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EXISTENCE AND REGULARITY OF SOLUTIONS FOR HYPERBOLIC FUNCTIONAL DIFFERENTIAL PROBLEMS

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Abstract. A generalized Cauchy problem for quasilinear hyperbolic functional differential systems is considered. A theorem on the local existence of weak solutions is proved. The initial problem is transformed into a system of functional integral equations for an unknown function and for their partial derivatives with respect to spatial variables. The existence of solutions for this system is proved by using a method of successive approximations. We show a theorem on the differentiability of solutions with respect to initial functions which is the main result of the paper.

Keywords: functional differential equations, weak solutions, Haar pyramid, differentiability with respect to initial functions.

Mathematics Subject Classification: 35R10, 35L60.

1. INTRODUCTION

For any metric space U and V we denote by $C(U,V)$ the class of all continuous functions from U into V. We use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Suppose that $M \in C([0,a],\mathbb{R}_+^n)$, $a>0$, $\mathbb{R}_+=[0,+\infty)$, is nondecreasing and $M(0)=0_{[n]}$, where $0_{[n]}=(0,\ldots,0)\in\mathbb{R}^n$. Let E be a generalized Haar pyramid

$$E=\{(t,x)\in\mathbb{R}^{1+n}:\ t\in[0,a],\ -b+M(t)\le x\le b-M(t)\},$$

where $b\in\mathbb{R}^n$ and $b>M(a)$. Write $E_0=[-b_0,0]\times[-b,b]$, where $b_0\in\mathbb{R}_+$ and

$$E_{0.i}=\big(E_0\cup E\big)\cap\big([-b_0,a_i]\times\mathbb{R}^n\big),\quad i=1,\ldots,k,$$

where $0\le a_i<a$ for $1\le i\le k$. For $(t,x)\in E$ define

$$D[t,x]=\{(\tau,y)\in\mathbb{R}^{1+n}:\ \tau\le 0,\ (t+\tau,x+y)\in E_0\cup E\}.$$

Set

$$D_0[t,x] = [-b_0 - t, -t] \times [-b - x, b - x],$$
$$D_\star[t,x] = \{(\tau,y):\ -t \le \tau \le 0,\ -b - x + M(\tau + t) \le y \le b - x - M(\tau + t)\}.$$

Then $D[t,x] = D_0[t,x] \cup D_\star[t,x)$ for $(t,x) \in E$. Write $r_0 = -b_0 - a$, $r = 2b$ and $B = [-r_0, 0] \times [-r, r]$. Then $D[t,x] \subset B$ for $(t,x) \in E$. Given $z : E_0 \cup E \to \mathbb{R}^k$ and $(t,x) \in E$, define $z_{(t,x)} : D[t,x] \to \mathbb{R}^k$ by

$$z_{(t,x)}(\tau,y) = z(t+\tau, x+y), \quad (\tau,y) \in D[t,x].$$

Then $z_{(t,x)}$ is the restriction of z to $\big(E_0 \cup E\big) \cap \big([-b_0, t] \times \mathbb{R}^n\big)$ and this restriction is shifted to $D[t,x]$.

Suppose that $\phi_0 : [0,a] \to \mathbb{R}$ and $\phi = (\phi_1, \ldots, \phi_n) : E \to \mathbb{R}^n$ are given functions such that $0 \le \phi_0(t) \le t$ and $(\phi_0(t), \phi(t,x)) \in E$ for $(t,x) \in E$. Write $\varphi(t,x) = (\phi_0(t), \phi(t,x))$ for $(t,x) \in E$.

Let $M_{k\times n}$ denote the set of all $k \times n$ matrices with real elements. If $X \in M_{k\times n}$ then X^T is the transpose matrix. The scalar product in $\mathbb{R}^n$ is denoted by "$\circ$". Put $\Omega = E \times C(B, \mathbb{R}^k)$ and suppose that

$$F : \Omega \to M_{k\times n},\ F = \big[F_{ij}\big]_{i=1,\ldots,k, j=1,\ldots,n},\ \ G : \Omega \to \mathbb{R}^k,\ G = (G_1, \ldots, G_k),$$

are given functions of the variables (t,x,w), $x = (x_1, \ldots, x_n)$, $w = (w_1, \ldots, w_k)$. For the above F we put $F_{[i]} = (F_{i1}, \ldots, F_{in})$ where $i = 1, \ldots, k$.

We will say that F and G satisfy condition (V) if for each $(t,x) \in E$ and for $w, \bar{w} \in C(B, \mathbb{R}^k)$ such that $w(\tau,y) = \bar{w}(\tau,y)$ for $(\tau,y) \in D[\varphi(t,x)]$ we have $F(t,x,w) = F(t,x,\bar{w})$ and $G(t,x,w) = G(t,x,\bar{w})$. The condition (V) means that the values of F and G at the point $(t,x,w) \in \Omega$ depend on $(t,x) \in E$ and on the restriction of w to the set $D[\varphi(t,x)]$ only. Let us denote by $z = (z_1, \ldots, z_k)$ an unknown function of the variables (t,x). Given $\psi_i : E_{0.i} \to \mathbb{R}$, $i = 1, \ldots, k$, we consider the system of functional differential equations

$$\partial_t z_i(t,x) + F_{[i]}(t,x,z_{\varphi(t,x)}) \circ \partial_x z_i(t,x) = G_i(t,x,z_{\varphi(t,x)}), \quad i = 1, \ldots, k, \tag{1.1}$$

with the initial conditions

$$z_i(t,x) = \psi_i(t,x) \text{ on } E_{0.i}, \quad i = 1, \ldots, k, \tag{1.2}$$

where $\partial_x z_i = (\partial_{x_1} z_i, \ldots, \partial_{x_n} z_i)$, $1 \le i \le k$. We assume that F and G satisfy the condition (V). System (1.1) with initial conditions (1.2) is called a generalized Cauchy problem. If $a_i = 0$ for $i = 1, \ldots, k$ then (1.1), (1.2) reduces to the classical Cauchy problem.

Write $\kappa = \min\{a_i :\ 1 \le i \le k\}$, $\tilde{\kappa} = \max\{a_i :\ 1 \le i \le k\}$ and

$$E_t = (E_0 \cup E) \cap ([-b_0, t] \times \mathbb{R}^n), \quad S_t = [-b + M(t), b - M(t)], \quad 0 \le t \le a,$$
$$I_{c.i}[x] = \{t \in [a_i, c] : -b + M(t) \le x \le b - M(t)\}, \quad i = 1, \ldots, k,$$
$$I[x] = \{t \in [0,a] : -b + M(t) \le x \le b - M(t)\},$$

where $x \in [-b,b]$ and $\tilde{\kappa} < c \le a$. We consider weak solutions of initial problems. A function $\tilde{z} : E_c \to \mathbb{R}^k$, $\tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_k)$, where $\tilde{a} < c \le a$, is a solution of (1.1), (1.2) provides that:

(i) $\tilde{z}$ is continuous and the derivatives $\partial_x \tilde{z}_i = (\partial_{x_1} \tilde{z}_i, \ldots, \partial_{x_n} \tilde{z}_i)$ exist on $E \cap ([a_i, c] \times \mathbb{R}^n)$ for $1 \leq i \leq k$,
(ii) for each i, $1 \leq i \leq k$, and $x \in [-b, b]$, the function $\tilde{z}_i(\cdot, x) : I_{c.i}[x] \to \mathbb{R}$ is absolutely continuous,
(iii) for each $x \in [-b, b]$ and for $1 \leq i \leq k$, the i-th equation in (1.1) is satisfied for almost all $t \in I_{c.i}[x]$ and conditions (1.2) hold.

The following problems are considered in the paper. Under natural assumptions on given functions we prove that there exists exactly one solution to (1.1), (1.2) defined on E_c and we give an estimate of the constant $c \in (\tilde{\kappa}, a]$.

Denote by $\mathbb{X}$ the class of all $\psi = (\psi_1, \ldots, \psi_k)$, $\psi_i : E_{0.i} \to \mathbb{R}$ for $1 \leq i \leq k$, such that there exists exactly one solution $\Xi[\psi] = (\Xi_1[\psi], \ldots, \Xi_k[\psi])$ to (1.1), (1.2). We give a construction of the space $\mathbb{X}$. Denote by $\Lambda_i[\psi]$, $1 \leq i \leq k$, the i-th bicharacteristic of (1.1) corresponding to $\Xi[\psi]$. We prove that for each $\psi \in \mathbb{X}$ there exists the Fréchet derivatives $\partial\Xi_i[\psi]$ and $\partial\Lambda_i[\psi]$ of Ξ_i and Λ_i at the point $\psi \in \mathbb{X}$, $1 \leq i \leq k$. Moreover, if $\psi, \vartheta \in \mathbb{X}$ then the functions $\big(\partial\Xi_1[\psi]\vartheta, \ldots, \partial\Xi_k[\psi]\vartheta\big)$ and $\partial\Lambda_i[\psi]\vartheta$, $1 \leq i \leq k$, are solutions of a linear integral functional system generated by (1.1), (1.2).

Numerous papers were published concerning various problems for first order partial functional differential equations or systems. The following questions have been considered: functional differential inequalities generated by initial or mixed problems and their applications, uniqueness of solutions and continuous dependence on initial or initial boundary conditions, existence theory of classical or weak solutions of equations or finite systems with initial or initial boundary conditions, approximate solutions of functional differential problems. It is not our aim to show a full review of papers concerning the above problems. We shall mention only those which contain such reviews. They are [1–3, 5–9, 12, 16, 18–21, 23] and the monograph [15].

In the paper we start investigations of the differentiability of solutions with respect to initial functions. The monographs [11, 17] contain results on the regularity of solutions of initial problems for ordinary functional differential equations. Differential systems with deviated variables and differential integral problems are particular cases of systems considered here.

2. INTEGRAL FUNCTIONAL EQUATIONS

For $x \in \mathbb{R}^n$, $p \in \mathbb{R}^k$, $X \in M_{k\times n}$, where $x = (x_1, \ldots, x_n)$, $p = (p_1, \ldots, p_k)$, $X = \big[x_{ij}\big]_{i=1,\ldots,k, j=1,\ldots,n}$, we define the norms

$$\|x\| = \sum_{i=1}^{n} |x_i|, \quad \|p\|_\infty = \max\{|p_i| : \ 1 \leq i \leq k\},$$

$$\|X\|_{k\times n} = \max\left\{ \sum_{j=1}^{n} |x_{ij}| : \ 1 \leq i \leq k \right\}.$$

For $z \in C(E_0 \cup E, \mathbb{R}^k)$, $v \in C(E_0 \cup E, \mathbb{R}^n)$, $u \in C(E_0 \cup E, M_{k\times n})$, we define the seminorms

$$\|z\|_{(t,\mathbb{R}^k)} = \max\{\|z(\tau,y)\|_\infty : (\tau,y) \in E_t\}, \quad \|v\|_{(t,\mathbb{R}^n)} = \max\{\|v(\tau,y)\| : (\tau,y) \in E_t\},$$

$$\|u\|_{(t,M_{k\times n})} = \max\{\|u(\tau,y)\|_{k\times n} : (\tau,y) \in E_t\},$$

where $t \in [0,a]$. The norm in the space $C(B,\mathbb{R}^k)$ is given by

$$\|w\|_B = \max\{\|w(\tau,y)\|_\infty : (\tau,y) \in B\}.$$

We denote by $CL(B,\mathbb{R})$ the set of all linear and continuous functions defined on $C(B,\mathbb{R})$ and taking values in $\mathbb{R}$. Let $\|\cdot\|_\star$ be the norm in $CL(B,\mathbb{R})$ generated by the maximum norm in the space $C(B,\mathbb{R})$. Let $CL(B,M_{k\times n})$ be the class of all $Y = \big[Y_{ij}\big]_{i=1,\dots,k,j=1,\dots,n}$ such that $Y_{ij} \in CL(B,\mathbb{R})$, $i = 1,\dots,k$, $j = 1,\dots,n$. For $Y \in CL(B,M_{k\times n})$ we put

$$\|Y\|_{k\times n;\star} = \max\left\{\sum_{j=1}^{n} \|Y_{ij}\|_\star : 1 \le i \le k\right\}.$$

In a similar way we define the space $CL(B,\mathbb{R}^k)$. Let $\mathbb{L}([\tau,t],M_{k\times n})$, $[\tau,t] \subset \mathbb{R}$, denote the class of all integrable functions $\Psi : [\tau,t] \to M_{k\times n}$. In a similar way we define the space $\mathbb{L}([\tau,t],\mathbb{R}^k_+)$.

