A TOTALLY MAGIC CORDIAL LABELING OF ONE-POINT UNION OF \( n \) COPIES OF A GRAPH

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Communicated by Dalibor Fronček

Abstract. A graph \( G \) is said to have a totally magic cordial (TMC) labeling with constant \( C \) if there exists a mapping \( f : V(G) \cup E(G) \rightarrow \{0, 1\} \) such that \( f(a) + f(b) + f(ab) \equiv C \pmod{2} \) for all \( ab \in E(G) \) and \( |n_f(0) - n_f(1)| \leq 1 \), where \( n_f(i) \) \((i = 0, 1)\) is the sum of the number of vertices and edges with label \( i \). In this paper, we establish the totally magic cordial labeling of one-point union of \( n \)-copies of cycles, complete graphs and wheels.

Keywords: totally magic cordial labeling, one-point union of graphs.

Mathematics Subject Classification: 05C78.

1. INTRODUCTION

All graphs considered here are finite, simple and undirected. The set of vertices and edges of a graph \( G \) is denoted by \( V(G) \) and \( E(G) \) respectively. Let \( p = |V(G)| \) and \( q = |E(G)| \). A general reference for graph theoretic ideas can be seen in [3]. The concept of cordial labeling was introduced by Cahit [1]. A binary vertex labeling \( f : V(G) \rightarrow \{0, 1\} \) induces an edge labeling \( f^* : E(G) \rightarrow \{0, 1\} \) defined by \( f^*(uv) = |f(u) - f(v)| \). Such labeling is called cordial if the conditions \( |v_f(0) - v_f(1)| \leq 1 \) and \( |e_{f^*}(0) - e_{f^*}(1)| \leq 1 \) are satisfied, where \( v_f(i) \) and \( e_{f^*}(i) \) \((i = 0, 1)\) are the number of vertices and edges with label \( i \) respectively. A graph is called cordial if it admits a cordial labeling. The cordiality of a one-point union of \( n \) copies of graphs is given in [6].

Kotzig and Rosa introduced the concept of edge-magic total labeling in [5]. A bijection \( f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, p + q\} \) is called an edge-magic total labeling of \( G \) if \( f(x) + f(xy) + f(y) \) is constant \((\text{called the magic constant of } f)\) for every edge \( xy \) of \( G \). The graph that admits this labeling is called an edge-magic total graph.

The notion of totally magic cordial (TMC) labeling was due to Cahit [2] as a modification of edge magic total labeling and cordial labeling. A graph \( G \) is said to have TMC labeling with constant \( C \) if there exists a mapping \( f : V(G) \cup E(G) \rightarrow \{0, 1\} \) such that...
such that \( f(a) + f(b) + f(ab) \equiv C \pmod{2} \) for all \( ab \in E(G) \) and \( |n_f(0) - n_f(1)| \leq 1 \), where \( n_f(i) \) \((i = 0, 1)\) is the sum of the number of vertices and edges with label \( i \).

A rooted graph is a graph in which one vertex is named as a special way so as to distinguish it from other nodes. The special node is called the root of the graph. Let \( G \) be a rooted graph. The graph obtained by identifying the roots of \( n \) copies of \( G \) is called the one-point union of \( n \) copies of \( G \) and is denoted by \( G^{(n)} \).

In this paper, we establish the TMC labeling of a one-point union of \( n \)-copies of cycles, complete graphs and wheels.

2. MAIN RESULTS

In this section, we present sufficient conditions for a one-point union of \( n \) copies of a rooted graph to be TMC and also obtain conditions under which a one-point union of \( n \) copies of graphs such as a cycle, complete graph and wheel are TMC graphs.

We relate the TMC labeling of a one-point union of \( n \) copies of a rooted graph to the solution of a system which involves an equation and an inequality.

**Theorem 2.1.** Let \( G \) be a graph rooted at a vertex \( u \) and for \( i = 1, 2, \ldots, k \), \( f_i : V(G) \cup E(G) \rightarrow \{0, 1\} \) be such that \( f_i(a) + f_i(b) + f_i(ab) \equiv C \pmod{2} \) for all \( ab \in E(G) \) and \( f_i(u) = 0 \). Let \( n_f(0) = \alpha_i, n_f(1) = \beta_i \) for \( i = 1, 2, \ldots, k \). Then the one-point union \( G^{(n)} \) of \( n \) copies of \( G \) is TMC if the system (2.1) has a nonnegative integral solution for the \( x_i \)'s:

\[
\left| \sum_{i=1}^{k} (\alpha_i - 1)x_i - \sum_{i=1}^{k} \beta_i x_i + 1 \right| \leq 1 \quad \text{and} \quad \sum_{i=1}^{k} x_i = n. \quad (2.1)
\]

