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ASYMPTOTICALLY ISOMETRIC COPIES OF c_0 IN MUSIELAK-ORLICZ SPACES

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Abstract. Criteria in order that a Musielak-Orlicz function space L^Φ as well as Musielak-Orlicz sequence space l^Φ contains an asymptotically isometric copy of c_0 are given. These results extend some results of [Y.A. Cui, H. Hudzik, G. Lewicki, *Order asymptotically isometric copies of c_0 in the subspaces of order continuous elements in Orlicz spaces*, Journal of Convex Analysis **21** (2014)] to Musielak-Orlicz spaces.

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1. INTRODUCTION

Let $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ stand for the sets of reals, nonnegative reals and natural numbers, respectively. Let (T, Σ, μ) be an arbitrary σ-finite and complete measure space that does not reduce to a finite number of atoms only. A mapping $\Phi : T \times \mathbb{R} \to [0, +\infty]$ is said to be a *Musielak-Orlicz function* if:

1. There is a null set $T_0 \in \Sigma$ such that $\Phi(t, \cdot)$ is an Orlicz function for any $t \in T \setminus T_0$, that is, $\Phi(t, \cdot)$ is convex, even, vanishing at zero, left continuous on $\mathbb{R}^+$ and not identically equal to zero.
2. For any $u \in \mathbb{R}$, the function $\Phi(\cdot, u)$ is Σ-measurable.

Let $L^0 = L^0(T, \Sigma, \mu)$ denote the space of all (equivalence classes of) Σ-measurable real functions defined on T. Given any Musielak-Orlicz function Φ, we define on L^0 a convex modular I_Φ by the formula

$$I_\Phi(x) = \int_T \Phi(t, x(t))d\mu.$$

The *Musielak-Orlicz space* L^Φ generated by a Musielak-Orlicz function Φ is defined by the formula

$$L^\Phi = \{x \in L^0 : I_\Phi(\lambda x) < \infty \quad \text{for some} \quad \lambda > 0\}.$$

We will considered this space under the Luxemburg norm (see [2, 9–12]):

$$\|x\|_\Phi = \inf\{\lambda > 0 : I_\Phi(x/\lambda) \le 1\}$$

Let Ω denote the nonatomic part of T and $\mathcal{N}$ denote the purely atomic part of T. Then the measure space (T, Σ, μ) can be written as the direct sum

$$(\Omega, \Sigma \cap \Omega, \mu/_\Omega) \oplus (\mathcal{N}, 2^{\mathcal{N}}, \mu/_{2^{\mathcal{N}}}).$$

In this paper we will consider two separate cases: μ nonatomic and μ purely atomic with $\mathcal{N} = \mathbb{N}$.

In a nonatomic case we say that Φ satisfies the growth *condition* $\triangle_2$ ($\Phi \in \triangle_2$ for short) if there exist a null set $B \in \Sigma \cap \Omega$, a constant $K > 0$ and a nonnegative Σ-measurable function h on Ω such that $\int_\Omega \Phi(t, h(t))d\mu < \infty$ and $\Phi(t, 2u) \le K\Phi(t, u)$ for all $t \in \Omega \setminus B$ and $u \ge h(t)$ (see [2] and [11]).

In the purely atomic case we assume that $(T, \Sigma, \mu) = (\mathbb{N}, 2^{\mathbb{N}}, \text{card})$ and we will write $\Phi_n(u)$, l^Φ and x_n in place of $\Phi(n, u)$, L^Φ and $x(n)$, respectively. Then l^Φ is called the Musielak-Orlicz sequence space.

We say that $\Phi \in \delta_2^0$ if there are $K > 0$, $a > 0$ and a sequence $(c_n)_{n=1}^\infty$ in $[0, +\infty]$ such that $\sum_{n=m}^\infty c_n < \infty$ for some $m \in \mathbb{N}$ and the inequality

$$\Phi_n(2u) \le K\Phi_n(u) + c_n$$

holds for all $n \in \mathbb{N}$ and $u \in \mathbb{R}$ satisfying $\Phi_n(u) \le a$ (see [11]).