We will say that $F : \Omega \to M_{k\times n}$ satisfies the Carathéodory conditions if $F(\cdot,x,w) \in \mathbb{L}(I[x],M_{k\times n})$, where $(x,w) \in [-b,b] \times C(B,\mathbb{R}^k)$ and $F(t,\cdot) : S_t \times C(B,\mathbb{R}^k) \to M_{k\times n}$ is continuous for $t \in [0,a]$. In a similar way we define Carathéodory conditions for $G : \Omega \to \mathbb{R}^k$ and for the derivatives

$$\partial_x G = \big[\partial_{x_\nu} G_\mu\big]_{\mu=1,\dots,k,\nu=1,\dots,n}, \quad \partial_x F_{[i]} = \big[\partial_{x_\nu} F_{i\mu}\big]_{\mu,\nu=1,\dots,n}, \quad i = 1,\dots,k.$$

Suppose that there exist the Fréchet derivatives

$$\partial_w F_{[i]}(P) = \big[\partial_{w_\nu} F_{i\mu}(P)\big]_{\mu=1,\dots,n,\nu=1,\dots,k} \;\; P = (t,x,w) \in \Omega, \quad 1 \le i \le k,$$

and $\partial_w F_{[i]}(P) \in CL(B,M_{n\times k})$ for $P \in \Omega$, $1 \le i \le k$. We will say that $\partial_w F_{[i]}$, $1 \le i \le k$, satisfy the Carathéodory conditions if the functions $\partial_w F_{[i]}(t,\cdot) : S_t \times C(B,\mathbb{R}^k) \to CL(B,M_{n\times k})$, $1 \le i \le k$, are continuous for $t \in [0,a]$ and for $(x,w) \in [-b,b] \times C(B,\mathbb{R}^k)$, $\tilde{w} \in C(B,\mathbb{R})$ we have

$$\partial_w F_{[i]}(\cdot,x,w)\tilde{w} \in \mathbb{L}(I[x],M_{n\times k}), \quad i = 1,\dots,k,$$

where

$$\partial_w F_{[i]}(t,x,w)\tilde{w} = \big[\partial_{w_\nu} F_{i\mu}(t,x,w)\tilde{w}\big]_{\mu=1,\dots,n,\nu=1,\dots,k}.$$

In a similar way we define Carathéodory conditions for the Fréchet derivatives

$$\partial_w G(P) = \big[\partial_{w_\nu} G_\mu(P)\big]_{\mu,\nu=1,\dots,k}, \quad P \in \Omega.$$

Assumption $\mathrm{H}_0[F]$. The function $F : \Omega \to M_{k\times n}$ satisfies the condition (V) and

1) the Carathéodory conditions for F hold and there is $L \in \mathbb{L}([0,a],\mathbb{R}^n_+)$, $L = (L_1,\dots,L_n)$, such that

$$\big(|F_{i1}(t,x,w)|,\dots,|L_{in}(t,x,w)|\big) \le L(t) \ \text{ on } \ \Omega \ \text{ for } \ 1 \le i \le k,$$

2) for $t \in [0,a]$ we have

$$M(t) = \int_0^t L(\tau)d\tau.$$

Assumption $\mathrm{H}[\varphi]$**.** The functions $\phi_0 : [0,a] \to \mathbb{R}$, $\phi : E \to \mathbb{R}^n$ are continuous and

1) $0 \le \phi_0(t) \le t$ for $t \in [0,a]$ and $\varphi(t,x) = (\phi_0(t),\phi(t,x)) \in E$ for $(t,x) \in E$,
2) there exist the derivatives

$$\partial_x\phi(t,x) = \big[\partial_{x_\nu}\phi_\mu(t,x)\big]_{\mu,\nu=1,\dots,n}$$

   and $\partial_x\phi \in C(E,M_{n\times n})$,
3) the constant $Q \ge 0$ is defined by

$$Q = \max\{\|\partial_x\phi(t,x)\|_{n\times n} : \ (t,x) \in E\}$$

   and there is $Q_0 \in \mathbb{R}_+$ such that

$$\|\partial_x\phi(t,x) - \partial_x\phi(t,y)\|_{n\times n} \le Q_0\|x-y\|, \quad (t,x),(t,y) \in E.$$

Given $\bar c = (c_0,c_1,c_2) \in \mathbb{R}^3_+$, we denote by $\mathbb{X}$ the set of all $\psi = (\psi_1,\dots,\psi_k)$ such that for each i, $1 \le i \le k$ we have:

(i) $\psi_i \in C(E_{0.i},\mathbb{R})$, the derivatives $\partial_x\psi_i = (\partial_{x_1}\psi_i,\dots,\partial_{x_n}\psi_i)$ exist on $E_{0.i}$ and $\partial_x\psi_i \in C(E_{0.i},\mathbb{R}^n)$,
(ii) the estimates

$$|\psi_i(t,x)| \le c_0, \ \ \|\partial_x\psi_i(t,x)\| \le c_1,$$
$$\|\partial_x\psi_i(t,x) - \partial_x\psi_i(t,y)\| \le c_2\|x-y\|$$

   are satisfied on $E_{0.i}$.

Let $\psi \in \mathbb{X}$, $\psi = (\psi_1,\dots,\psi_k)$, be given and $\tilde\kappa < c \le a$. We denote by $C_{\psi.c}$ the class of all $z \in C(E_c,\mathbb{R}^k)$, $z = (z_1,\dots,z_k)$, such that $z_i(t,x) = \psi_i(t,x)$ on $E_{0.i}$ for $1 \le i \le k$. For the above ψ and $c \in (\tilde\kappa,a]$ we denote by $C_{\partial\psi_i.c}$, $1 \le i \le k$, the class of all $v \in C(E_c,\mathbb{R}^n)$ such that $v(t,x) = \partial_x\psi_i(t,x)$ on $E_{0.i}$.

Suppose that Assumption $\mathrm{H}[\varphi]$, $\mathrm{H}_0[F]$ are satisfied and $\psi \in X$, $z \in C_{\psi.c}$, $\tilde\kappa < c \le a$. Let us denote by $g_{[i]}[z](\cdot,t,x)$ the solution of the Cauchy problem

$$\eta'(\tau) = F_{[i]}(\tau,\eta(\tau),z_{\varphi(\tau,\eta(\tau))}), \quad \eta(t) = x, \tag{2.1}$$

where $(t,x) \in E_c$ and $a_i < t \le c$. The function $g_{[i]}[z](\cdot,t,x)$ is the i-th bicharactersitic of (1.1) corresponding to z.

Lemma 2.1. *If Assumptions* $\mathrm{H}[\varphi]$, $\mathrm{H}_0[F]$ *are satisfied and* $\psi \in \mathbb{X}$, $z \in C_{\psi.c}$, $\tilde{\kappa} < c \leq a$, $1 \leq i \leq k$, *then the bicharacteristic* $g_{[i]}[z](\cdot, t, x)$ *is defined on* $[a_i, t]$.

Proof. The local existence of a solution to (2.1) follows from classical theorems on Carathéodory solutions for ordinary differential equations. Suppose that $[t_0, t]$ is the interval on which the bicharacteristic $g_{[i]}[z](\cdot, t, x)$ is defined. Then

$$-L(\tau) \leq \frac{d}{d\tau} g_{[i]}[z](\tau, t, x) \leq L(\tau) \quad \text{for} \quad \tau \in [t_0, t]$$

and consequently

$$-b + M(\tau) \leq g_{[i]}[z](\tau, t, x) \leq b - M(\tau) \quad \text{for} \quad \tau \in [t_0, t].$$

Then the bicharacteristic $g_{[i]}[z](\cdot, t, x)$ is defined on $[a_i, t]$. This is the desired conclusion. □

Write $\mathbb{F}[z] = (\mathbb{F}_1[z], \ldots, \mathbb{F}_k[z])$, where

$$\mathbb{F}_i[z](t, x) = \psi_i(t, x) \;\text{ on }\; E_{0.i}, \tag{2.2}$$

and

$$\begin{aligned} \mathbb{F}_i[z](t, x) = \psi_i(a_i, g_{[i]}[z](a_i, t, x)) + \\ + \int_{a_i}^{t} G_i \tau, g_{[i]}[z](\tau, t, x), z_{\varphi(\tau, g_{[i]}[z](\tau, t, x))}) \, d\tau \;\text{ on }\; E_c \setminus E_{0.i}, \end{aligned} \tag{2.3}$$

where $i = 1, \ldots, k$. We consider the functional integral equation

$$z = \mathbb{F}[z]. \tag{2.4}$$

It is easy to give sufficient conditions for the existence and uniqueness of a continuous solution $\tilde{z} : E_c \to \mathbb{R}^k$, $\tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_k)$, of (2.4). We consider solutions to a functional differential problem (1.1), (1.2). Then the main question in our investigations is to prove that there exist the derivatives $\partial_t \tilde{z}_i$, $\partial_x \tilde{z}_i = (\partial_{x_1} \tilde{z}_i, \ldots, \partial_{x_n} \tilde{z}_i)$ on $E_c \setminus E_{0.i}$ for $1 \leq i \leq k$. We show that under natural assumptions on given functions there exists $\tilde{u} : E_c \to M_{k\times n}$, $\tilde{u} = \big[\tilde{u}_{ij}\big]_{i=1,\ldots,k\, j=1,\ldots,n}$ such that $\tilde{u}_{[i]} = (\tilde{u}_{i1}, \ldots, \tilde{u}_{in}) \in C_{\partial\psi_i.c}$ and $\tilde{u}_{[i]} = \partial_x \tilde{z}_i$ on $E_c \setminus E_{0.i}$, where $1 \leq i \leq k$.

