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ALL GRAPHS WITH PAIRED-DOMINATION NUMBER TWO LESS THAN THEIR ORDER

Włodzimierz Ulatowski

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Abstract. Let $G = (V, E)$ be a graph with no isolated vertices. A set $S \subseteq V$ is a paired-dominating set of G if every vertex not in S is adjacent with some vertex in S and the subgraph induced by S contains a perfect matching. The paired-domination number $\gamma_p(G)$ of G is defined to be the minimum cardinality of a paired-dominating set of G. Let G be a graph of order n. In [*Paired-domination in graphs*, Networks **32** (1998), 199–206] Haynes and Slater described graphs G with $\gamma_p(G) = n$ and also graphs with $\gamma_p(G) = n - 1$. In this paper we show all graphs for which $\gamma_p(G) = n - 2$.

Keywords: paired-domination, paired-domination number.

Mathematics Subject Classification: 05C69.

1. INTRODUCTION

All graphs considered in this paper are finite, undirected, without loops, multiple edges and isolated vertices. Let $G = (V, E)$ be a graph with the vertex set V and the edge set E. Then we use the convention $V = V(G)$ and $E = E(G)$. Let G and G' be two graphs . If $V(G) \subseteq V(G')$ and $E(G) \subseteq E(G')$ then G is a *subgraph* of G' (and G' is a *supergraph* of G), written as $G \subseteq G'$. The number of vertices of G is called the *order* of G and is denoted by $n(G)$. When there is no confusion we use the abbreviation $n(G) = n$. Let C_n and P_n denote the cycle and the path of order n, respectively. The *open neighborhood* of a vertex $v \in V$ in G is denoted $N_G(v) = N(v)$ and defined by $N(v) = \{u \in V : vu \in E\}$ and the *closed neighborhood* $N[v]$ of v is $N[v] = N(v) \cup \{v\}$. For a set S of vertices the *open neighborhood* $N(S)$ is defined as the union of open neighborhoods $N(v)$ of vertices $v \in S$, the *closed neighborhood* $N[S]$ of S is $N[S] = N(S) \cup S$. The *degree* $d_G(v) = d(v)$ of a vertex v in G is the number of edges incident to v in G; by our definition of a graph, this is equal to $|N(v)|$.

A *leaf* in a graph is a vertex of degree one. A *subdivided star* $K^*_{1,t}$ is a star $K_{1,t}$, where each edge is subdivided exactly once.

In the present paper we continue the study of paired-domination. Problems related to paired-domination in graphs appear in [1–5]. A set M of independent edges in a graph G is called a *matching* in G. A *perfect matching* M in G is a matching in G such that every vertex of G is incident to an edge of M. A set $S \subseteq V$ is a *paired-dominating set*, denoted PDS, of a graph G if every vertex in $V - S$ is adjacent to a vertex in S and the subgraph $G[S]$ induced by S contains a perfect matching M. Therefore, a paired-dominating set S is a dominating set $S = \{u_1, v_1, u_2, v_2, \ldots, u_k, v_k\}$ with matching $M = \{e_1, e_2, \ldots, e_k\}$, where $e_i = u_i v_i$, $i = 1, \ldots, k$. Then we say that u_i and v_i are *paired* in S. Observe that in every graph without isolated vertices the end-vertices of any maximal matching form a PDS. The *paired-domination number* of G, denoted $\gamma_p(G)$, is the minimum cardinality of a PDS of G. We will call a set S a $\gamma_p(G)$-set if S is a paired-dominating set of cardinality $\gamma_p(G)$. The following statement is an immediate consequence of the definition of PDS.

Observation 1.1 ([4]). *If u is adjacent to a leaf of G, then u is in every PDS.*

Haynes and Slater [4] show that for a connected graph G of order at least six and with minimum degree $\delta(G) \geq 2$, two-thirds of its order is the bound for $\gamma_p(G)$.

Theorem 1.2 ([4]). *If a connected graph G has $n \geq 6$ and $\delta(G) \geq 2$, then*

$$\gamma_p(G) \leq 2n/3.$$

Henning in [5] characterizes the graphs that achieve equality in the bound of Theorem 1.2.

In [4] the authors give the solutions of the graph-equations $\gamma_p(G) = n$ and $\gamma_p(G) = n - 1$, where G is a graph of order n.