**Proof.** Suppose \( x_i = \delta_i, i = 1, 2, \ldots, k \), is a nonnegative integral solution of system (2.1). Then we label the \( \delta_i \) copies of \( G \) in \( G^{(n)} \) with \( f_i \) \((i = 1, 2, \ldots, k)\). As each of these copies has the property \( f_i(a) + f_i(b) + f_i(ab) \equiv C \pmod{2} \) and \( f_i(u) = 0 \) for all \( i = 1, 2, \ldots, k \), \( G^{(n)} \) is TMC.

**Corollary 2.2.** Let \( G \) be a graph rooted at a vertex \( u \) and \( f \) be a labeling such that \( f(a) + f(b) + f(ab) \equiv C \pmod{2} \) for all \( ab \in E(G) \) and \( f(u) = 0 \). If \( n_f(0) = n_f(1) + 1 \), then \( G^{(n)} \) is TMC for all \( n \geq 1 \).

**Example 2.3.** One point union of a path is TMC.

**Corollary 2.4.** Let \( G \) be a graph rooted at \( u \). Let \( f_i, i = 1, 2, 3 \) be labelings of \( G \) such that \( f_i(a) + f_i(b) + f_i(ab) \equiv C \pmod{2} \) for all \( ab \in E(G) \), \( f_i(u) = 0 \) and \( \gamma_i = \alpha_i - \beta_i \).

1. If \( \gamma_1 = -2 \) and \( \gamma_2 = 2 \), then \( G^{(n)} \) is TMC for all \( n \neq 1 \pmod{4} \).
2. If either
   a) \( \gamma_1 = -1 \) and \( \gamma_2 = 3 \), or
   b) \( \gamma_1 = 4, \gamma_2 = 2 \) and \( \gamma_3 = -4 \), or
   c) \( \gamma_1 = -3, \gamma_2 = 3 \) and \( \gamma_3 = 5 \),
   then \( G^{(n)} \) is TMC for all \( n \geq 1 \).
3. If \( \gamma_1 = 0 \) and \( \gamma_2 = 4 \), then \( G^{(n)} \) is TMC for all \( n \neq 3 \pmod{4} \).
Theorem 3.1. We consider \( K \) be a cycle of order \( m \) and \( m \) as a rooted graph with the vertex \( v_1 \) as its root.

We consider \( C \) be a TMC labeling of \( K \) as its root.

(2a). The system (2.1) in Theorem 2.1 becomes \( \{−2x_1 + 2x_2 + 1\} \leq 1, x_1 + x_2 = n. \) When \( n = 2t, x_1 = t \) and \( x_2 = t \) is the solution. When \( n = 2t + 1, x_1 = t + 1 \) and \( x_2 = t \) is the solution. Hence, by Theorem 2.1, \( G^{(n)} \) is TMC for all \( n \geq 1 \).

3. ONE-POINT UNION OF CYCLES

Let \( C_m \) be a cycle of order \( m \). Let

\[
V(C_m) = \{v_i|1 \leq i \leq m\}
\]

and

\[
E(C_m) = \{v_iv_{i+1}|1 \leq i < m\} \cup \{v_mv_1\}.
\]

We consider \( C_m \) as a rooted graph with the vertex \( v_1 \) as its root.

**Theorem 3.1.** Let \( C_m^{(n)} \) be the one-point union of \( n \) copies of a cycle \( C_m \). Then \( C_m^{(n)} \) is TMC for all \( m \geq 3 \) and \( n \geq 1 \).