Assumption $\mathrm{H}_\star[F, G]$. The function $G : \Omega \to \mathbb{R}^k$ satisfies condition (V) and

1) the Carathéodory conditions for G hold and there is $\alpha \in \mathbb{L}([0, a], \mathbb{R}_+)$ such that $\|G(t, x, \theta)\|_\infty \leq \alpha(t)$ on E, where $\theta \in C(B, \mathbb{R}^k)$ is given by $\theta(\tau, y) = 0_{[k]}$ on B and $0_{[k]} = (0, \ldots, 0) \in \mathbb{R}^k$,
2) Assumption $\mathrm{H}_0[F]$ is satisfied and there exist the derivatives

$$\partial_x G = \big[\, \partial_{x_\nu} G_\mu \,\big]_{\mu=1,\ldots,k,\, \nu=1,\ldots,n}, \;\; \partial_x F_{[i]} = \big[\, \partial_{x_\nu} F_{i\mu} \,\big]_{\mu,\, \nu=1,\ldots,n}$$

and the functions $\partial_x G$, $\partial_x F_{[i]}$, $i = 1, \ldots, k$, satisfy the Carathéodory conditions,

3) for $P = (t, x, w) \in \Omega$ there exist the Fréchet derivatives

$$\partial_w G(P) = \left[\, \partial_{w_\nu} G_\mu(P) \,\right]_{\mu,\, \nu=1,\dots,k}, \;\; \partial_w F_{[i]}(P) = \left[\, \partial_{w_\nu} F_{i\,\mu}(P) \,\right]_{\mu=1,\dots,n,\, \nu=1,\dots,k}$$

and the functions $\partial_w G$, $\partial_w F_{[i]}$, $i = 1, \dots, k$, satisfy the Carathéodory conditions,

4) there is $\beta \in \mathbb{L}([0, a], \mathbb{R}_+)$ such that

$$\|\partial_x G(t, x, w)\|_{k\times n}, \;\|\partial_w G(t, x, w)\|_{k\times k;\star} \le \beta(t) \;\text{ on }\; \Omega$$

and

$$\|\partial_x F_{[i]}(t, x, w)\|_{n\times n}, \;\|\partial_w F_{[i]}(t, x, w)\|_{n\times k;\star} \le \beta(t) \;\text{ on }\; \Omega \text{ for } \; i = 1, \dots, k.$$

For $\Theta \in CL(B, \mathbb{R})$ and $\bar{w} \in C(B, \mathbb{R}^n)$, $\bar{w} = (\bar{w}_1, \dots, \bar{w}_n)$ we define $\Theta \star \bar{w} = (\Theta\,\bar{w}_1, \dots, \Theta\,\bar{w}_n)$. For a function $u : E_c \to M_{k\times n}$, $u = \left[u_{ij}\right]_{i=1,\dots,k,\, j=1,\dots,n}$ we put $u_{[i]} = (u_{i1}, \dots, u_{in})$ and

$$u_{(t,x)} = \left[(u_{ij})_{(t,x)}\right]_{i=1,\dots,k,\, j=1,\dots,n}, \; (\, u_{[i]}\,)_{(t,x)} = (\,(u_{i1})_{(t,x)}, \dots, (u_{in})_{(t,x)}\,), \; 1 \le i \le k.$$

Let us denote by $V_{[i\,\mu]}[z, u]$, $W_{[i]}[z, u]$, $i = 1, \dots, k$, $\mu = 1, \dots, n$, the functions given by

$$V_{[i\,\mu]}[z, u](t, x) = \partial_x F_{i\,\mu}(t, x, z_{\varphi(t,x)}) + \sum_{\nu=1}^{k} \partial_{w_\nu} F_{i\,\mu}(t, x, z_{\varphi(t,x)}) \star (\, u_{[\nu]}\,)_{\varphi(t,x)}\; \partial_x \phi(t, x),$$

$$W_{[i]}[z, u](t, x) = \partial_x G_i(t, x, z_{\varphi(t,x)}) + \sum_{\nu=1}^{k} \partial_{w_\nu} G_i(t, x, z_{\varphi(t,x)}) \star (\, u_{[\nu]}\,)_{\varphi(t,x)}\; \partial_x \phi(t, x).$$

The functions $(\, u_{[\nu]}\,)_{\varphi(t,x)}\; \partial_x \phi(t, x) : D[\varphi(t, x)] \to \mathbb{R}^n$, $\nu = 1, \dots, k$, are defined by

$$(\, u_{[\nu]}\,)_{\varphi(t,x)} = \left(\sum_{j=1}^{n} (\, u_{\nu\, j}\,)_{\varphi(t,x)}\; \partial_{x_1} \phi_j(t, x), \dots, \sum_{j=1}^{n} (\, u_{\nu\, j}\,)_{\varphi(t,x)}\; \partial_{x_n} \phi_j(t, x) \right).$$

Let $\psi \in \mathbb{X}$, $\psi = (\psi_1, \dots, \psi_k)$, be given and $\tilde{\kappa} < c \le a$. Write

$$\mathbb{G}[z, u] = \left[\, \mathbb{G}_{ij}[z, u] \,\right]_{i=1,\dots,k,\, j=1,\dots,n}, \;\; \mathbb{G}_{[i]}[z, u] = \left(\, \mathbb{G}_{i\,1}[z, u], \dots, \mathbb{G}_{i\,n}[z, u] \,\right),$$

where

$$\mathbb{G}_{[i]}[z, u](t, x) = \partial_x \psi_i(t, x) \;\text{ on }\; E_{0.i} \tag{2.5}$$

and

$$\begin{aligned} \mathbb{G}_{[i]}[z, u](t, x) = \partial_x \psi_i(a_i, g_{[i]}[z](a_i, t, x)) + \int_{a_i}^{t} W_{[i]}[z, u](\tau, g_{[i]}[z](\tau, t, x))\, d\tau - \\ - \sum_{\mu=1}^{n} \int_{a_i}^{t} V_{[i\mu]}[z, u](\tau, g_{[i]}[z](\tau, t, x))\, u_{i\mu}(\tau, g_{[i]}[z](\tau, t, x))\, d\tau. \end{aligned} \tag{2.6}$$

We put $i = 1, \ldots, k$ in (2.5), (2.6). We consider the system of integral functional equations consisting of (2.4) and

$$u = \mathbb{G}[z, u]. \tag{2.7}$$

We first give estimates of solutions to (2.4), (2.7).

Lemma 2.2. *Suppose that Assumptions* $\mathrm{H}[\varphi]$, $\mathrm{H}_\star[F, G]$ *are satisfied and*

1) $\psi \in \mathbb{X}$ *and* $\tilde{\kappa} < c \le a$,
2) *the functions* $\tilde{z} : E_c \to \mathbb{R}^k$, $\tilde{u} : E_c \to M_{k\times n}$, *are continuous and they satisfy* (2.4), (2.7).

Then

$$\|\tilde{z}\|_{(t,\mathbb{R}^k)} \le \zeta(t), \quad \|\tilde{u}\|_{(t,M_{k\times n})} \le \chi(t), \quad t \in [\kappa, c],$$

where

$$\zeta(t) = c_0 \exp\Big[\int_\kappa^t \beta(\tau)\, d\tau\Big] + \int_\kappa^t \alpha(\xi) \exp\Big[\int_\xi^t \beta(\tau) d\tau\Big]\, d\xi, \tag{2.8}$$

$$\chi(t) = \Big[c_1 + (1 + Q_\star c_1) \int_\kappa^t \beta(\tau)\, d\tau\Big]\Big[1 - Q_\star(1 + Q_\star c_1) \int_\kappa^t \beta(\tau)\, d\tau\Big]^{-1} \tag{2.9}$$

and $Q_\star = \max\{1, Q\}$.

Proof. Write

$$\tilde{\zeta}(t) = \|\tilde{z}\|_{(t,\mathbb{R}^k)}, \quad \tilde{\chi}(t) = \|\tilde{u}\|_{(t,M_{k\times n})}, \quad t \in [\kappa, c].$$

It follows from Assumption $\mathrm{H}_\star[F, G]$ and from (2.2), (2.3), (2.5), (2.6) that $(\tilde{\zeta}, \tilde{\chi})$ satisfy the integral inequalities

$$\tilde{\zeta}(t) \le c_0 + \int_\kappa^t \big[\alpha(\tau) + \beta(\tau)\, \tilde{\zeta}(\tau)\big]\, d\tau,$$

$$\tilde{\chi}(t) \le c_1 + \int_\kappa^t \beta(\tau) \big[1 + Q_\star \tilde{\chi}(\tau)\big]^2\, d\tau, \quad t \in [\kappa, c].$$

The functions (ζ, χ) satisfy integral equations corresponding to the above inequalities. This proves the lemma. □

In the next part of the paper we assume that $c \in (\kappa, a]$ is such a small constant that

$$Q_\star(1 + Q_\star c_1) \int_\kappa^c \beta(\tau)\, d\tau < 1. \tag{2.10}$$

3. EXISTENCE AND UNIQUENESS OF SOLUTIONS

Suppose that $\psi \in \mathbb{X}$ and $\zeta,\ \chi : [\kappa, c] \to \mathbb{R}_+$ are defined by (2.8), (2.9). Given $d,\ h \in \mathbb{R}_+$, $d \ge c_1$, $h \ge c_2$. We denote by $C_{\psi.c}[\zeta, d]$ the class of all $z \in C_{\psi.c}$, such that

$$\|z\|_{(t,\mathbb{R}^k)} \le \zeta(t) \ \text{ for } \ t \in [\kappa, c] \ \text{ and } \ \|z(t,x) - z(t,y)\|_\infty \le d\,\|x - y\| \ \text{ on } \ E_c.$$

We denote by $C_{\partial\psi_i.c}[\chi, h]$, $1 \le i \le k$, the set of all $v \in C_{\partial\psi_i.c}$ satisfying the conditions:

$$\|v\|_{(t,\mathbb{R}^n)} \le \chi(t) \ \text{ for } \ t \in [a_i, c] \ \text{ and } \ \|v(t,x) - v(t,y)\| \le h\,\|x - y\| \ \text{ on } \ E_c \setminus E_{0.i}.$$

Write $A = \zeta(a)$, $C = \chi(a)$ and $\Omega[A] = E \times K_{C(B,\mathbb{R}^k)}[A]$, where

$$K_{C(B,\mathbb{R}^k)}[A] = \{w \in C(B, \mathbb{R}^k) : \ \|w\|_B \le A\}.$$

Assumption H$[F,G]$. The functions $F : \Omega \to M_{k\times n}$, $G : \Omega \to \mathbb{R}^k$ satisfy Assumption H$_\star[F,G]$ and there is $\gamma \in \mathbb{L}([0,a], \mathbb{R}_+)$ such that the terms

$$\begin{aligned} &\|\partial_x F_{[i]}(t,x,w) - \partial_x F_{[i]}(t,\bar{x},\bar{w})\|_{n\times n}, \\ &\|\partial_w F_{[i]}(t,x,w) - \partial_w F_{[i]}(t,\bar{x},\bar{w})\|_{n\times k;\star}, \quad i = 1,\ldots,k, \end{aligned}$$

and

$$\|\partial_x G(t,x,w) - \partial_x G(t,\bar{x},\bar{w})\|_{k\times n}, \quad \|\partial_w G(t,x,w) - \partial_w G(t,\bar{x},\bar{w})\|_{k\times k;\star}$$

are bounded form above on $\Omega[A]$ by $\gamma(t)\big[\|x - \bar{x}\| + \|w - \bar{w}\|_B\big]$.