Theorem 1.3 ([4]). *A graph G with no isolated vertices has $\gamma_p(G) = n$ if and only if G is mK_2.*

Let $\mathcal{F}$ be the collection of graphs C_3, C_5, and the subdivided stars $K^*_{1,t}$. Now, we can formulate the following statements.

Theorem 1.4 ([4]). *For a connected graph G with $n \geq 3$, $\gamma_p(G) \leq n-1$ with equality if and only if $G \in \mathcal{F}$.*

Corollary 1.5 ([4]). *If G is a graph with $\gamma_p(G) = n - 1$, then $G = H \cup rK_2$ for $H \in \mathcal{F}$ and $r \geq 0$.*

In the present paper we consider the graph-equation

$$\gamma_p(G) = n - 2, \tag{1.1}$$

where $n \geq 4$ is the order of a graph G.

Our aim in this paper is to find all graphs G satisfying (1.1). For this purpose we need the following definition and statements.

Definition 1.6. A subgraph G of a graph G' is called a *special subgraph* of G', and G' is a *special supergraph* of G, if either $V(G) = V(G')$ or the subgraph $G'[V(G') - V(G)]$ has a perfect matching.

It is clear that if $V(G) = V(G')$ then the concepts "subgraph" and "special subgraph" are equivalent. Now we can formulate the following fact.

Fact 1.7. *Let G be a special subgraph of G'.*

a) *If S is a PDS in G then $S' = S \cup (V(G') - V(G))$ is a PDS in G'.*
b) *If $\gamma_p(G) = n - r$ then $\gamma_p(G') \leq n' - r$, where $n = |V(G)|$, $n' = |V(G')|$ and $0 \leq r \leq n - 2$.*

Proof. a) Assume that

$$S = \{u_1, v_1, u_2, v_2, \ldots, u_t, v_t\} \quad \text{and} \quad V(G') - V(G) = \{u_{t+1}, v_{t+1}, \ldots, u_k, v_k\},$$

where u_i and v_i are paired in S (for $i = 1, \ldots, t$) and in $V(G') - V(G)$ (for $i = t+1, \ldots, k$). Hence $M = \{e_1, e_2, \ldots, e_k\}$, where $e_i = u_i v_i$, for $i = 1, \ldots, k$, is a perfect matching in $G'[S']$. By definition of a PDS and by $V(G) - S = V(G') - S'$ we obtain the statement of a).

b) Let S be a γ_p-set in G, thus $|V(G) - S| = r$. It follows from a) that $S' = S \cup (V(G') - V(G))$ is a PDS in G'. Moreover, we have the equality

$$|S'| = n' - |V(G') - S'| = n' - |V(G) - S| = n' - r.$$

Therefore we obtain $\gamma_p(G') \leq |S'| = n' - r$. □

Now assume that G is a connected graph of order $n \geq 4$ satisfying (1.1). Let $S = \{u_1, v_1, u_2, v_2, \ldots, u_k, v_k\}$ be a $\gamma_p(G)$-set with a perfect matching $M = \{e_1, e_2, \ldots, e_k\}$, where $e_i = u_i v_i$ for $i = 1, 2, \ldots, k$, and $V - S = \{x, y\}$. By letting $\alpha(S)$ denote the minimum cardinality of a subset of S wich dominates $V - S$, i.e.

$$\alpha(S) = \min\{|S'| : S' \subseteq S, V - S \subseteq N(S')\}.$$

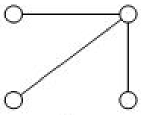
Let S_i be any set of size $\alpha(S)$ such that $S_i \subseteq S$ and $V - S \subseteq N(S_i)$. For S, M and S_i we define a graph H as follows:

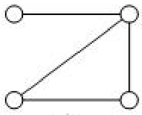
$$V(H) = V(G) \quad \text{and} \quad E(H) = M \cup \{uv : u \in S_i, v \in \{x, y\}\}.$$

It is clear that H is a spanning forest of G; we denote it as $G_{sf}(S, M, S_i)$.

2. THE MAIN RESULT

The main purpose of this paper is to construct all graphs G of order n for which $\gamma_p(G) = n - 2$. At first consider the family $\mathcal{G}$ of graphs in Fig. 1. We shall show that only the graphs in family $\mathcal{G}$ are connected and satisfy condition (1.1).