**Proof.** Define the labelings \( f_1 \) and \( f_2 \) from \( V(C_m) \cup E(C_m) \) into \( \{0, 1\} \) as follows: \( f_1(v_i) = 0 \) for \( 1 \leq i \leq m, f_1(v_iv_{i+1}) = 1 \) for \( 1 \leq i < m, f_1(v_mv_1) = 1, 1 \leq i \leq m \) and

\[
f_2(v_i) = \begin{cases} 1 & \text{if } i = m, \\ 0 & \text{if } i \neq m, \end{cases}
\]

and \( f_2(v_mv_1) = 0 \). Then \( \alpha_1 = m, \beta_1 = m, \alpha_2 = m + 1 \) and \( \beta_2 = m - 1 \). Thus system (2.1) in Theorem 2.1 becomes \( \{−x_1 + x_2 + 1\} \leq 1, x_1 + x_2 = n. \) When \( n = 2t, x_1 = t \) and \( x_2 = t \) is the solution. When \( n = 2t + 1, x_1 = t + 1 \) and \( x_2 = t \) is the solution. Hence, by Theorem 2.1, \( C_m^{(n)} \) is TMC for all \( m \geq 3 \) and \( n \geq 1 \).

\[
\square
\]

4. ONE-POINT UNION OF COMPLETE GRAPHS

Let \( K_m \) be a complete graph of order \( m \). Let

\[
V(K_m) = \{v_i|1 \leq i \leq m\}
\]

and

\[
E(K_m) = \{v_iv_j|i \neq j, 1 \leq i \leq m, 1 \leq j \leq m\}.
\]

We consider \( K_m \) as a rooted graph with the vertex \( v_1 \) as its root. Let \( f : V(K_m) \cup E(K_m) \rightarrow \{0, 1\} \) be a TMC labeling of \( K_m \). Without loss of generality, assume \( C = 1. \)
Then for any edge \( e = uv \in E(K_m) \), we have either \( f(e) = f(u) = f(v) = 1 \) or \( f(e) = f(u) = 0 \) and \( f(v) = 1 \) or \( f(e) = f(v) = 0 \) and \( f(u) = 1 \) or \( f(u) = f(v) = 0 \) and \( f(e) = 1 \). Thus, under the labeling \( f \), the graph \( K_m \) can be decomposed as \( K_m = K_p \cup K_r \cup K_{p,r} \) where \( K_p \) is the sub-complete graph in which all the vertices and edges are labeled with 1, \( K_r \) is the sub-complete graph in which all the vertices are labeled with 0 and edges are labeled with 1 and \( K_{p,r} \) is the complete bipartite subgraph of \( K_m \) with the bipartition \( V(K_p) \cup V(K_r) \) and its edges are labeled with 0. Then we find \( n_f(0) = r + pr \) and \( n_f(1) = \frac{p^2 + r^2 + p - r}{2} \).

**Table 1.** Possible values of \( \alpha_i \) and \( \beta_i \) for distinct labelings of \( K_m \)

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<thead>
<tr>
<th>( i )</th>
<th>( p )</th>
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<th>( \alpha_i )</th>
<th>( \beta_i )</th>
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<tr>
<td>1</td>
<td>0</td>
<td>m</td>
<td>( m )</td>
<td>( \frac{m^2 - m}{2} )</td>
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<td>2</td>
<td>1</td>
<td>m-1</td>
<td>2 ( (m - 1) )</td>
<td>( \frac{m^2 - 3m + 4}{2} )</td>
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<tr>
<td>3</td>
<td>2</td>
<td>m-2</td>
<td>3 ( (m - 2) )</td>
<td>( \frac{m^2 - 5m + 12}{2} )</td>
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<tr>
<td>4</td>
<td>3</td>
<td>m-3</td>
<td>4 ( (m - 3) )</td>
<td>( \frac{m^2 - 7m + 24}{2} )</td>
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Table 1 gives the possible values of \( \alpha_i \) and \( \beta_i \) for distinct labelings \( f_i \) of \( K_m \) such that \( f_i(a) + f_i(b) + f_i(ab) \equiv 1 \text{ (mod 2)} \) for all \( ab \in E(K_m) \).

**Theorem 4.1.** Let \( K_m^{(n)} \) be the one-point union of \( n \) copies of a complete graph \( K_m \). If \( \sqrt{m - 1} \) has an integer value, then \( K_m^{(n)} \) is TMC for \( m \equiv 1, 2 \text{ (mod 4)} \).

**Proof.** Let \( f : V(K_m) \cup E(K_m) \rightarrow \{0, 1\} \) be a TMC labeling of \( K_m \). Under the labeling \( f \), the graph \( K_m \) can be decomposed as \( K_m = K_p \cup K_r \cup K_{p,r} \). Then we have, \( n_f(0) = r + pr \) and \( n_f(1) = \frac{p^2 + r^2 + p - r}{2} \). By Corollary 2.2, \( K_m^{(n)} \) is TMC if \( n_f(0) = n_f(1) + 1 \). Whenever, \( n_f(0) = n_f(1) + 1 \), \( p^2 + p(1 - 2r) + r^2 - 3r + 2 = 0 \). This implies that \( r = \frac{1}{2} [(m + 1) \pm \sqrt{m - 1}] \) as \( p \equiv m - r \). Also, \( n_f(0) = n_f(1) + 1 \) is possible only when \( m \equiv 1, 2 \text{ (mod 4)} \). Therefore, \( K_m^{(n)} \) is TMC for \( m \equiv 1, 2 \text{ (mod 4)} \), if \( \sqrt{m - 1} \) has an integer value.