Remark 3.1. It is important in our considerations that we assume the Lipschitz condition for $\partial_x G$, $\partial_w G$, $\partial_x F_{[i]}$, $\partial_w F_{[i]}$, $1 \le i \le k$, on the bounded domain $\Omega[A]$. It is clear that there are functional differential systems such that Assumption H$[F,G]$ holds and the functions $\partial_x G$, $\partial_w G$, $\partial_x F_{[i]}$, $\partial_w F_{[i]}$, $1 \le i \le k$, do not satisfy the Lipschitz condition with respect to w on Ω.

Remark 3.2. Note that the theorems on the existence of solutions to hyperbolic functional differential systems presented in [13,14] are not applicable to (1.1), (1.2).

Lemma 3.3. *Suppose that Assumptions* H$[\varphi]$, H$[F,G]$ *are satisfied and* $\psi,\ \tilde{\psi} \in \mathbb{X}$, $z \in C_{\psi.c}[\zeta, d]$, $\tilde{z} \in C_{\tilde{\psi}.c}[\zeta, d]$. *Then the bicharacteristics* $g_{[i]}[z](\cdot, t, x)$ *and* $g_{[i]}[\tilde{z}](\cdot, t, x)$ *exist on intervals* $[a_i, \delta[z; t, x]\,]$ *and* $[a_i, \delta[\tilde{z}; t, x]\,]$ *such that for* $\xi = \delta[z; t, x]$, $\tilde{\xi} = \delta[\tilde{z}; t, x]$ *we have:* $(\xi, g_{[i]}[z](\xi, t, x)) \in \partial E_c$, $(\tilde{\xi}, g_{[i]}[z](\tilde{\xi}, t, x)) \in \partial E_c$, *where* $i = 1,\ldots,k$ *and* ∂E_c *is the boundary of* E_c. *For each* i, $1 \le i \le k$, *the solution of* (2.1) *is unique and we have the estimates*

$$\|g_{[i]}[z](\tau,t,x) - g_{[i]}[z](\tau,t,y)\| \le \|x - y\| \,\exp\left\{\bar{C}\left|\int_\tau^t \beta(\xi)\,d\xi\right|\right\}, \tag{3.1}$$

and

$$\|g_{[i]}[z](\tau,t,x) - g_{[i]}[\tilde{z}](\tau,t,x)\| \le \left|\int_\tau^t \beta(\xi)\,\|z - \tilde{z}\|_{(\xi,\mathbb{R}^k)}\right| \exp\left\{\bar{C}\left|\int_\tau^t \beta(\xi)\,d\xi\right|\right\}, \tag{3.2}$$

where $\bar{C} = 1 + dQ$, $(t,x),\ (t,y) \in E_c \setminus E_{0.i}$, $1 \le i \le k$.

Proof. The existence and uniqueness of the solution to (2.1) follows from classical theorems on Carathéodory solutions of ordinary differential equations. We prove that the integral inequalities

$$\|g_{[i]}[z](\tau,t,x)-g_{[i]}[z](\tau,t,y)\| \leq \|x-y\|+\bar{C}\left|\int_\tau^t \beta(\xi)\,\|g_{[i]}[z](\xi,t,x)-g_{[i]}[z](\xi,t,y)\|\,d\xi\right| \tag{3.3}$$

and

$$\begin{aligned}&\|g_{[i]}[z](\tau,t,x)-g_{[i]}[\tilde{z}](\tau,t,x)\| \leq \\ &\leq \bar{C}\left|\int_\tau^t \beta(\xi)\,\|g_{[i]}[z](\xi,t,x)-g_{[i]}[\tilde{z}](\xi,t,x)\|d\,\xi\right| + \left|\int_\tau^t \beta(\xi)\,\|z-\tilde{z}\|_{(\xi,\mathbb{R}^k)}\,d\,\xi\right|\end{aligned} \tag{3.4}$$

are satisfied for $1 \leq i \leq k$. It follows from (2.1) that

$$g_{[i]}[z](\tau,t,x) = x + \int_t^\tau F_{[i]}(\xi, g_{[i]}[z](\xi,t,x), z_{\varphi(\xi,g_{[i]}[z](\xi,t,x))})\,d\xi.$$

Note that the functions $z_{\varphi(\xi,g_{[i]}[z](\xi,t,x))}$ and $z_{\varphi(\xi,g_{[i]}[z](\xi,t,y))}$ have different domains. We need the following construction. Write $E_\star = [-b_0, a] \times [-b-r, b+r]$. There is $Z : E_\star \to \mathbb{R}^k$ such that:

(i) Z is continuous and $\|Z(t,x)-Z(t,y)\|_\infty \leq d\,\|x-y\|$ on $E_\star$,
(ii) $Z(t,x) = z(t,x)$ for $(t,x) \in E$.

Then we have $Z_{\varphi(\xi,g_{[i]}[z](\xi,t,x))},\ Z_{\varphi(\xi,g_{[i]}[z](\xi,t,y))} : B \to \mathbb{R}^k$. We conclude from Assumptions H[φ], H[F, G] that

$$\begin{aligned}&\|g_{[i]}[z](\tau,t,x)-g_{[i]}[z](\tau,t,y)\| \leq \\ &\leq \|x-y\| + \\ &\quad + \left|\int_t^\tau \|F_{[i]}(\xi, g_{[i]}[z](\xi,t,x), Z_{\varphi(\xi,g_{[i]}[z](\xi,t,x))}) - \right. \\ &\qquad\qquad \left. - F_{[i]}(\xi, g_{[i]}[z](\xi,t,y), Z_{\varphi(\xi,g_{[i]}[z](\xi,t,y))})\|\,d\xi\right| \leq \\ &\leq \|x-y\| + \\ &\quad + \left|\int_t^\tau \beta(\xi)\big[\|g_{[i]}[z](\xi,t,x)-g_{[i]}[z](\xi,t,y)\| + \right. \\ &\qquad\qquad \left. + \|Z_{\varphi(\xi,g_{[i]}[z](\xi,t,x))} - Z_{\varphi(\xi,g_{[i]}[z](\xi,t,y))}\|_B\big]\,d\xi\right|.\end{aligned}$$

This gives (3.3). In a similar way we prove (3.4). Then we obtain (3.1), (3.2) from the Gronwall inequality. □

Write

$$\Gamma_0(t) = \exp\Big\{\bar{c}\int_\kappa^t \beta(\xi)d\xi\Big\}\Big[\bar{C}(1+CQ)\int_\kappa^t \gamma(\xi)\,d\xi + (hQ^2+CQ_0)\int_\kappa^t \beta(\xi)\,d\xi\Big],$$

$$\tilde{\Gamma}(t) = (1+C)\Gamma_0(t) + \big[c_2 + h(1+CQ)\big]\,\exp\Big\{\bar{C}\int_\kappa^t \beta(\xi)\,d\xi\Big\},$$

$$\Gamma(t) = \exp\Big\{\bar{C}\int_\kappa^t \beta(\xi)\,d\xi\Big\}\Big[c_1 + \bar{C}\int_\kappa^t \beta(\xi)\,d\xi\Big].$$

Assumption H$[c]$. The constant $c \in (\kappa, a]$ is small enough to satisfy (2.10) and $\tilde{\Gamma}(c) \le h$, $\Gamma(c) \le d$.

Theorem 3.4. *Suppose that Assumptions* H$[\varphi]$, H$[F,G]$, H$[c]$ *are satisfied and* $\psi \in \mathbb{X}$. *Then there exists a solution* $\bar{z} : E_c \to \mathbb{R}^k$ *of* (1.1), (1.2). *If* $\tilde{\psi} \in \mathbb{X}$, $\tilde{\psi} = (\tilde{\psi}_1, \dots, \tilde{\psi}_k)$, *and* $\tilde{z} : E_c \to \mathbb{R}^k$ *is a solution of* (1.1) *with the initial conditions*

$$z_i(t,x) = \tilde{\psi}_i(t,x) \ \text{ on } \ E_{0.i} \ \text{ for } \ 1 \le i \le k,$$

then there is $\Phi \in \mathbb{L}([\kappa, c], \mathbb{R}_+)$ *such that*

$$\|\bar{z} - \tilde{z}\|_{(t,\mathbb{R}^k)} \le [|\psi - \tilde{\psi}|]_0 \ \exp\Big\{\int_\kappa^t \Phi(\xi)\,d\xi\Big\}, \quad t \in [\kappa, c], \tag{3.5}$$

where

$$[|\psi - \tilde{\psi}|]_0 = \max_{1\le i\le k} \ \max\big\{|\psi_i(t,x) - \tilde{\psi}(t,x)| : \ (t,x) \in E_{0.i}\big\}.$$

Proof. The proof falls into three parts.
Part I. We define the sequences $\{z^{(m)}\}$, $\{u^{(m)}\}$, where

$$z^{(m)} : E_c \to \mathbb{R}^k, \ \ z^{(m)} = (z_1^{(m)}, \dots, z_k^{(m)}), \ u^{(m)} : E_c \to M_{k\times n},$$

$$u^{(m)} = \big[u_{ij}^{(m)}\big]_{i=1,\dots,k,\, j=1,\dots,n}, u_{[i]}^{(m)} = (u_{i1}^{(m)}, \dots, u_{in}^{(m)}) \ \text{ for } \ 1 \le i \le k,$$

in the following way. Write

$$z_i^{(0)}(t,x) = \psi_i(t,x) \ \text{ on } \ E_{0.i}, \ \ z_i^{(0)}(t,x) = \psi_i(a_i,x) \ \text{ on } \ E_c \setminus E_{0.i},$$

$$u_{[i]}^{(0)}(t,x) = \partial_x\psi_i(t,x) \ \text{ on } \ E_{0.i}, \ \ u_{[i]}^{(0)}(t,x) = \partial_x\psi_i(a_i,x) \ \text{ on } \ E_c \setminus E_{0.i},$$

where $i = 1, \ldots, k$. If $z^{(m)} : E_c \to \mathbb{R}^k$ and $u^{(m)} : E_c \to M_{k\times n}$ are already defined, then $u^{(m+1)}_{[i]}$ is a solution of the equation

$$v = \mathbb{Q}^{(m)}_{[i]}[v], \tag{3.6}$$

where $v = (v_1, \ldots, v_n)$ and

$$\mathbb{Q}^{(m)}_{[i]}[v](t,x) = \partial_x \psi_i(t,x) \ \text{ on } \ E_{0.i}, \tag{3.7}$$

$$\begin{aligned} &\mathbb{Q}^{(m)}_{[i]}[v](t,x) = \\ &= \partial_x \psi_i(a_i, g_{[i]}[z^{(m)}(a_i,t,x))+ \\ &+ \int_{a_i}^{t} W_{[i]}[z^{(m)}, u^{(m)}](\tau, g_{[i]}[z^{(m)}](\tau,t,x))\, d\tau- \\ &- \sum_{\mu=1}^{n} \int_{a_i}^{t} V_{[i\mu]}[z^{(m)}, u^{(m)}](\tau, g_{[i]}[z^{(m)}](\tau,t,x))\; v_\mu(\tau, g_{[i]}[z^{(m)}](\tau,t,x)\, d\tau \ \text{ on } \ E_c \setminus E_{0.i}. \end{aligned} \tag{3.8}$$