G_1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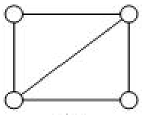
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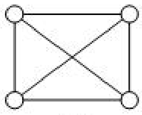
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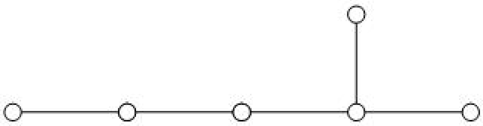
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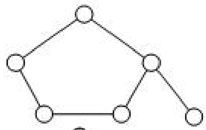
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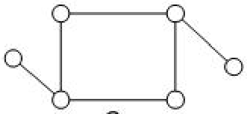
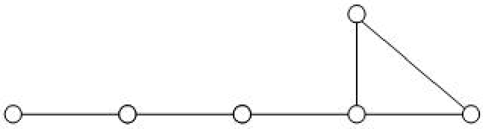

G_7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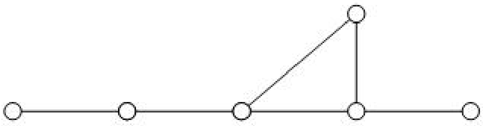
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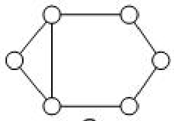
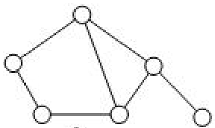

G_{10}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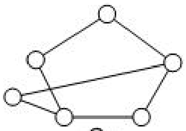
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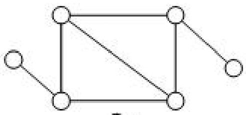
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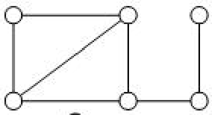
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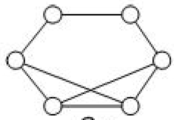
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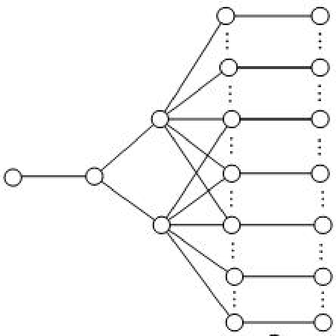
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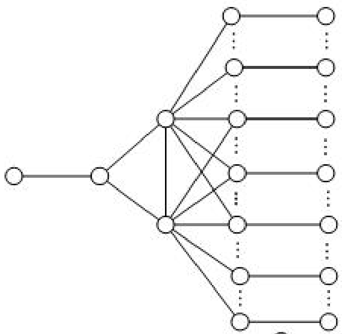
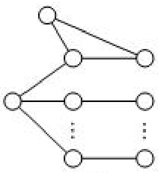
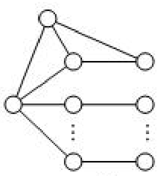
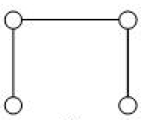
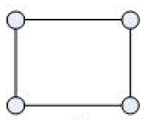