**Theorem 4.2 ([4]).** Let \( G \) be an odd graph with \( p + q \equiv 2 \text{ (mod 4)} \). Then \( G \) is not TMC.

**Theorem 4.3.** Let \( K_m^{(n)} \) be the one-point union of \( n \) copies of a complete graph \( K_m \).

(i) If \( m \equiv 0 \text{ (mod 8)} \), then \( K_m^{(n)} \) is not TMC for \( n \equiv 3 \text{ (mod 4)} \).

(ii) If \( m \equiv 4 \text{ (mod 8)} \), then \( K_m^{(n)} \) is not TMC for \( n \equiv 1 \text{ (mod 4)} \).

**Proof.** Clearly, \( p = |V(K_m^{(n)})| = n(m - 1) + 1 \) and \( q = |E(K_m^{(n)})| = \frac{nm(m - 1)}{2} \) so that \( p + q = \frac{n(m - 1)(m + 2)}{2} + 1 \).

Part (i) Assume \( m = 8k \) and \( n = 4l + 3 \). Since the degree of every vertex is odd and
p + q \equiv 2(\text{mod} \ 4), \text{it follows from Theorem } 4.2 \text{ that } K_{6n}^{(n)} \text{ is not TMC.}

Part (ii) can similarly be proved. \qed

**Theorem 4.4.** $K_4^{(n)}$ is TMC if and only if $n \not\equiv 1(\text{mod} \ 4)$.

**Proof.** Necessity follows from Theorem 4.3 and for sufficiency we define the labelings $f_1$ and $f_2$ as follows: $f_1(v_i) = 0$ for $1 \leq i \leq 4$, $f_1(v_i, v_j) = 1$ for $1 \leq i, j \leq 4$ and under the labeling $f_2$ decompose $K_4$ as $K_1 \cup K_3 \cup K_{1,3}$. From Table 1, we observe that $\alpha_1 = 4, \beta_1 = 6, \alpha_2 = 6$ and $\beta_2 = 4$. Therefore, by Corollary 2.4 (1), $K_4^{(n)}$ is TMC if $n \not\equiv 1(\text{mod} \ 4)$. \qed

**Theorem 4.5.** $K_5^{(n)}$ is TMC for all $n \geq 1$.

**Proof.** Define $f : V(K_5^{(n)}) \cup E(K_5^{(n)}) \rightarrow \{0, 1\}$ as follows:

$$f(v_i) = \begin{cases} 
0 & \text{if } i \not= 5, \\
1 & \text{if } i = 5 
\end{cases}$$

and

$$f(v_i, v_j) = \begin{cases} 
1 & \text{if } 1 \leq i, j \leq 4, \\
0 & \text{if } i = 5 \text{ or } j = 5. 
\end{cases}$$

Clearly, $\alpha = \beta + 1 = 8$. Therefore, by Corollary 2.2, $K_5^{(n)}$ is TMC for all $n \geq 1$. \qed

**Theorem 4.6.** $K_6^{(n)}$ is TMC for all $n \geq 1$.

**Proof.** Let $f_1$ and $f_2$ be the labelings from $V(K_6^{(n)}) \cup E(K_6^{(n)})$ into $\{0, 1\}$. Then, under the labelings $f_1$ and $f_2$ the graph $K_6$ can be decomposed as $K_1 \cup K_5 \cup K_{1,5}$ and $K_2 \cup K_4 \cup K_{2,4}$ respectively. Clearly, $\alpha_1 = 10, \beta_1 = 11, \alpha_2 = 12$ and $\beta_2 = 9$. Hence, by Corollary 2.4 (2a), $K_6^{(n)}$ is TMC for all $n \geq 1$. \qed