Recall that if X is a Banach function lattice and $x \in X$, then x is said to be *order continuous* if $\|x_n\| \to 0$ for any sequence (x_n) in X such that $0 \le x_n \le |x|$ and $x_n \to 0$ μ-a.e. The subspace of all order continuous elements in X is denoted by X_a. It is possible that $X_a = \{0\}$. This is the case when X is equal to L^∞ or $L^1 \cap L^\infty$ for example. If the measure space (T, Σ, μ) is purely atomic, then $(L^\Phi)_a \ne \{0\}$ for any Musielak-Orlicz function Φ. However, if the measure space (T, Σ, μ) is nonatomic, we have $(L^\Phi)_a \ne \{0\}$ if and only if the set $\{t \in T : \Phi(t, \cdot)$ is finitely valued$\}$ has a positive measure, actually $\text{supp}(L^\Phi)_a = \{t \in T : \Phi(t, \cdot)$ is finitely valued$\}$ in this case. Consequently, if Φ does not depend on the parameter t and the measure μ is nonatomic, then $(L^\Phi)_a \ne \{0\}$ if and only if Φ is finitely valued (this is of course the case for Orlicz spaces).

A Banach function lattice X is said to be order continuous ($X \in OC$ for short) if $X_a = X$. It is well known that order continuity of a Banach function lattice X as well as of an element $x \in X$ is preserved if we change a norm $\|\cdot\|$ in X into another one $|||\cdot|||$ which is equivalent to $\|\cdot\|$. It is also well known that $(L^\Phi)_a = E^\Phi$, where $E^\Phi = \{x \in L^0 : I_\Phi(\lambda x) < \infty$ for any $\lambda > 0\}$, when the measure space is nonatomic and that in the purely atomic case, we have $(l^\Phi)_a = h^\Phi$, where

$$h^\Phi = \Big\{x = (x_n)_{n=1}^\infty : \forall_{\lambda>0} \exists_{n_\lambda \in \mathbb{N}} \sum_{n=n_\lambda}^\infty \Phi_n(\lambda x_n) < \infty\Big\}.$$

It is also known that h^Φ is the closure (in the norm topology in l^Φ) of the space of all real sequences $x = (x_n)$ with a finite number of coordinates different from zero. Moreover (see [2] and [11]), for a nonatomic measure, we have $L^\Phi = E^\Phi$ if and only if $\Phi \in \triangle_2$ and for the purely atomic measure the equality $l^\Phi = h^\Phi$ holds if and only if $\Phi \in \delta_2^0$.

We say that a Banach space $(X, \|\cdot\|)$ contains asymptotically isometric copy of c_0 if there exists a sequence (ϵ_n) of numbers in $(0,1)$ such that $\lim_{n\to\infty} \epsilon_n = 0$ and there exists a linear operator $P: c_0 \to X$ such that

$$\sup_n (1-\epsilon_n)|x_n| \le \|Px\| \le \sup_n |x_n|$$

for every element $x = (x_n)$ of c_0.

The notion of asymptotically isometric copy of c_0 was introduced in [6], where it is shown that if a Banach space X contains such a copy, then X fails the fixed-point property for nonexpansive self-mappings on closed bounded convex subsets of X.

2. RESULTS

Theorem 2.1. *h^Φ equipped with the Luxemburg norm contains an asymptotically isometric copy of c_0 if and only if Φ does not satisfy the δ_2^0 condition.*

Proof. Let $\Phi \notin \delta_2^0$ and for $\varepsilon > 0, k \in \mathbb{N}, i \in \mathbb{N}$ define the numbers

$$d_i^k = \sup\{\Phi_i((1+\frac{1}{k})x) : \Phi_i(x) \le \frac{1}{2^{k+1}} \text{ and } \Phi_i((1+\varepsilon)x) \ge 2^{k+1}\Phi_i(x)\}.$$