We put $i = 1, \ldots, k$ in (3.6)–(3.8). The function $z^{(m+1)}$ is given by

$$z^{(m+1)}(t,x) = \mathbb{F}[z^{(m)}](t,x) \ \text{ on } \ E_c. \tag{3.9}$$

We prove that:
(I_m) the sequences $\{z^{(m)}\}$ and $\{u^{(m)}\}$ are defined on E_c and for $m \ge 0$ we have

$$z^{(m)} \in C_\psi[\zeta, d], \ \ u^{(m)}_{[i]} \in C_{\partial\psi_i.c}[\chi, h] \ \text{ for } \ 1 \le i \le k,$$

(II_m) there exist the sequences $\{\partial_x z^{(m)}_i\}$, $1 \le i \le k$, and for $m \ge 0$ we have

$$\partial_x z^{(m)}_i(t,x) = u^{(m)}_{[i]}(t,x) \ \text{ on } \ E_{0.i} \ \text{ for } \ 1 \le i \le k.$$

We prove (I_m) and (II_m) by induction. It is clear that conditions (I_0) and (II_0) are satisfied. Suppose that (I_m) and (II_m) hold for a given $m \ge 0$. We first prove that there is

$$u^{(m+1)} : E_c \to M_{k\times n}, \ \ u^{(m+1)} = \big[u^{(m+1)}_{ij}\big]_{i=1,\ldots,k,\, j=1,\ldots,n},$$
$$u^{(m+1)}_{[i]} = (u^{(m+1)}_{i1}, \ldots, u^{(m+1)}_{in}) \ \text{ for } \ 1 \le i \le k,$$

and $u^{(m+1)}_{[i]} \in C_{\partial\psi_i.c}[\chi, h]$ for $1 \le i \le k$. We claim that

$$\mathbb{Q}^{(m)}_{[i]} : C_{\partial\psi_i.c}[\chi, h] \to C_{\partial\psi_i.c}[\chi, h]. \tag{3.10}$$

Suppose that $v \in C_{\partial\psi_i.c}[\chi, h]$. It is easily seen that the terms

$$\int_{a_i}^{t} \big\| \, W_{[i]}[z^{(m)}, u^{(m)}](\tau, g_{[i]}(\tau,t,x)) - W_{[i]}[z^{(m)}, u^{(m)}](\tau,t,y)) \, \big\| \, d\tau,$$

$$\int_{a_i}^{t} \| V_{[i\mu]}[z^{(m)}, u^{(m)}](\tau, g_{[i]}(\tau, t, x)) - V_{[i\mu]}[z^{(m)}, u^{(m)}](\tau, t, y)) \| \, d\tau$$

can be bounded from above by $\Gamma_0(t)\, \|x - y\|$. This gives

$$\big\|\mathbb{Q}^{(m)}_{[i]}[v](t,x) - \mathbb{Q}^{(m)}_{[i]}[v](t,y)\big\| \le \tilde{\Gamma}(t)\, \|x-y\| \text{ on } E_c \setminus E_{0.i} \quad \text{for } i = 1, \ldots, k. \tag{3.11}$$

It follows from Assumptions H$[\varphi]$, H$[F, G]$ that

$$\big\|\mathbb{Q}^{(m)}_{[i]}[v](t,x)\big\| \le c_1 + \int_{\kappa}^{t} \beta(\tau) \left[1 + Q_\star\, \chi(t)\right]^2 d\tau, \quad (t,x) \in E_c \setminus E_{0.i},$$

and consequently

$$\big\|\mathbb{Q}^{(m)}_{[i]}[v]\big\|_{(t,\mathbb{R}^k)} \le \chi(t) \quad \text{for} \quad t \in [a_i, c]. \tag{3.12}$$

Estimates (3.11), (3.12) and (3.7) imply (3.10).

It follows that there is $K \in \mathbb{L}([\kappa, c], \mathbb{R}_+)$ such that for v, $\tilde{v} \in C_{\partial\psi_i.c}[\chi, h]$ we have

$$\big\|\mathbb{Q}^{(m)}_{[i]}[v](t,x) - \mathbb{Q}^{(m)}_{[i]}[\tilde{v}]\| \le \int_{a_i}^{t} K(\tau)\, \|v - \tilde{v}\|_{(\tau,\mathbb{R}^n)}\, d\tau, \quad (t,x) \in E_c \setminus E_{0.i}.$$

For the above $v, \tilde{v}$ we put

$$[|v - \tilde{v}|] = \max \left\{ \|v - \tilde{v}\|_{(t,\mathbb{R}^n)} \exp\left[-2 \int_{a_i}^{t} K(\tau) d\tau \right] : t \in [a_i, c] \right\}.$$

Then we have

$$\|\mathbb{Q}^{(m)}_{[i]}[v](t,x) - \mathbb{Q}^{(m)}_{[i]}[\tilde{v}]\| \le [|v - \tilde{v}|] \int_{a_i}^{t} K(\tau) \exp\left\{ 2 \int_{a_i}^{\tau} K(\xi) d\xi \right\} d\tau \le$$

$$\le \frac{1}{2} [|v - \tilde{v}|] \exp\left\{ 2 \int_{a_i}^{t} K(\xi) d\xi \right\}, \quad (t,x) \in E_c \setminus E_{0.i},$$

and consequently

$$\big[\big|\mathbb{Q}^{(m)}_{[i]}[v] - \mathbb{Q}^{(m)}_{[i]}[\tilde{v}\big|\big] \le \frac{1}{2} [|v - \tilde{v}|].$$

From the Banach fixed point theorem it follows that there exists exactly one $u^{(m+1)}_{[i]} \in C_{\partial\psi_i.c}[\chi, h]$ satisfying (3.6). Then $u^{(m+1)}$ is defined on E_c. It is easily seen that $z^{(m+1)}$ given by (3.9) satisfies the conditions

$$\|z^{(m+1)}\|_{(t,\mathbb{R}^k)} \le \zeta(t), \quad t \in [\kappa, c],$$

$$\|z^{(m+1)}(t,x) - z^{(m+1)}(t,y)\|_\infty \le \Gamma(t)\,\|x-y\| \quad \text{on } \ E_c.$$

It follows from the above estimates and from (2.2) that $z^{(m+1)} \in C_{\psi.c}[\zeta, d]$ which completes the proof of (I_{m+1}). Write

$$W_i^{(m+1)}(t,x,y) = z_i^{(m+1)}(t,y) - z_i^{(m+1)}(t,x) - u_{[i]}^{(m+1)}(t,x) \circ (y-x),$$
$$(t,x),(t,y) \in E_c, \ 1 \le i \le k,$$

and $W^{(m+1)} = \big(W_1^{(m+1)}, \ldots, W_k^{(m+1)}\big)$. It follows that there is $C^{(m+1)} \in \mathbb{R}_+$ such that

$$\|W^{(m+1)}(t,x,y)\|_\infty \le C^{(m+1)} \|x-y\|^2, \quad (t,x),(t,y) \in E_c. \tag{3.13}$$

We conclude from (3.13) that there exists the derivatives $\partial_x z_i^{(m+1)}$, $1 \le i \le k$, and $\partial_x z_i^{(m+1)}(t,x) = u_{[i]}^{(m+1)}(t,x)$ on E_c. This proves (II_{m+1}).

Part II. We prove that the sequences $\{z^{(m)}\}$ and $\{u^{(m)}\}$ are uniformly convergent on E_c. Write

$$Z^{(m)}(t) = \|z^{(m)} - z^{(m-1)}\|_{(t,\mathbb{R}^k)}, \ \ U^{(m)}(t) = \|u^{(m)} - u^{(m-1)}\|_{(t,M_{k\times n})},$$

where $t \in [\kappa, c]$, $m \ge 1$. We conclude from Assumptions H$[\varphi]$, H$[F,G]$ and from (3.6)–(3.9) that there are K_0, K_1, $K_2 \in \mathbb{L}([\kappa,c], \mathbb{R}_+)$ such that

$$Z^{(m+1)}(t) \le \int_\kappa^t K_0(\tau) Z^{(m)}(\tau) d\tau \tag{3.14}$$

and

$$U^{(m+1)}(t) \le \int_\kappa^t K_1(\tau)\big[Z^{(m)}(\tau) + U^{(m)}(\tau)\,\big]\,d\tau + \int_\kappa^t K_2(\tau)\,U^{(m+1)}(\tau)\,d\tau, \tag{3.15}$$

where $m \ge 1$, $t \in [\kappa, c]$. From (3.15) it may be concluded that

$$U^{(m+1)}(t) \le \int_\kappa^t K_1(\tau)\,\big[Z^{(m)}(\tau) + U^{(m)}(\tau)\big]\,d\tau \ \exp\Big\{\int_\kappa^t K_2(\tau)\,d\tau\Big\} \tag{3.16}$$

where $m \ge 1$, $t \in [\kappa, c]$. It follows from (3.14), (3.16) that there is $K \in \mathbb{L}([\kappa,c],\mathbb{R}_+)$ such that

$$Z^{(m+1)}(t) + U^{(m+1)}(t) \le \int_\kappa^t K(\tau)\,\big[Z^{(m)}(\tau) + U^{(m)}(\tau)\big]\,d\tau, \quad t \in [\kappa, c],\, m \ge 1. \tag{3.17}$$

Write $V^{(m)}(t) = Z^{(m)}(t) + U^{(m)}(t)$, $t \in [\kappa, c]$, $m \ge 1$, and

$$[|V^{(m)}|] = \max\Big\{V^{(m)}(t)\exp\Big\{-2\int_\kappa^t K(\tau)\,d\tau\Big\} : t \in [\kappa, c]\Big\}.$$

We conclude from (3.17) that

$$[|V^{(m+1)}|] \le \frac{1}{2}[|V^{(m)}|] \text{ for } m \ge 1.$$

There is $C_1 \in \mathbb{R}_+$ such that $[|V^{(1)}|] \le C_1$. From the above recurrent inequality we conclude that

$$\lim_{m\to\infty}[|V^{(m)}|] = 0$$

and consequently there are

$$\bar{z} : E_c \to \mathbb{R}^k, \quad \bar{z} = (\bar{z}_1, \ldots, \bar{z}_k),$$

$$\bar{u} : E_c \to M_{k\times n}, \;\; \bar{u} = \left[\bar{u}_{ij}\right]_{i=1,\ldots,k,\, j=1,\ldots,n}, \;\; \bar{u}_{[i]} = (\bar{u}_{i1}, \ldots, \bar{u}_{in}), \quad 1 \le i \le k,$$

such that

$$\bar{z}(t,x) = \lim_{m\to\infty} z^{(m)}(t,x), \;\; \bar{u}(t,x) = \lim_{m\to\infty} u^{(m)}(t,x) \text{ uniformly on } E_c.$$

We conclude from (I_m), (II_m) that there exist the derivatives $\partial_x \bar{z}_i$, $1 \le i \le k$, and $\partial_x \bar{z}_i(t,x) = \bar{u}_{[i]}(t,x)$ on E_c for $1 \le i \le k$. It follows from (3.9) that

$$\bar{z}_i(t, g_{[i]}[\bar{z}](t,a_i,x)) = \psi_i(a_i,x) + \int_{a_i}^{t} G_i\big(\tau, g_{[i]}[\bar{z}](\tau,a_i,x),\, \bar{z}_{\varphi(\tau, g_{[i]}[\bar{z}](\tau,a_i,x))}\big)\, d\tau$$

and $\bar{z}(t,x) = \psi_i(t,x)$ on $E_{0.i}$, $1 \le i \le k$. It is easily seen that $\bar{z}$ is a solution to (1.1), (1.2).