G_{17}


G_{18}


G_{19}


G_{20}


G_{21}

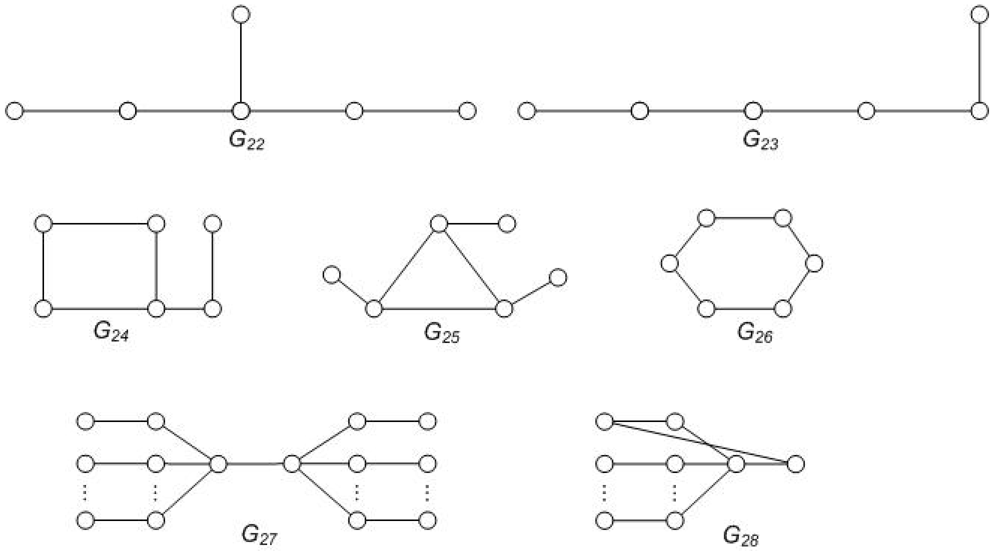


Fig. 1. Graphs in family $\mathcal{G}$

Theorem 2.1. *Let G be a connected graph of order $n \geq 4$. Then $\gamma_p(G) = n - 2$ if and only if $G \in \mathcal{G}$.*

Proof. Our aim is to construct all connected graphs G for which (1.1) holds. Let G be a connected graph of order $n \geq 4$ satisfying (1.1). We shall prove that $G \in \mathcal{G}$.

Let us consider the following cases.

Case 1. There exists a $\gamma_p(G)$-set S such that $\alpha(S) = 1$.

Case 1.1. $k = 1$. Then we have the graphs shown in Fig. 2. It is clear that the graphs H_i satisfy (1.1) and $H_i = G_i$ for $i = 1, 2, 3, 4$.

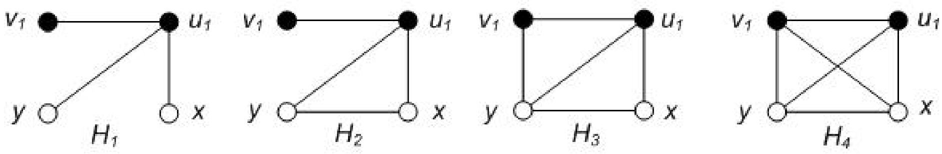


Fig. 2. The graphs for Case 1.1.

Figure 2 illustrates the graphs H_i, where the shaded vertices form a γ_p-set. We shall continue to use this convention in our proof.

At present for $k \geq 2$ we shall find all connected graphs G satisfying (1.1) and having a $\gamma_p(G)$-set S with $\alpha(S) = 1$. It is clear that in Case 1 any graph $G_{sf}(S, M, S_i)$ is independent of the choice of S, M and S_i, so we can write $G_{sf}(S, M, S_i) = G_{sf}$. The spanning forest G_{sf} consists of k components $G^{(1)}, G^{(2)}, \ldots, G^{(k)}$, where $G^{(1)} = K_{1,3}$

with $V(K_{1,3}) = \{x, y, u_1, v_1\}$, where u_1 is the central vertex, while $G^{(i)} = K_2$ for $i = 2, \ldots, k$ (see Fig. 3). Now by adding suitable edges to G_{sf} we are able to reconstruct G.

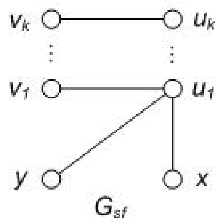


Fig. 3. The spanning forest of G

Case 1.2. $k = 2$. Now we start with the graph H_5 (Fig. 4). In our construction of the

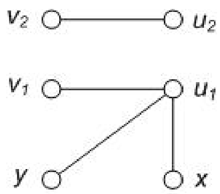


Fig. 4. The spanning forest H_5

desired connected graphs we add one or more edges to H_5. Thus, let us consider the following cases regarding the number of these edges.
Case 1.2.1. One edge (Fig. 5). One can see that $H_6 = G_5$ satisfies (1.1) but H_7 does not.

Fig. 5. The graphs obtained by adding one edge to H_5

Case 1.2.2. Two edges. For H_7 we have $\gamma_p(H_7) = 6 - 4 = |V(H_7)| - 4$. Thus, by Fact 1.7 b) for any special supergraph G' of H_7 we obtain $\gamma_p(G') \leq |V(G')| - 4$. Hence, we deduce that it suffices to add one edge to H_6. Since adding the edges u_1u_2 or u_1v_2

leads to H_7, we shall omit these edges in our construction. Now consider the graphs of Fig. 6.