It is known (see [1, 4, 7, 8]) that

$$\sum_{i=1}^{\infty} d_i^k = \infty \text{ for every } k \in \mathbb{N}.$$

Define i_1 as the largest natural number such that

$$\sum_{i=1}^{i_1} d_i^1 \le 1,$$

whenever $d_1^1 \le 1$ and $i_1 = 0$ otherwise. Then

$$\sum_{i=1}^{i_1+1} d_i^1 > 1.$$

Put $N_1 = \{1, 2, \ldots, i_1 + 1\}$. Next define i_2 as the largest natural number such that

$$\sum_{i=i_1+2}^{i_2} d_i^2 \le 1,$$

if $d^2_{i_1+2} \leq 1$ and $i_2 = i_1 + 2$ otherwise. Then

$$\sum_{i=i_1+2}^{i_2+1} d_i^2 > 1.$$

Put $N_2 = \{i_1 + 2, \ldots, i_2 + 1\}$. By induction we can construct the sets

$$N_k = \{i_{k-1} + 2, \ldots, i_k, i_k + 1\} \quad (k \in \mathbb{N}, i_0 = -1)$$

such that

$$\sum_{i \in N_k \setminus \{i_k+1\}} d_i^k \leq 1 \text{ and } \sum_{i \in N_k} d_i^k > 1.$$

For every $k \in \mathbb{N}$ and $i \in N_k$ there exist such numbers x_i that

$$\sum_{i \in N_k} \Phi_i((1 + \frac{1}{k})x_i) > 1,\ \Phi_i(x_i) \leq \frac{1}{2^{k+1}} \text{ and } \Phi_i((1 + \frac{1}{k})x_i) \geq 2^{k+1}\Phi_i(x_i).$$

Hence

$$\sum_{i \in N_k} \Phi_i(x_i) \leq \sum_{i \in N_k \setminus \{i_k+1\}} \tfrac{1}{2^{k+1}} \Phi_i((1 + \tfrac{1}{k})x_i) + \tfrac{1}{2^{k+1}} \leq$$
$$\leq \tfrac{1}{2^{k+1}} \sum_{i \in N_k \setminus \{i_k+1\}} d_i^k + \tfrac{1}{2^{k+1}} \leq \tfrac{1}{2^k}.$$

Define $y_k = \sum_{i \in N_k} x_i e_i$ for $k \in \mathbb{N}$. Then

$$I_\Phi(y_k) = \sum_{i \in N_k} \Phi_i(x_i) \leq \tfrac{1}{2^k},$$
$$I_\Phi((1 + \tfrac{1}{k})y_k) = \sum_{i \in N_k} \Phi((1 + \tfrac{1}{k})x_i) > 1.$$

for any $k \in \mathbb{N}$. Now define an operator $P : c_0 \to h^\Phi$ by the formula

$$Pu = \sum_{k=1}^{\infty} u_k y_k \text{ for } u = (u_k) \in c_0.$$

We will show that P is well defined, i.e. $Pu \in h^\Phi$ for any $u \in c_0$. Take any $\lambda > 0$ and $l \in \mathbb{N}$ such that $\lambda |u_k| \leq 1$ for every $k \geq l$. Then

$$I_\Phi(\lambda \cdot Pu \cdot \chi_{N_l \cup N_{l+1} \cup \ldots}) = I_\Phi(\lambda \sum_{k=l}^{\infty} u_k y_k) = \sum_{k=l}^{\infty} I_\Phi(\lambda u_k y_k) \leq$$
$$\leq \sum_{k=l}^{\infty} I_\Phi(y_k) \leq \sum_{k=l}^{\infty} \tfrac{1}{2^k} < \infty.$$

Consequently, $Pu \in h^{\Phi}$.

Next we will show that $\|Pu\| \leq \|u\|_{\infty}$. For any nonzero $u \in c_0$ we have

$$I_{\Phi}\left(\tfrac{Pu}{\|u\|_{\infty}}\right) = I_{\Phi}\left(\tfrac{1}{\|u\|_{\infty}} \sum_{k=1}^{\infty} u_k y_k\right) \leq \sum_{k=1}^{\infty} I_{\Phi}\left(\tfrac{1}{\|u\|_{\infty}} u_k y_k\right) \leq$$
$$\leq \sum_{k=1}^{\infty} I_{\Phi}(y_k) \leq \sum_{k=1}^{\infty} \tfrac{1}{2^k} = 1.$$

Consequently, $\|Pu\| \leq \|u\|_{\infty}$.