Part III. We prove (3.5). It follows that the function $\bar{z} - \tilde{z}$ satisfies the integral inequality

$$\|\bar{z} - \tilde{z}\|_{(t,\mathbb{R}^k)} \le [|\psi - \tilde{\psi}|]_0 + \int_{\kappa}^{t} \Phi(\tau)\, \|\bar{z} - \tilde{z}\|_{(\tau,\mathbb{R}^k)}\, d\tau, \quad t \in [\kappa, c],$$

where

$$\Phi(\tau) = \beta(\tau)\Big\{1 + \exp\Big[\bar{C}\int_{\kappa}^{c} \beta(\xi)\, d\xi\Big]\, \Big[c_1 + \bar{C}\int_{\kappa}^{c} \beta(\xi)\, d\xi\Big]\Big\}.$$

It follows from the Gronwall inequality that (3.5) is satisfied with the above given Φ.

This completes the proof of the theorem. □

Remark 3.5. Note that results presented in [4, 10, 22] are not applicable to our generalized Cauchy problem.

4. DIFFERENTIABILITY OF SOLUTIONS

Given $\bar{c} = (c_0, c_1, c_2) > (0,0,0)$. In this part of the paper we denote by $\mathbb{X}$ the set of all $\psi = (\psi_1, \ldots, \psi_k)$ such that for each i, $1 \le i \le k$ we have:

(i) $\psi_i \in C(E_{0.i}, \mathbb{R})$, the derivatives $\partial_x \psi_i = (\partial_{x_1}\psi_i, \ldots, \partial_{x_n}\psi_i)$ exist on $E_{0.i}$ and $\partial_x \psi_i \in C(E_{0.i}, \mathbb{R}^n)$,

(ii) the estimates
$$|\psi_i(t,x)| < c_0, \quad \|\partial_x \psi_i(t,x)\| < c_1,$$
$$\|\partial_x \psi_i(t,x) - \partial_x \psi_i(t,y)\| < c_2 \|x - y\|$$
are satisfied on $E_{0.i}$.

Suppose the Assumptions H$[\varphi]$, H$[F,G]$, H$[c]$ are satisfied and $\psi \in \mathbb{X}$. Let us denote by $z(\cdot,\psi)$ the solution of (1.1), (1.2). Let $g_{[i]}[z(\cdot,\psi)]$, $1 \le i \le k$, denote the i-th bicharacteristic of (1.1) corresponding to $z(\cdot,\psi)$. Write
$$\Sigma_{c.i} = \{(\tau,t,x) :\ (t,x) \in E :\ a_i \le t \le c,\ a_i \le \tau \le t\}, \quad i = 1, \ldots, k.$$

We will use the symbols $\Xi = (\Xi_1, \ldots, \Xi_k)$ and $\Lambda = (\Lambda_1, \ldots, \Lambda_k)$ to denote the operators defined on $\mathbb{X}$ in the following way:
$$\Xi[\psi] = z(\cdot,\psi), \ \ \Lambda_i[\psi] = g_{[i]} z(\cdot,\psi), \quad i = 1, \ldots, k.$$

Then we have: $\Xi : \mathbb{X} \to C(E_c, \mathbb{R}^k)$ and $\Lambda_i : \mathbb{X} \to C(\Sigma_{c.i}, \mathbb{R}^n)$ for $1 \le i \le k$.

We prove that for each $\psi \in \mathbb{X}$ there exist the Fréchet derivatives $\partial\Xi_i[\psi]$, $\partial\Lambda_i[\psi]$ of Ξ_i and Λ_i at the point $\psi \in X$, $1 \le i \le k$. Moreover, if $\psi, \vartheta \in \mathbb{X}$ then the functions $(\partial\Xi_1[\psi]\vartheta, \ldots, \partial\Xi_k[\psi]\vartheta)$ and $\partial\Lambda_i[\psi]\vartheta$, $1 \le i \le k$, are solutions of linear integral functional systems generated by (1.1), (1.2).

The following notations will be needed throughout the paper. For $Y \in CL(B, M_{n\times k})$, $U \in CL(B, M_{k\times n})$, $\tilde{Y} \in CL(B, \mathbb{R}^k)$, $\tilde{w} \in C(B, \mathbb{R}^k)$, $q \in \mathbb{R}^n$, where
$$Y = \big[Y_{ij}\big]_{i=1,\ldots,n,\ j=1,\ldots,k}, \quad U = \big[U_{ij}\big]_{i=1,\ldots,k,\ j=1,\ldots,n},$$
$$\tilde{Y} = (\tilde{Y}_1, \ldots, \tilde{Y}_k), \quad \tilde{w} = (\tilde{w}_1, \ldots, \tilde{w}_k)^T, \quad q = (q_1, \ldots, q_n)^T,$$
we write
$$Y\,\tilde{w} = \left(\sum_{j=1}^{k} Y_{1j}\tilde{w}_j, \ldots, \sum_{j=1}^{k} Y_{nj}\tilde{w}_j \right)^T,$$
$$\tilde{Y} \diamond \tilde{w} = \sum_{i=1}^{k} \tilde{Y}_i\,\tilde{w}_i,$$
$$U\,q = \left(\sum_{j=1}^{n} U_{1j} q_j, \ldots, \sum_{j=1}^{n} U_{kj} q_j \right)^T.$$

Let $z = (z_1, \ldots, z_k)^T$ and $\Delta_{[i]} = (\Delta_{i1}, \ldots, \Delta_{in})$, $1 \le i \le k$, denote unknown functions of the variables (t,x) and (τ,t,x) respectively. Suppose that ψ, $\vartheta \in \mathbb{X}$. We construct a linear system of integral functional equations for $\partial\Xi_i[\psi]\vartheta$, $\partial\Lambda_i[\psi]\vartheta$, $1 \le i \le k$. Write $u(\cdot,\psi) = \partial_x z(\cdot,\psi)$, and

$$U^{(i)}(\tau,t,x,\Delta_{[i]}) = \big(u(\cdot,\psi)\big)_{\varphi(\tau,g_{[i]}[z(\cdot,\psi)](\tau,t,x))}\big[\partial_x\phi(\tau,g_{[i]}[z(\cdot,\psi)](\tau,t,x))\Delta_{[i]}(\tau,t,x)\big],$$

$$\theta_i(\tau,t,x) = \big(\tau, g_{[i]}[z(\cdot,\psi)](\tau,t,x), (z(\cdot,\psi))_{\varphi(\tau,g_{[i]}[z(\cdot,\psi)](\tau,t,x))}\big),$$

where $1 \le i \le k$. It follows from Theorem 3.4 that

$$z(\cdot,\psi) : E_c \to \mathbb{R}^k,\ u(\cdot,\psi) : E_c \to M_{k\times n},\ g_{[i]}[z(\cdot,\psi)] : \Sigma_{c.i} \to \mathbb{R}^n, \quad 1 \le i \le k,$$

are known functions. We consider the system of integral functional equations

$$\begin{aligned} z_i(t,x) = {} & \vartheta(a_i, g_{[i]}[z(\cdot,\psi)](a_i,t,x)) + \\ & + \partial_x\psi_i(a_i, g_{[i]}[z(\cdot,\psi)](a_i,t,x)) \circ \Delta_{[i]}(a_i,t,x) + \\ & + \int_{a_i}^{t} \partial_x G_i(\theta_i(\xi,t,x)) \circ \Delta_{[i]}(\xi,t,x)\, d\xi + \\ & + \int_{a_i}^{t} \partial_w G_i(\theta_i(\xi,t,x)) \diamond z_{\varphi(\xi,g_{[i]}[z(\cdot,\psi)](\xi,t,x))}\, d\xi + \\ & + \int_{a_i}^{t} \partial_w G_i(\theta_i(\xi,t,x)) \diamond U^{(i)}(\xi,t,x,\Delta_{[i]})\, d\xi \end{aligned} \tag{4.1}$$

with the initial conditions

$$z_i(t,x) = \vartheta_i(t,x) \ \text{ on } \ E_{0.i} \tag{4.2}$$

and

$$\begin{aligned} \Delta_{[i]}(\tau,t,x) = {} & \int_{t}^{\tau} \partial_x F_{[i]}(\theta_i(\xi,t,x))\Delta_{[i]}(\xi,t,x)\, d\xi + \\ & + \int_{t}^{\tau} \partial_w F_{[i]}(\theta_i(\xi,t,x))\, z_{\varphi(\xi,g_{[i]}[z(\cdot,\psi)](\xi,t,x))}\, d\xi + \\ & + \int_{t}^{\tau} \partial w F_{[i]}(\theta_i(\xi,t,x))\, U^{(i)}(\xi,t,x,\Delta_{[i]})\, d\xi. \end{aligned} \tag{4.3}$$

We put $i = 1,\dots,k$ in (4.1)–(4.3).

We formulate the main theorem of the paper.

Theorem 4.1. *If Assumptions* $\mathrm{H}[\varphi]$, $\mathrm{H}[F,G]$, $\mathrm{H}[\mathrm{c}]$ *are satisfied and* $\psi \in X$, *then:*

(i) *there exist the Fréchet derivatives* $\partial\Xi_i[\psi]$, $\partial\Lambda_i[\psi]$, $1 \le i \le k$,
(ii) *if* $\vartheta \in X$ *then the functions* $\partial\Xi_i[\psi]\vartheta$, $\partial\Lambda_i[\psi]\vartheta$, $1 \le i \le k$, *satisfy* (4.1)–(4.3).

Proof. The proof will be divided into three parts.
Part I. We prove that there exists exactly one solution $(\tilde{z}, \tilde{\Delta}_{[1]}, \ldots, \tilde{\Delta}_{[k]})$ of system (4.1)–(4.3) and $\tilde{z} \in C_{\vartheta.c}$, $\tilde{\Delta}_{[i]} \in C(\Sigma_{c.i}, \mathbb{R}^n)$ for $1 \le i \le k$.

