
Then 
\[ f_i(v) = \sum_{e \in E(v)} f_i(e) = k \text{ for each } v \in V(H) \text{ and each } 1 \leq i \leq k + 2 \text{ and } \sum_{i=1}^{k+2} f_i(e) = k \text{ for every } e \in E(H). \]
Therefore \( \{f_1, f_2, \ldots, f_{k+2}\} \) is a signed star \((k,k)\)-dominating family on \( H \), and consequently 
\( d^{(k,k)}_{SS}(H) = k + 2 \). This family of examples demonstrates that 
\( d^{(k,k)}_{SS}(G) = k + 2 \) in Corollary 3.5 is possible when the order of \( G \) is even.

Theorem 3.6. Let \( k \geq 1 \) and \( p \geq 2 \) be integers, and let \( G \) be an \( r \)-regular graph with 
\( r = k + 2p + 1 \). If \( p < k + 1 \), then 
\( d^{(k,k)}_{SS}(G) \leq r - 4 \).
Proof. According to Theorem 3.1 (2), we have $d_{SS}^{(k,k)}(G) \leq r - 3$. We suppose to
the contrary that $d_{SS}^{(k,k)}(G) = r - 3 = k + 2p - 2$. Let $\{f_1, f_2, \ldots, f_{k+2p-2}\}$ be a
signed star $(k,k)$-dominating family of $G$. If $e \in E(G)$ is an arbitrary edge, then
$\sum_{i=1}^{k+2p-2} f_i(e) \leq k$. If $v \in V(G)$ is an arbitrary vertex, then $\sum_{e \in E(v)} f_i(e) \geq k$ for each
$1 \leq i \leq k+2p-2$. As in the proof of Theorem 3.1 (2), we see that $\sum_{e \in E(v)} f_i(e) \geq k+1$
for each $1 \leq i \leq k + 2p - 2$. Using again the identity $2|E(G)| = |V(G)|(k + 2p + 1)$, we
deduce that

$$|V(G)|(k + 2p + 1)k = 2|E(G)|k \geq 2 \sum_{e \in E(G)} \sum_{i=1}^{r-3} f_i(e) =$$

$$\geq \sum_{v \in V(G)} \sum_{i=1}^{r-3} \sum_{e \in E(v)} f_i(e) \geq \sum_{v \in V(G)} \sum_{i=1}^{r-3} (k + 1) =$$

$$= |V(G)|(k + 2p - 2)(k + 1).$$

It follows that $(k + 2p + 1)k \geq (k + 2p - 2)(k + 1)$. This yields $k + 1 \geq p$, a contradiction
to the hypothesis $p < k + 1$. \qed

**Theorem 3.7.** Let $k \geq 1$ and $p \geq 2$ be integers, and let $G$ be an $r$-regular graph with
$r = k + 2p + 1$. If $k + 1 < 2p$, then $d_{SS}^{(k,k)}(G) \neq r - 4$.

Proof. Suppose to the contrary that $d_{SS}^{(k,k)}(G) = r - 4 = k + 2p - 3$. Let $\{f_1, f_2, \ldots, f_{k+2p-3}\}$ be a signed star $(k,k)$-dominating family of $G$. If $e \in E(G)$
is an arbitrary edge, then $\sum_{i=1}^{k+2p-3} f_i(e) \leq k$. If $k$ is odd, then on the left-hand side
of this inequality a sum of an even number of odd summands occurs. Therefore it is
an even number, and as $k$ is odd, it follows that $\sum_{i=1}^{k+2p-3} f_i(e) \leq k - 1$. If $k$ is even, then we obtain analogously the same bound
$\sum_{i=1}^{k+2p-3} f_i(e) \leq k - 1$. If $v \in V(G)$ is an arbitrary vertex, then we obtain as above that $\sum_{e \in E(v)} f_i(e) \geq k + 1$ for each
$1 \leq i \leq k + 2p - 3$. Using the identity $2|E(G)| = |V(G)|(k + 2p + 1)$, we deduce that

$$|V(G)|(k + 2p + 1)(k - 1) = 2|E(G)|(k - 1) \geq 2 \sum_{e \in E(G)} \sum_{i=1}^{r-4} f_i(e) =$$

$$\geq \sum_{v \in V(G)} \sum_{i=1}^{r-4} \sum_{e \in E(v)} f_i(e) \geq \sum_{v \in V(G)} \sum_{i=1}^{r-4} (k + 1) =$$

$$= |V(G)|(k + 2p - 3)(k + 1).$$

It follows that $(k + 2p + 1)(k - 1) \geq (k + 2p - 3)(k + 1)$ and hence $k + 1 \geq 2p$. This is
a contradiction to the hypothesis $k + 1 < 2p$, and the proof is complete. \qed

Combining Theorems 3.1, 3.6 and 3.7, we obtain the next bounds on $d_{SS}^{(k,k)}(G)$
immediately.

**Corollary 3.8.** Let $k \geq 1$ and $p \geq 2$ be integers, and let $G$ be an $r$-regular graph with
$r = k + 2p + 1$. If $k + 1 < 2p < 2k + 2$, then $k \leq d_{SS}^{(k,k)}(G) \leq r - 5$. 

The special case $k = p = 2$ in Corollary 3.8 leads to the following result.

**Corollary 3.9.** If $G$ is a 7-regular graph, then $d^{(2,2)}_{SS}(G) = 2$.

**REFERENCES**


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