**Theorem 4.7.** $K_7^{(n)}$ is TMC for all $n \geq 1$.

**Proof.** Let $f_1, f_2$ and $f_3$ be the labelings from $V(K_7^{(n)}) \cup E(K_7^{(n)})$ into $\{0, 1\}$. Then, under the labelings $f_1, f_2$ and $f_3$ the graph $K_7$ can be decomposed as $K_3 \cup K_4 \cup K_{3,4}$, $K_4 \cup K_3 \cup K_{4,3}$ and $K_5 \cup K_2 \cup K_{5,2}$ respectively. We observe that $\alpha_1 = 16, \beta_1 = 12, \alpha_2 = 15, \beta_2 = 13, \alpha_3 = 12$ and $\beta_3 = 16$. Hence, by Corollary 2.4 (2b), $K_7^{(n)}$ is TMC for all $n \geq 1$. \qed

**Theorem 4.8.** $K_8^{(n)}$ is TMC if and only if $n \not\equiv 3(\text{mod} \ 4)$.

**Proof.** Necessity follows from Theorem 4.3 and for sufficiency we define the labelings $f_1$ and $f_2$ as follows: under the labelings $f_1$ and $f_2$ the graph $K_8$ can be decomposed as $K_2 \cup K_6 \cup K_{2,6}$ and $K_3 \cup K_5 \cup K_{3,5}$ respectively. Clearly, $\alpha_1 = 18, \beta_1 = 18, \alpha_2 = 20$ and $\beta_2 = 16$. Hence, by Corollary 2.4 (3), $K_8^{(n)}$ is TMC if $n \not\equiv 3(\text{mod} \ 4)$. \qed

**Theorem 4.9.** $K_9^{(n)}$ is TMC for all $n \geq 1$. 
Define the labelings $f_1$, $f_2$ and $f_3$ the graph $K_9$ can be decomposed as $K_2 \cup K_7 \cup K_{2,7}$, $K_3 \cup K_6 \cup K_{3,6}$ and $K_4 \cup K_5 \cup K_{4,5}$ respectively. We observe that $\alpha_1 = 21$, $\beta_1 = 24$, $\alpha_2 = 24$, $\beta_2 = 21$, $\alpha_3 = 25$ and $\beta_3 = 20$. Therefore, by Corollary 2.4 (2c), the graph $K_9^{(n)}$ is TMC for all $n \geq 1$.

5. ONE-POINT UNION OF WHEELS

A wheel $W_m$ is obtained by joining the vertices $v_1, v_2, \ldots, v_m$ of a cycle $C_m$ to an extra vertex $v$ called the centre. We consider $W_m$ as a rooted graph with $v$ as its root.

**Theorem 5.1.** Let $W_m^{(n)}$ be the one-point union of $n$ copies of a wheel $W_m$.

(i) If $m \equiv 0 \pmod{4}$, then $W_m^{(n)}$ is TMC for all $n \geq 1$.

(ii) If $m \equiv 1 \pmod{4}$, then $W_m^{(n)}$ is TMC for $n \neq 3 \pmod{4}$.

(iii) If $m \equiv 2 \pmod{4}$, then $W_m^{(n)}$ is TMC for all $n \geq 1$.

(iv) If $m \equiv 3 \pmod{4}$, then $W_m^{(n)}$ is TMC for $n \neq 1 \pmod{4}$.

**Proof.** Define the labelings $f_1, f_2, f_3, f_4$ and $f_5$ as follows: $f_j(v) = 0$ for $j = 1, 2, 3, 4, 5$. $f_1(v_m v_1) = 0$,

$$f_1(v_i) = \begin{cases} 1 & \text{if } i \equiv 0 \pmod{4}, \\ 0 & \text{if } i \not\equiv 0 \pmod{4}, \end{cases} \quad f_1(v_i v_{i+1}) = \begin{cases} 1 & \text{if } i \equiv 1, 2 \pmod{4}, \\ 0 & \text{if } i \equiv 0, 3 \pmod{4} \end{cases}$$

and

$$f_2(v_i) = f_2(v_i v_{i+1}) = 1, \quad f_2(v_v) = 0 \text{ for } i = 1, 2, \ldots, m \text{ and } f_2(v_m v_1) = 1.$$ 

$$f_3(v_i) = f_3(v_i), \quad f_3(v_i v_{i+1}) = f_1(v_i v_{i+1}), \quad f_3(v_v) = f_1(v_v) \text{ for } i = 1, 2, \ldots, m \text{ and } f_3(v_m v_1) = 1. \quad f_4(v_1) = f_4(v_1 v_2) = f_4(v_m v_1) = 0, \quad f_4(v_i) = f_3(v_i), \quad f_4(v_i v_{i+1}) = f_3(v_i v_{i+1}), \quad f_4(v_v) = f_3(v_v) \text{ for } i = 2, 3, \ldots, m \text{ and } f_4(v_1) = 0.$$ 