Suppose that $z \in C_{\vartheta.c}$ is given. Let us consider system (4.3) with the above fixed z. It follows that for each $i \in \{1, \ldots, k\}$ there exists exactly one solution $\Delta_{[i]}[z]$ of (4.3) and $\Delta_{[i]}[z] \in C(\Sigma_{c.i}, \mathbb{R}^n)$. Moreover, there is $\gamma_\star \in \mathbb{L}([\kappa, c], \mathbb{R}_+)$ such that

$$\|\Delta_{[i]}[z](\tau,t,x) - \Delta_{[i]}[\bar{z}](\tau,t,x)\| \le \int_\tau^t \gamma_\star(\xi)\, \|z - \bar{z}\|_{(\xi,\mathbb{R}^k)}\, d\xi, \tag{4.4}$$

where $(\tau,t,x) \in \Sigma_{c.i}$ and z, $\bar{z} \in C_{\vartheta.c}$. Denote $\tilde{\mathbb{F}}[z] = (\tilde{\mathbb{F}}_1[z], \ldots, \tilde{\mathbb{F}}_k[z])$, where

$$\begin{aligned}
\tilde{\mathbb{F}}_i[z] = {} & \vartheta(a_i, g_{[i]}[z(\cdot,\psi)](a_i,t,x)) + \\
& + \partial_x \psi_i(a_i, g_{[i]}[z(\cdot,\psi)](a_i,t,x)) \circ \Delta_{[i]}[z](a_i,t,x) + \\
& + \int_{a_i}^t \partial_x G_i(\theta_i(\xi,t,x)) \circ \Delta_{[i]}[z](\xi,t,x)\, d\xi + \\
& + \int_{a_i}^t \partial_w G_i(\theta_i(\xi,t,x)) \diamond z_{\varphi(\xi, g_{[i]}[z(\cdot,\psi)](\xi,t,x))}\, d\xi + \\
& + \int_{a_i}^t \partial_w G_i(\theta_i(\xi,t,x)) \diamond U^{(i)}(\xi,t,x,\Delta_{[i]}[z])\, d\xi
\end{aligned}$$

for $(t,x) \in E_c \setminus E_{0.i}$ and $\tilde{\mathbb{F}}_i[z](t,x) = \vartheta_i(t,x)$ on $E_{0.i}$. We put $i = 1, \ldots, k$ in the above definitions. We consider the integral functional equation

$$z = \tilde{\mathbb{F}}[z]. \tag{4.5}$$

It follows from Assumption $\mathrm{H}[F,G]$ and from (4.4) that there is $\gamma_0 \in \mathbb{L}([\kappa,c],\mathbb{R}_+)$ such that

$$\|\tilde{\mathbb{F}}[z](t,x) - \tilde{\mathbb{F}}[\bar{z}](t,x)\|_\infty \le \int_\kappa^t \gamma_0(\xi)\, \|z - \bar{z}\|_{(\xi,\mathbb{R}^k)}\, d\xi, \quad (t,x) \in E_c,\ \kappa \le t \le c,$$

where z, $\bar{z} \in C_\vartheta$. For the above z, $\bar{z}$ we put

$$[|z - \bar{z}|] = \max \left\{ \|z - \bar{z}\|_{(t,\mathbb{R}^k)}\, \exp\left[-2 \int_\kappa^t \gamma_0(\xi)\, d\xi \right] \,:\, t \in [\kappa, c] \right\}.$$

Then we have

$$[|\,\tilde{\mathbb{F}}[z] - \tilde{\mathbb{F}}[\bar{z}]\,|] \le \frac{1}{2}[|z - \bar{z}|].$$

From the Banach fixed point theorem it follows that there exists exactly one solution $\tilde{z} \in C_{\vartheta.c}$ of (4.5). Then $(\tilde{z}, \Delta_{[1]}[\tilde{z}], \ldots, \Delta_{[k]}[\tilde{z}])$ is the desired solution of (4.1)–(4.3).

Part II. Write

$$z^{(s)}(t,x) = \frac{1}{s}\big[z(t,x,\psi + s\vartheta) - z(t,x,\psi)\big]^T,$$

$$\Delta^{(s)}_{[i]}(\tau,t,x) = \frac{1}{s}\big[g_{[i]}[z(\cdot,\psi + s\vartheta)](\tau,t,x) - g_{[i]}[z(\cdot,\psi)](\tau,t,x)\big]^T,$$

where $s \in \mathbb{R}$, $s \neq 0$, $i = 1,\ldots,k$. There is $\varepsilon_0 > 0$ such that for $s \in (-\varepsilon_0, \varepsilon_0)$, $s \neq 0$, we have: $\psi + s\vartheta \in \mathbb{X}$.

We write integral functional equations for $\big(z^{(s)}, \Delta^{(s)}_{[1]}, \ldots, \Delta^{(s)}_{[k]}\big)$. More precisely, we prove that the above functions are approximate solutions to (4.1)–(4.3). We use the Hadamard mean value theorem. We need the following intermediate points:

$$P^{(s)}_i(\lambda,\xi,t,x) = \Big(\xi, (1-\lambda)g_{[i]}[z(\cdot,\psi)](\xi,t,x) + \lambda g_{[i]}[z(\cdot,\psi + s\vartheta)](\xi,t,x),$$

$$(1-\lambda)(z(\cdot,\psi))_{\varphi(\xi,g_{[i]}[z(\cdot,\psi)](\xi,t,x))} + \lambda(z(\cdot,\psi + s\vartheta))_{\varphi(\xi,g_{[i]}[z(\cdot,\psi+\vartheta)](\xi,t,x))}\Big),$$

$$Q^{(s)}_i(\lambda,\xi,t,x) = \big((1-\lambda)\varphi(\xi, g_{[i]}[z(\cdot,\psi)](\xi,t,x)) + \lambda\,\varphi(\xi, g_{[i]}[z(\cdot,\psi + s\vartheta)](\xi,t,x))\big),$$

$$S^{(s)}_i(\lambda,\xi,t,x) = \big(\xi, (1-\lambda)g_{[i]}[z(\cdot,\psi)](\xi,t,x) + \lambda g_{[i]}[z(\cdot,\psi + s\vartheta)](\xi,t,x)\big),$$

where $\lambda \in [0,1]$ and $i = 1,\ldots,k$. Write

$$U^{(i,s)}(\tau,t,x,\Delta^{(s)}_{[i]}) =$$
$$= \int_0^1 (\,u(\cdot,\psi))_{Q^{(s)}_i(\lambda,\tau,t,x)}\,d\lambda \int_0^1 \partial_x\phi(\tau,\,,S^{(s)}_i(\lambda,\tau,t,x))\,d\lambda\,\Delta^{(s)}_{[i]}(\tau,t,x),$$

where the function

$$\int_0^1 (\,u(\cdot,\psi))_{Q^{(s)}_i(\lambda,\tau,t,x)}\,d\lambda : B[Q^{(s)}_i(\lambda,\tau,t,x)] \to M_{k\times n}$$

is defined by

$$\Big(\int_0^1 (\,u(\cdot,\psi))_{Q_i(\lambda,s,\tau,t,x)}\,d\lambda\Big)(\tau,y) = \int_0^1 (\,u(\cdot,\psi))_{Q_i(\lambda,s,\tau,t,x)}(\tau,y)\,d\lambda.$$

It follows from Assumption H$[F,G]$ and from the Hadamard mean value theorem that

$$\begin{aligned} z_i^{(s)}(t,x) = {} & \vartheta(a_i, g_{[i]}[z(\cdot,\psi+s\vartheta)](a_i,t,x)) + \\ & + \int_0^1 \partial_x \psi_i(S_i^{(s)}(\lambda,a_i,t,x))\,d\lambda \circ \Delta_{[i]}^{(s)}(a_i,t,x) + \\ & + \int_{a_i}^t \int_0^1 \partial_x G_i(P_i^{(s)}(\lambda,\xi,t,x))\,d\lambda \circ \Delta_{[i]}^{(s)}(\xi,t,x)\,d\xi + \\ & + \int_{a_i}^t \int_0^1 \partial_w G_i(P_i^{(s)}(\lambda,\xi,t,x))\,d\lambda \diamond \big(z^{(s)}\big)_{\varphi(\xi,g_{[i]}[z(\cdot,\psi+\vartheta)](\xi,t,x))}\,d\xi + \\ & + \int_{a_i}^t \int_0^1 \partial_w G_i(P_i^{(s)}(\lambda,\xi,t,x)0\,d\lambda \diamond U^{(i,s)}(\xi,t,x,\Delta_{[i]}^{(s)})\,d\xi \ \text{ on } E_c \setminus E_{0.i}, \end{aligned}$$

$$z_i^{(s)}(t,x) = \vartheta(t,x) \ \text{ on } \ E_{0.i}$$

and

$$\begin{aligned} \Delta_i^{(s)}(\tau,t,x) = {} & \int_t^\tau \int_0^1 \partial_x F_{[i]}(P_i^{(s)}(\lambda,\xi,t,x))\,d\lambda\ \Delta_{[i]}^{(s)}(\xi,t,x)\,d\xi + \\ & + \int_t^\tau \int_0^1 \partial_w F_{[i]}(P_i^{(s)}(\lambda,\xi,t,x))\,d\lambda\ \big(z^{(s)}\big)_{\varphi(\xi,g_{[i]}[z(\cdot,\psi+\vartheta)](\xi,t,x))}\,d\xi + \\ & + \int_t^\tau \int_0^1 \partial_w F_{[i]}(P_i^{(s)}(\lambda,\xi,t,x))\,d\lambda\ U^{(i,s)}(\xi,t,x,\Delta_{[i]}^{(s)})\,d\xi \quad \text{on } E_{c.i}. \end{aligned}$$

It is clear that integral functional equations (4.1)–(4.3) are generated by the above relations.

Part III. We prove that

$$\lim_{s\to 0} z^{(s)}(t,x) = \tilde{z}(t,x) \quad \text{uniformly on} \quad E_c, \tag{4.6}$$

$$\lim_{s\to 0} \Delta^{(s)}_{[i]}(\tau,t,x) = \tilde{\Delta}_{[i]}(\tau,t,x) \quad \text{uniformly on} \quad \Sigma_{c.i},\ i=1,\ldots,k. \tag{4.7}$$