Finally, we will show that there exists a sequence (ε_n) such that

$$\varepsilon_n \downarrow 0 \text{ and } \sup_n (1-\varepsilon_n)|u_n| \leq \|Pu\|.$$

Define for every $k \in \mathbb{N}$ the number $\varepsilon_k = \frac{1}{k+1}$. Observe that $\frac{1}{1-\varepsilon_k} = 1 + \frac{1}{k}$. Take any $\lambda > 1$. For every nonzero $u = (u_k) \in c_0$ there exists $m \in \mathbb{N}$ such that

$$\frac{(1-\varepsilon_m)\lambda |u_m|}{\sup_n (1-\varepsilon_n)|u_n|} \geq 1,$$

equivalently

$$\frac{\lambda |u_m|}{\sup_n (1-\varepsilon_n)|u_n|} \geq \tfrac{1}{1-\varepsilon_m}.$$

Then we have

$$I_{\Phi}\left(\tfrac{\lambda Pu}{\sup_n (1-\varepsilon_n)|u_n|}\right) = I_{\Phi}\left(\tfrac{\sum_{k=1}^{\infty} \lambda u_k y_k}{\sup_n (1-\varepsilon_n)|u_n|}\right) \geq I_{\Phi}\left(\tfrac{\lambda u_m y_m}{\sup_n (1-\varepsilon_n)|u_n|}\right) \geq$$
$$\geq I_{\Phi}\left(\tfrac{1}{1-\varepsilon_m} y_m\right) = I_{\Phi}\left(\tfrac{1}{1-\varepsilon_m} \sum_{i \in N_m} x_i e_i\right) =$$
$$= \sum_{i \in N_m} \Phi_i\left(\left(1 + \tfrac{1}{m}\right) x_i\right) > 1,$$

whence

$$\tfrac{1}{\lambda} \sup_n (1-\varepsilon_n)|u_n| \leq \|Pu\|$$

and from arbitrariness of $\lambda > 1$, we get the thesis.

Now assume that $\Phi \in \delta_2^0$. Then $h^{\Phi} = l^{\Phi}$ is the dual space of h^{Ψ}, where Ψ is the Orlicz function complementary in the sense of Young to Φ. Assume that h^{Φ} contains an asymptotically isometric copy of c_0. Then it contains, as a dual space, an isometric copy of l^{∞} (see [5]). But this contradicts the fact that h^{Φ} is order continuous. □

Theorem 2.2. *If Φ takes only finite values then: E^{Φ} contains an asymptotically isometric copy of c_0 if and only if Φ does not satisfy the Δ_2 condition.*

Proof. If $\Phi < \infty$ and $\Phi \notin \Delta_2^0$ then there exist sequences of measurable functions (x_n) and measurable sets (E_n) such that:

$$\begin{aligned}
&E_m \cap E_n = \emptyset \text{ for } m \neq n,\\
&x_n(t) < \infty \text{ for every } t \in E_n, n \in \mathbb{N},\\
&\int_{E_n} \Phi(t, x_n(t))d\mu = \frac{1}{2^n},\\
&\Phi(t, (1+\tfrac{1}{n})x_n(t)) \geq 2^{n+2}\Phi(t, x_n(t)) \text{ for every } t \in E_n, n \in \mathbb{N}.
\end{aligned}$$

For details see [2].

Take any $n \in \mathbb{N}$ and define for every $k \in \mathbb{N}$ the set

$$E_{n,k} = \{t \in E_n : |x_n(t)| \leq k\} \cap T_k,$$

where (T_k) is a sequence of measurable sets satisfying: $T_1 \subset T_2 \subset \ldots$, $\bigcup T_{n=1}^{\infty} = T$ and $\mu(T_k) < \infty$ for every $k \in \mathbb{N}$. Such sets exist by the assumption of σ-finiteness of the measure μ. Then, we have

$$\begin{aligned}
&E_{n,1} \subset E_{n,2} \subset \ldots,\\
&\bigcup_{k=1}^{\infty} E_{n,k} = E_n,\\
&\mu(E_{n,k}) < \infty \text{ for every } k \in \mathbb{N}.
\end{aligned}$$