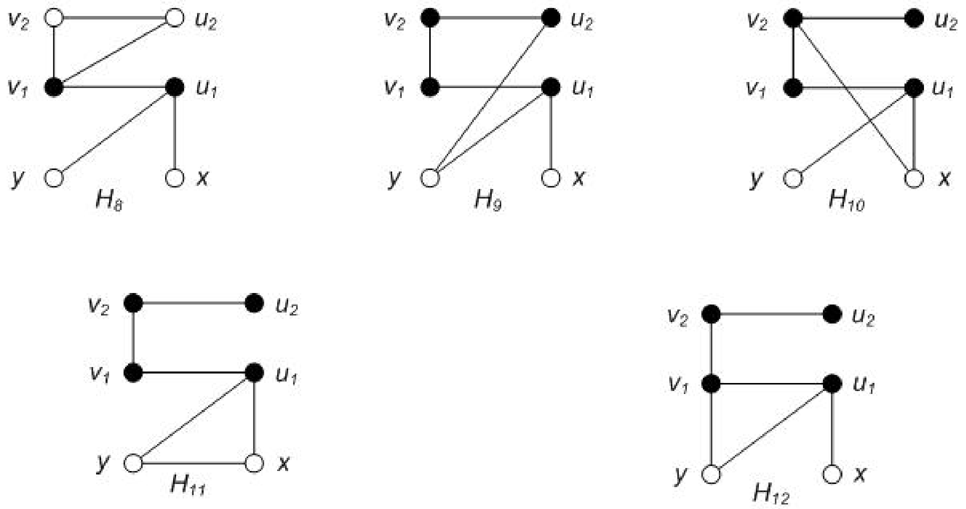


Fig. 6. Adding a new edge to H_6

Certainly, $\gamma_p(H_8) = n-4$, $\gamma_p(H_i) = n-2$ and $H_i = G_{i-3}$ for $i = 9, \ldots, 12$. Using the above argument for H_8 we do not take v_1u_2. Let us consider the following cases.
Case 1.2.3. Three edges. It follows from Fact 1.7 b) that it suffices to add one edge to H_i for $i = 9, \ldots, 12$.
Case 1.2.3.1. H_9. Observe that $H_i = G_{i-3}$, $i = 13, 14, 15$, satisfy (1.1). Moreover, the graphs depicted in Fig. 7 are the unique graphs for which (1.1) holds in this case. Indeed, the edge v_2y leads to a supergraph of H_8, and joining u_2 to x we have H_{15}.

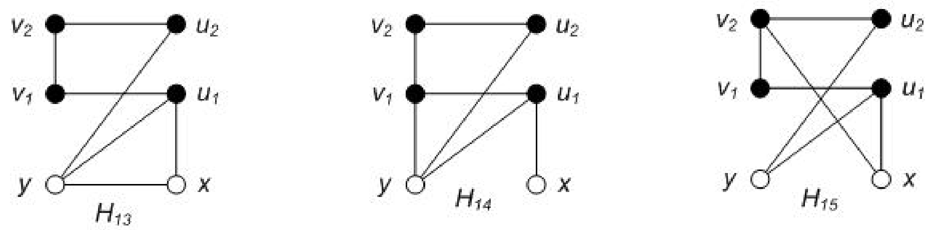


Fig. 7. $H_9 + e$

Case 1.2.3.2. H_{10}. Then we obtain a supergraph of H_7 by means of edge v_2y, a supergraph of H_8 by means of xy, u_2x, instead by adding u_2y we return to H_{15}. Therefore, it remains to research the graph of Fig. 8. It obvious that (1.1) holds for $H_{16} = G_{13}$.

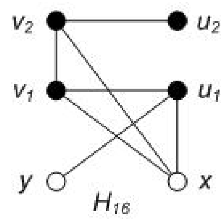


Fig. 8. The graph obtained from H_{10} by adding an edge

Case 1.2.3.3. H_{11}. Then it suffices to consider the graph of Fig. 9. Really, edges v_2x, v_2y lead to a supergraph of H_8 and u_2x, u_2y lead to H_{13}. Observe that for $H_{17} = G_{14}$ equality (1.1) is true.