It follows from (4.1)–(4.3) that

$$\begin{aligned}
&\tilde{z}_i(t,x) - z^{(s)}_i(t,x) = \\
&= \partial_x\psi_i(a_i, g_{[i]}[z(\cdot,\psi)](a_i,t,x) \circ \big[\tilde{\Delta}_{[i]}(a_i,t,x) - \Delta^{(s)}_{[i]}(a_i,t,x)\big] + \\
&\quad + \int_{a_i}^{t} \partial_x G_i(\theta_i(\xi,t,x)) \circ [\tilde{\Delta}_{[i]}(\xi,t,x) - \Delta^{(s)}_{[i]}(\xi,t,x)\big] + \\
&\quad + \int_{a_i}^{t} \partial_w G_i(\theta_i(\xi,t,x)) \diamond \big[\tilde{z}_{\varphi(\xi,g_{[i]}[z(\cdot,\psi)](\xi,t,x))} - \\
&\qquad\qquad - (z^{(s)})_{\varphi(\xi,g_{[i]}[z(\cdot,\psi+s\vartheta)](\xi,t,x))}\big]\, d\xi + \\
&\quad + \int_{a_i}^{t} \partial_w G_i(\theta_i(\xi,t,x) \diamond U^{(i)}(\xi,t,x,\tilde{\Delta}_{[i]} - \Delta^{(s)}_{[i]})\, d\xi + A_i(s,t,x)
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
&\tilde{\Delta}_{[i]}(\tau,t,x) - \Delta^{(s)}_{[i]}(\tau,t,x) = \\
&= \int_{t}^{\tau} \partial_x F_{[i]}(\theta_i(\xi,t,x)) \circ [\tilde{\Delta}_{[i]}(\xi,t,x) - \Delta^{(s)}_{[i]}(\xi,t,x)\big] + \\
&\quad + \int_{t}^{\tau} \partial_w F_{[i]}(\theta_i(\xi,t,x)) \diamond \big[\tilde{z}_{\varphi(\xi,g_{[i]}[z(\cdot,\psi)](\xi,t,x))} - \\
&\qquad\qquad - (z^{(s)})_{\varphi(\xi,g_{[i]}[z(\cdot,\psi+s\vartheta)](\xi,t,x))}\big]\, d\xi + \\
&\quad + \int_{t}^{\tau} \partial_w F_{[i]}(\theta_i(\xi,t,x) \diamond U^{(i)}(\xi,t,x,\tilde{\Delta}_{[i]} - \Delta^{(s)}_{[i]})\, d\xi + B_i(s,\tau,t,x),
\end{aligned} \tag{4.9}$$

where

$$
\begin{aligned}
&A_i(s,t,x) = \\
&= \vartheta_i(a_i, g_{[i]}[z(\cdot,\psi)](a_i,t,x) - \vartheta_i(a_i, g_{[i]}[z(\cdot,\psi+s\vartheta)](a_i,t,x) + \\
&\quad + \int_0^1 \big[\partial_x \psi_i(a_i, g_{[i]}[z(\cdot,\psi)](a_i,t,x)) - \partial_x \psi_i(S_i^{(s)}(\lambda,a_i,t,x))\big]\, d\lambda \circ \Delta_{[i]}^{(s)}(a_i,t,x) + \\
&\quad + \int_{a_i}^t \int_0^1 \big[\partial_x G_i(\theta_i(\xi,t,x)) - \partial_x G_i(P_i^{(s)}(\lambda,\xi,t,x))\big]\, d\lambda \circ \Delta_{[i]}^{(s)} + \\
&\quad + \int_{a_i}^t \int_0^1 \big[\partial_w G_i(\theta_i(\xi,t,x)) - \\
&\qquad\qquad - \partial_w G_i(P_i^{(s)}(\lambda,\xi,t,x))\big]\, d\lambda \diamond (z^{(s)})_{\varphi(\xi, g_{[i]}[z(\cdot,\psi+s\vartheta)](\xi,t,x))}\, d\xi + \\
&\quad + \int_{a_i}^t \Big[\partial_w G_i(\theta_i(\xi,t,x)) \diamond U^{(i)}(\xi,t,x,\Delta_{[i]}^{(s)}) - \\
&\qquad\quad - \int_0^1 \partial_w G_i(P_i^{(s)}(\lambda,\xi,t,x)) d\lambda \diamond U^{(i,s)}(\xi,t,x,\Delta_{[i]}^{(s)})\Big]\, d\xi
\end{aligned}
\tag{4.10}
$$

and

$$
\begin{aligned}
&B_i(s,\tau,t,x) = \\
&= \int_t^\tau \int_0^1 \big[\partial_x F_{[i]}(\theta_i(\xi,t,x)) - \partial_x F_{[i]}(P_i^{(s)}(\lambda,\xi,t,x))\big]\, d\lambda\ \Delta_{[i]}^{(s)}(\xi,t,x)\, d\xi + \\
&\quad + \int_t^\tau \int_0^1 \big[\partial_w F_{[i]}(\theta_i(\xi,t,x)) - \\
&\qquad\qquad - \partial_w F_{[i]}(P_i^{(s)}(\lambda,\xi,t,x))\big]\, d\lambda (z^{(s)})_{\varphi(\xi, g_{[i]}[z(\cdot,\psi+s\vartheta)](\xi,t,x))}\, d\xi + \\
&\quad + \int_t^\tau \Big[\partial_w F_{[i]}(\theta_i(\xi,t,x)) U^{(i)}(\xi,t,x,\Delta_{[i]}^{(s)}) - \\
&\qquad\quad - \int_0^1 \partial_w F_{[i]}(P_i^{(s)}(\lambda,\xi,t,x)) d\lambda U^{(i,s)}(\xi,t,x,\Delta_{[i]}^{(s)})\Big] d\xi.
\end{aligned}
\tag{4.11}
$$

We put $i = 1, \ldots, k$ in (4.8)–(4.11) It follows from (4.9), (4.11) that there are $f_0 \in C([0,1], \mathbb{R}_+)$ and γ_0, $\gamma_1 \in \mathbb{L}([\kappa, c], \mathbb{R}_+)$ such that

$$\|\tilde{\Delta}_{[i]}(\tau,t,x) - \Delta^{(s)}_{[i]}(\tau,t,x)\| \le f_0(s) + \left| \int_t^\tau \gamma_0(\xi) \|\tilde{z} - z^{(s)}\|_{(\xi,\mathbb{R}^k)}\, d\xi \right| + \\ + \left| \int_t^\tau \gamma_1(\xi) \|\tilde{\Delta}_{[i]}(\xi,t,x) - \Delta^{(s)}_{[i]}(\xi,t,x)\|\, d\xi \right|, \quad (\tau,t,x) \in E_{c,i},$$

and $f_0(0) = 0$. We conclude from the Gronwall inequality that

$$\|\tilde{\Delta}_{[i]}(\tau,t,x) - \Delta^{(s)}_{[i]}(\tau,t,x)\| \le \left[f_0(s) + \left| \int_t^\tau \gamma_0(\xi)\, \|\tilde{z} - z^{(s)}\|_{(\xi,\mathbb{R}^k)}\, d\xi \right| \right] \exp\left\{ \left| \int_t^\tau \gamma_1(\xi)\, d\xi \right| \right\}. \tag{4.12}$$

From (4.8), (4.10), (4.12) we deduce that there are $f \in C([0,1], \mathbb{R}_+)$ and $\tilde{\gamma} \in \mathbb{L}([\kappa, c], \mathbb{R}_+)$ such that

$$\|\tilde{z} - z^{(s)}\|_{(t,\mathbb{R}^k)} \le f(s) + \int_\kappa^t \tilde{\gamma}(\xi)\, \|\tilde{z} - z^{(s)}\|_{(\xi,\mathbb{R}^k)}\, d\xi, \quad t \in [\kappa, c],$$

and $f(0) = 0$. Then we have

$$\|\tilde{z} - z^{(s)}\|_{(t,\mathbb{R}^k)} \le f(s) \exp\left\{ \int_\kappa^t \tilde{\gamma}(\xi)\; d\xi \right\}, \quad t \in [\kappa, c]. \tag{4.13}$$

We conclude from (4.12), (4.13) that relations (4.6), (4.7) hold. This completes the proof of the theorem. □

We give comments on particular cases of problem (1.1), (1.2). Suppose that there is $\tilde{M} = (\tilde{M}_1, \ldots, \tilde{M}_n) \in \mathbb{R}^n_+$ such that $M(t) = \tilde{M}t$ for $t \in [0, a]$. Then E is the classical Haar pyramid. Suppose that $\kappa > 0$, $0 \le \kappa_0 \le \kappa$ and $h \le \tilde{M}\kappa_0$, $h = (h_1, \ldots, h_n) \in \mathbb{R}^n_+$. Consider the functions

$$\tilde{F} : E \times \mathbb{R}^k \to M_{k\times n},\ \tilde{F} = \left[\, \tilde{F}_{ij}\, \right]_{i=1,\ldots,k,\, j=1,\ldots,n},\ \ \tilde{F}_{[i]} = (\tilde{F}_{i1}, \ldots, \tilde{F}_{in}), \quad 1 \le i \le k,$$

$$\tilde{G} : E \times \mathbb{R}^k \to \mathbb{R}^k,\ \ \tilde{G} = (\tilde{G}_1, \ldots, \tilde{G}_k).$$

Write

$$F_{[i]}(t,x,w) = \tilde{F}_{[i]}\Big(t, x, \int_{-h}^{h} w(-\kappa_0, y)\, dy\Big), \quad G_i(t,x,w) = \tilde{G}_i\Big(t, x, \int_{-h}^{h} w(-\kappa_0, y)\, dy\Big),$$

where $(t, x, w) \in \Omega$ and $a_i \le t \le a$, and

$$F_{[i]}(t, x, w) = \tilde{F}_{[i]}\Big(t, x, \int_{-h}^{h} w(0, y)\, dy\Big), \quad G_i(t, x, w) = \tilde{G}_i\Big(t, x, \int_{-h}^{h} w(0, y)\, dy\Big),$$

where $(t, x, w) \in \Omega$ and $0 \le t < a_i$. We put $i = 1, \dots, k$ in the above definitions. Note that the i-th equation in (1.1) is considered for $a_i \le t \le a$ and $(t, x) \in E$.

Suppose that $\varphi(t, x) = (t, x)$ for $(t, x) \in E$. Then (1.1) reduces to the differential integral system

$$\begin{aligned} \partial_t z_i(t, x) + \tilde{F}_{[i]}\Big(t, x, \int_{x-h}^{x+h} z(t - \kappa_0, y)\, dy\Big) \circ \partial_x z_i(t, x) = \\ = \tilde{G}_i\Big(t, x, \int_{x-h}^{x+h} z(t - \kappa_0, y)\, dy\Big), \quad i = 1, \dots, k. \end{aligned} \tag{4.14}$$

It follows easily that Theorem 4.1 can be applied to (4.14), (1.2).

For the above $\tilde{F}$ and $\tilde{G}$ we put

$$F(t, x, w) = \tilde{F}(t, x, w(0, 0_{[n]})), \quad G(t, x, w) = \tilde{G}(t, x, w(0, 0_{[n]})).$$

Then (1.1) is a system of quasilinear differential equations with deviated variables

$$\partial_t z_i(t, x) + \tilde{F}_{[i]}(t, x, z(\varphi(t, x)) \circ \partial_x z_i(t, x) = \tilde{G}_i(t, x, z(\varphi(t, x)), \quad i = 1, \dots, k. \tag{4.15}$$

It is clear that Theorem 4.1 can be applied to (4.15), (1.2).

Let us consider the quasilinear system

$$\partial_t z_i(t, x) + F_{[i]}(t, x, z_{(t,x)}) \circ \partial_x z_i(t, x) = G_i(t, x, z_{(t,x)}), \quad i = 1, \dots, k, \tag{4.16}$$

which is a particular case of (1.1). The functional differential problem consisting of (4.16) and (1.2) is a generalized Cauchy problem.

This is the following motivation for investigation of (1.1), (1.2) instead of (4.16), (1.2). Quasilinear systems with deviated variables are obtained from (4.16) in the following way. Write

$$\begin{aligned} F(t, x, w) &= \tilde{F}(t, x, w(\varphi(t, x) - (t, x))), \\ G(t, x, w) &= \tilde{G}(t, x, w(\varphi(t, x) - (t, x))). \end{aligned} \tag{4.17}$$

Then system (4.16) is equivalent to (4.15). Note that the functions F and G given by (4.17) do not satisfy Assumption H$[F, G]$. More precisely, the derivatives $\partial_x G$, $\partial_x F_{[i]}$, $1 \le i \le k$, do not exist on Ω.

With the above motivation we have considered problem (1.1), (1.2).

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