Consequently, we get that $|x_n|\chi_{E_{n,k}} \uparrow |x_n|\chi_{E_n}$ as $k \to \infty$. By the Beppo Levi monotone convergence theorem, we get

$$\lim_{k\to\infty} \int_{E_n} \Phi(t, x_n(t)\chi_{E_{n,k}}(t))d\mu = \int_{E_n} \Phi(t, x_n(t))d\mu.$$

Now, for every $n \in \mathbb{N}$ we can fix $k \in \mathbb{N}$ such that

$$\tfrac{1}{2^{n+1}} \leq \int_{E_n} \Phi(t, x_n(t)\chi_{E_{n,k}}(t))d\mu = \int_{E_{n,k}} \Phi(t, x_n(t))d\mu \leq \int_{E_n} \Phi(t, x_n(t))d\mu = \tfrac{1}{2^n}.$$

Let us denote $E_{n,k}$ by F_k.

Summarizing, for now we have constructed a sequence of measurable functions (x_n) and a sequence of measurable sets (F_n) satisfying the following conditions:

$$\begin{aligned}
&F_m \cap F_n = \emptyset \text{ for } m \neq n,\\
&x_n \text{ is bounded on } F_n,\ n \in \mathbb{N},\\
&\tfrac{1}{2^{n+1}} \leq \int_{F_n} \Phi(t, x_n(t))d\mu \leq \tfrac{1}{2^n},\\
&\Phi(t, (1+\tfrac{1}{n})x_n(t)) \geq 2^{n+2}\Phi(t, x_n(t)) \text{ for every } t \in F_n,\ n \in \mathbb{N}.
\end{aligned}$$

Define an operator $P : c_0 \to E^\Phi$ by the formula

$$Pu = \sum_{n=1}^{\infty} u_n x_n \chi_{F_n} \text{ for } u = (u_n) \in c_0.$$

We will show that $I_\Phi(\lambda Pu) < \infty$ for any $u \in c_0$ and $\lambda > 0$. Fix any $u = (u_n) \in c_0$ and take any $\lambda > 0$. There exists $l_0 \in \mathbb{N}$ such that $\lambda |u_n| \le 1$ for every $n \ge l_0$. We have

$$\begin{aligned} I_\Phi(\lambda Pu) &= \int_T \Phi\left(t, \lambda \sum_{n=1}^{\infty} u_n x_n(t) \chi_{F_n}(t)\right) = \sum_{n=1}^{\infty} \int_{F_n} \Phi(t, \lambda u_n x_n(t)) d\mu = \\ &= \sum_{n=1}^{l_0 - 1} \int_{F_n} \Phi(t, \lambda u_n x_n(t)) d\mu + \sum_{n=l_0}^{\infty} \int_{F_n} \Phi(t, x_n(t)) d\mu \le \\ &\le \sum_{n=1}^{l_0 - 1} \int_{F_n} \Phi(t, \lambda u_n x_n(t)) d\mu + \sum_{n=l_0}^{\infty} \tfrac{1}{2^n} < \infty, \end{aligned}$$

since $\int_{F_n} \Phi(t, \lambda u_n x_n(t)) d\mu$ is finite for every $n \in \mathbb{N}$. Consequently, $Pu \in E^\Phi$.

Next we will show that $\|Pu\| \le \|u\|_\infty$. For any nonzero $u \in c_0$ we have

$$\begin{aligned} I_\Phi\left(\tfrac{Pu}{\|u\|_\infty}\right) &= \int_T \Phi\left(t, \sum_{n=1}^{\infty} \tfrac{1}{\|u\|_\infty} u_n x_n(t) \chi_{F_n}(t)\right) d\mu \le \\ &\le \int_T \Phi\left(t, \sum_{n=1}^{\infty} x_n(t) \chi_{F_n}(t)\right) d\mu = \sum_{n=1}^{\infty} \int_{F_n} \Phi\left(t, x_n(t)\right) d\mu \le \sum_{n=1}^{\infty} \tfrac{1}{2^n} = 1. \end{aligned}$$

Consequently, $\|Pu\| \le \|u\|_\infty$.