$$f_5(v_i) = \begin{cases} 1 & \text{if } i \equiv 1 \pmod{2}, \\ 0 & \text{if } i \not\equiv 0 \pmod{2}, \end{cases} \quad f_5(v_v) = \begin{cases} 0 & \text{if } i \equiv 1 \pmod{2}, \\ 1 & \text{if } i \not\equiv 0 \pmod{2}, \end{cases}$$

**Case 1.** $m \equiv 0 \pmod{4}$.

If we consider the labeling $f_1$ we have, $n_{f_1}(0) = n_{f_1}(1) + 1$. Then, by Corollary 2.2, $W_m^{(n)}$ is TMC for all $n \geq 1$.

**Case 2.** $m \equiv 1 \pmod{4}$.

If we consider the labelings $f_2, f_3$ and $f_4$. We have $\alpha_2 = \frac{3m+1}{2}, \beta_2 = \frac{3m+1}{2}, \alpha_3 = \frac{3m+5}{2}, \beta_3 = \frac{3m-1}{2}, \alpha_4 = m + 1, \beta_4 = 2m$. Then, system (2.1) in Theorem 2.1 becomes $| - x_2 + 3x_3 - (m+1)x_4 + 1 | \leq 1$, $x_2 + x_3 + x_4 = n$. When $n = 4t + 1$, $x_2 = 3t$, $x_3 = t$, $x_4 = 0$ is a solution. When $n = 4t + 2$, $x_2 = 3t + 2$, $x_3 = t$, $x_4 = 0$ is a solution. When $n = 4t + 3$, the system has no solution. Hence, by Theorem 2.1, $W_m^{(n)}$ is TMC if $n \not\equiv 3 \pmod{4}$. 


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Case 3. \( m \equiv 2 \) (mod 4).
If we consider the labelings \( f_2, f_3, f_4 \) and \( f_5 \), we have \( \alpha_2 = m + 1, \beta_2 = 2m, \alpha_3 = \frac{3m}{2}, \beta_3 = \frac{3m+2}{2}, \alpha_4 = \frac{3m+4}{2}, \beta_4 = \frac{3m-2}{2}, \alpha_5 = 2m + 1, \beta_5 = m \). Thus, system (2.1) in Theorem 2.1 becomes \(| -mx_2 - 2x_3 + 2x_4 + mx_5 + 1 | \leq 1, x_2 + x_3 + x_4 + x_5 = n \).

When \( n = 4t \), \( x_2 = x_3 = x_4 = x_5 = t \) is a solution. When \( n = 4t + 1 \), \( x_2 = t, x_3 = t + 1, x_4 = t, x_5 = t \) is a solution. When \( n = 4t + 2 \), \( x_2 = t + 1, x_2 = t, x_4 = t, x_5 = t + 1 \) is a solution. When \( n = 4t + 3 \), \( x_2 = t + 1, x_3 = t + 1, x_4 = t, x_5 = t + 1 \) is a solution. Hence, by Theorem 2.1, \( W_m^{(n)} \) is TMC for all \( n \geq 1 \).

Case 4. \( m \equiv 3 \) (mod 4).
If we consider the labelings \( f_3 \) and \( f_4 \). We have \( \alpha_3 = \frac{3m-1}{2}, \beta_3 = \frac{3m+3}{2}, \alpha_4 = \frac{3m+1}{2}, \beta_4 = \frac{3m-1}{2} \). Therefore, system (2.1) in Theorem 2.1 becomes \(| -3x_3 + x_4 + 1 | \leq 1, x_3 + x_4 = n \). When \( n = 4t \), \( x_3 = t, x_4 = 3t \) is a solution. When \( n = 4t + 1 \), the system has no solution. When \( n = 4t + 2 \), \( x_3 = t + 1, x_4 = 3t + 1 \) is a solution. When \( n = 4t + 3 \), \( x_3 = t + 1, x_4 = 3t + 2 \) is a solution. Hence, by Theorem 2.1, \( W_m^{(n)} \) is TMC if \( n \not\equiv 1 \) (mod 4).

Acknowledgments
The authors sincerely thank the referee for the valuable suggestions which were used in this paper.

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