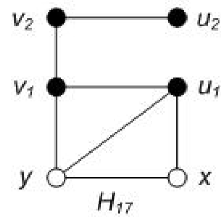


Fig. 9. $H_{11} + e$

Case 1.2.3.4. H_{12}. Here we do not obtain any new graph satisfying (1.1). Indeed, we obtain: a supergraph of H_7 (by adding v_1x), a supergraph of H_8 (by v_2x), H_{13} (by u_2x), H_{14} (by u_2y) and H_{16} (by v_2y).
Case 1.2.4. Four edges.
Case 1.2.4.1. H_{13}. Let G be a graph obtained by adding a new edge e to H_{13}. If $e = v_1y$ then $H_7 \subseteq G$; if $e = v_2y, v_2x$, then $H_8 \subseteq G$ and for $e = v_1x, u_2x$ we have the graph $G_{15} \in \mathcal{G}$ (Fig. 10).

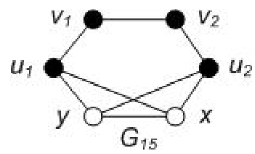


Fig. 10. $H_{15} + e$

Case 1.2.4.2. H_{14}. Keeping the above convention we note: if $e = xy$ then $H_7 \subseteq G$; if $e = v_2y, v_2x, u_2x$ then $H_8 \subseteq G$.

Case 1.2.4.3. H_{15}. If $e = v_2y$ then $H_7 \subseteq G$; if $e = xy, v_1y, u_2x$ then $H_8 \subseteq G$; if $e = v_1x$ then $G = G_{15}$. It is easy to see that (1.1) is true for G_{15}.
Case 1.2.4.4. H_{16}. In this case we conclude: if $e = xy$ then $H_7 \subseteq G$; if $e = v_2y, u_2x$ then $H_8 \subseteq G$; if $e = u_2y$ then $G = G_{15}$.
Case 1.2.4.5. H_{17}. Then we obtain the following results: if $e = v_1x, v_2y, u_2y$ then $H_7 \subseteq G$; if $e = v_2x$ then $H_8 \subseteq G$; if $e = u_2x$ then we have the graph H_{18} depicted in Fig. 11. It is clear that $H_{18} = G_{15}$.

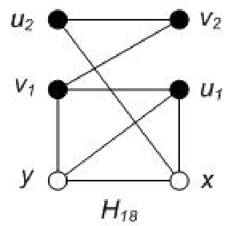


Fig. 11. $H_{17} + e$, where $e = u_2x$

Case 1.2.5. Five edges.
Case 1.2.5.1. G_{15}. Then it suffices to consider the following: if $e = v_1y$ then $H_7 \subseteq G$; if $e = v_1x$ then $H_8 \subseteq G$. Therefore, Case 1.2 is complete.
For case $k \geq 3$ we only consider graphs satisfying the condition $G[S'] = G_{sf}[S'] = K_{1,3}$ for $S' = \{x, y, u_1, v_1\}$. In other words, G contains the induced star $K_{1,3}$, where $V(K_{1,3}) = \{x, y, u_1, v_1\}$ and u_1 is the central vertex.
Case 1.3. $k = 3$. Then we start with the basic graph of Fig. 12. To obtain connected graphs we add two or more edges to H_{19} and investigate whether (1.1) holds for the resulting graphs. At first we find a forbidden subgraph $H \subseteq G$ i.e. such that $\gamma_p(H) = n - 4$. We have already shown two forbidden special subgraphs H_7, H_8, and we now present the other one in Fig. 13. For a while we return to the general case $k \geq 3$. The forbidden special subgraphs H_7 and H_{20} determine a means of construction of graphs G from G_{sf}.

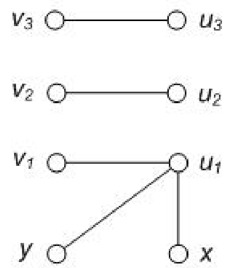


Fig. 12. The spanning forest $G_{sf} = H_{19}$

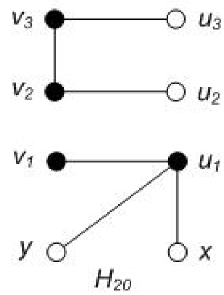


Fig. 13. The forbidden special subgraph

Claim 1. *Let G be a connected graph satisfying* (1.1) *and obtained from $G_{sf} = H_{19}$. Then vertex u_i or v_i, $i = 2, \ldots, k$, can be adjacent to the vertices v_1, x, y, only.*

Now we add at least two edges to H_{19}. We consider the following cases.
Case 1.3.1. Two edges. Then we obtain the graphs H_{21} and H_{22} for which (1.1) holds (Fig. 14).