Finally, we will show that there exists a sequence (ε_n) such that $\varepsilon_n \downarrow 0$ and $\sup_n (1 - \varepsilon_n)|u_n| \le \|Pu\|$ for every $u \in c_0$. Define $\varepsilon_n = \frac{1}{n+1}$ for every $n \in \mathbb{N}$ and notice that $\frac{1}{1-\varepsilon_n} = 1 + \frac{1}{n}$ $(n \in \mathbb{N})$. Fix any nozero $u \in c_0$ and $\lambda > 1$. Since $1 - \varepsilon_n \to 1$ as $n \to \infty$, then there exists $m \in \mathbb{N}$ such that

$$\frac{\lambda(1 - \varepsilon_m)|u_m|}{\sup_n (1 - \varepsilon_n)|u_n|} \ge 1$$

and equivalently

$$\frac{\lambda |u_m|}{\sup_n (1 - \varepsilon_n)|u_n|} \ge \frac{1}{(1 - \varepsilon_m)}.$$

Now we have

$$\begin{aligned}
I_\Phi\left(\tfrac{\lambda Pu}{\sup_n(1-\varepsilon_n)|u_n|}\right) &= \int_T \Phi\left(t, \sum_{n=1}^{\infty} \lambda u_n x_n(t) \tfrac{1}{\sup_n(1-\varepsilon_n)|u_n|} \chi_{F_n}(t)\right) d\mu \geq \\
&\geq \int_T \Phi\left(t, \lambda u_m x_m(t) \tfrac{1}{\sup_n(1-\varepsilon_n)|u_n|} \chi_{F_m}(t)\right) d\mu = \\
&= \int_{F_m} \Phi\left(t, \lambda u_m x_m(t) \tfrac{1}{\sup_n(1-\varepsilon_n)|u_n|}\right) d\mu \geq \\
&\geq \int_{F_m} \Phi\left(t, \tfrac{1}{1-\varepsilon_m} x_m(t)\right) d\mu = \int_{F_m} \Phi\left(t, (1+\tfrac{1}{m}) x_m(t)\right) d\mu \geq \\
&\geq 2^{m+2} \int_{F_m} \Phi(t, x_m(t)) d\mu \geq 2^{m+2} \cdot \tfrac{1}{2^{m+1}} = 2 > 1.
\end{aligned}$$

Consequently, $\sup_n(1-\varepsilon_n)|u_n| \leq \|Pu\|$.

The proof of the conversion is similar to in the previous theorem. □

REFERENCES

[1] G. Alherk, H. Hudzik, *Copies of l^1 and c_0 in Musielak-Orlicz sequence spaces*, Comment. Math. Univ. Carolinae **35** (1994) 1, 9–19.

[2] S. Chen, *Geometry of Orlicz spaces*, Dissertationes Math. **356** (1996), 1–204.

[3] Y.A. Cui, H. Hudzik, G. Lewicki, *Order asymptotically isometric copies of c_0 in the subspaces of order continuous elements in Orlicz spaces*, Journal of Convex Analysis **21** (2014).

[4] M. Denker, H. Hudzik, *Uniformly non-$l_n^{(1)}$ Musielak-Orlicz sequence spaces*, Proc. Indian Acad. Sci. **101** (1991), 71–86.

[5] P.N. Dowling, *Isometric Copies of c_0 and l^∞ in Duals of Banach Spaces*, J. Math. Anal. Appl. **244** (2000), 223–227.

[6] P.N. Dowling, C.J. Lennard, B. Turett, *Reflexivity and the fixed point property for nonexpansive maps*, J. Math. Anal. Appl. **200** (1996), 653–662.

[7] H. Hudzik, *On some equivalent condition in Musielak-Orlicz spaces*, Comment. Math. **24** (1984), 57–64.

[8] A. Kamińska, *Flat Orlicz-Musielak sequence spaces*, Bull. Acad. Polon. Sci. Math. **30** (1982), 347–352.

[9] M.A. Krasnoselskiĭ, Ya.B. Rutickiĭ, *Convex functions and Orlicz spaces*, Groningen, 1961 (translation).

[10] W.A.J. Luxemburg, *Banach function spaces*, Thesis, Delft, 1955.

[11] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes Math. 1034, Springer--Verlag, 1983.

[12] M.M. Rao, Z.D. Ren, *Theory of Orlicz spaces*, Marcel Dekker, New York, 1991.

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