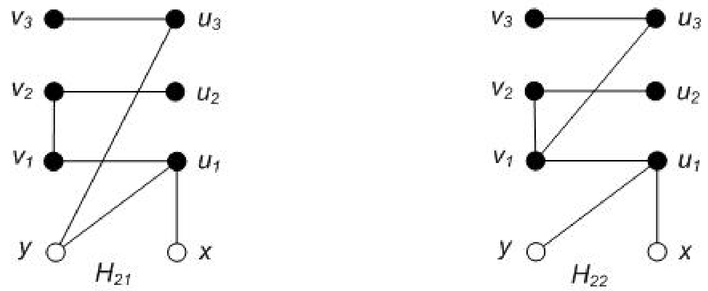


Fig. 14. Adding two edges to H_{19}

Case 1.3.2. Three edges. At present it suffices to add one edge in H_{21}, H_{22}. This way we obtain the graphs depicted in Figure 15.
Observe that (1.1) fails for H_{24} since $H_{20} \subseteq H_{24}$. Thus, H_{23} satisfies (1.1) but H_i, $i = 24, 25, 26$, do not.
Case 1.3.3. Four edges. By adding one edge to H_{23} we obtain the unique graph for which (1.1) holds (see Fig. 16). One can verify that in the remaining options we have special supergraphs of H_7, H_8, H_{20}, H_{25} or H_{26}.
Case 1.3.4. Five edges. Each new edge in H_{27} leads to a special supergraph of H_7, H_8, H_{20}, H_{25} or H_{26}. But the following statement is obvious.
Claim 2. *The graphs H_7, H_8, H_{20}, H_{25} and H_{26} are forbidden special subgraphs for* (1.1).

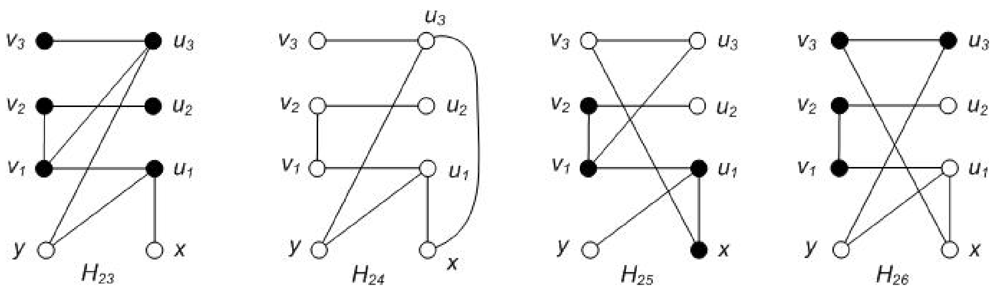


Fig. 15. $H_{21}+e$ and $H_{22}+e$

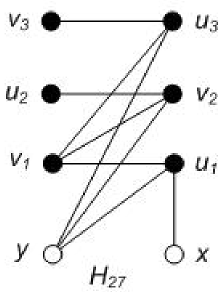


Fig. 16. The graph obtained from H_{23} by adding one edge

We now study a generalization of the case $k = 3$. We keep our earlier assumption regarding the induced star $K_{1,3}$ with vertex set $\{u_1, v_1, x, y\}$.

Case 1.4. $k \geq 3$. Then we give one property of graphs satisfying (1.1).

Claim 3. *Let G be a connected graph for which* (1.1) *holds and $k \geq 3$. If G contains the induced star $K_{1,3}$ with $V(K_{1,3}) = \{x, y, u_1, v_1\}$ then at least one vertex of $K_{1,3}$ is a leaf in G.*

Proof. Consider some cases.

Case A. $k = 3$. It follows from our earlier investigations that H_{21}, H_{22}, H_{23} and H_{27} are the unique connected graphs satisfying (1.1) in this case. Thus, we have the desired result.

Case B. $k \geq 4$. Claim 1 and Fact 1.7 b) imply that a special subgraph $G[S]$ induced by $S = \{x, y, u_1, v_1, u_2, v_2, u_3, v_3\}$ is connected and satisfies (1.1), i.e. it must be one of the graphs H_{21}, H_{22}, H_{23}, H_{27}.

Case B.1. $G[S] = H_{21}$. We show that x is a leaf in G. Suppose not and let x be adjacent to v_i, where $i \geq 4$. Then we obtain the graph H_{28} in Fig. 17, for which (1.1) does not hold.

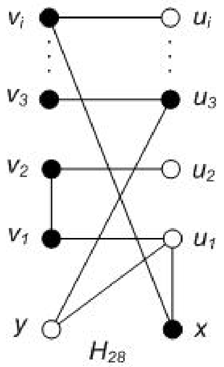


Fig. 17. x is adjacent to v_i for $i \geq 4$

Case B.2. $G[S] = H_{22}$.
Case B.2.1. Suppose that in G vertex v_i, $i \geq 4$, is adjacent to x and y. Then for graph H_{29} depicted in Fig. 18 equality (1.1) is false since $H_{20} \subseteq H_{29}$.

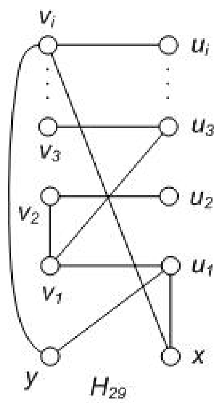


Fig. 18. v_i, for $i \geq 4$, is adjacent to x and y

Case B.2.2. Assume that in G vertices v_i and u_i, $i \geq 4$ are adjacent to x and y, respectively (see Fig. 19). In this way we obtain graph H_{30} which does not satisfy (1.1) since $H_{26} \subseteq H_{30}$.
Case B.2.3. Now, in G let vertices v_i and u_j, $4 \leq i < j$, be adjacent to x and y, respectively (Fig. 20). As can be seen, (1.1) fails for H_{31}, furthermore u_j is paired with y, u_i with v_i, u_3 with v_3 and v_1 with v_2. It follows from the above consideration that we omit the cases: $G[S] = H_{23}$ and $G[S] = H_{27}$, since H_{21}, H_{22} are subgraphs of H_{23}, H_{27}. In all cases we obtain special subgraphs of G for which (1.1) fails, therefore G does not satisfy (1.1), a contradiction. $\square$

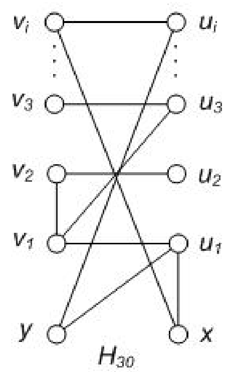


Fig. 19. v_i is adjacent to x and u_i to y, where $i \geq 4$

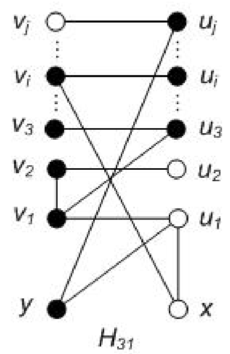


Fig. 20. v_i is adjacent to x and u_j to y for $4 \leq i < j$

We are now in a position to construct the desired graphs for $k \geq 3$. Let G be a connected graph satisfying the following conditions:

a) (1.1) holds,
b) $k \geq 3$,
c) G contains the induced $K_{1,3}$ with $V(K_{1,3}) = \{x, y, u_1, v_1\}$.

According to Claims 1–3 we can reconstruct G based on G_{sf}. By Claim 3, at least one vertex of $K_{1,3}$, say x, is a leaf in G. Hence, by Claim 1, a vertex u_i or v_i, $i = 2, \ldots, k$, can be adjacent to v_1, y, only. Observe that one vertex among u_i, v_i, for $i = 2, \ldots, k$ is a leaf. Indeed, if $v_i y$ and $u_i y$ ($v_i v_1$ and $u_i v_1$) are edges of G then H_8 is a special subgraph of G, but if $v_i y$, $u_i v_1 \in E(G)$ then H_{25} is a special subgraph of G (Fig. 21). From the above investigations we obtain the desired graph in Fig. 22. One can see that (1.1) holds for $H_{32} = G_{16}$. We emphasize that the numbers of edges yw_i or $v_1 w_i$, yp_j, $v_1 z_m$ can be zero here.

Fig. 21. Impossible edges in G

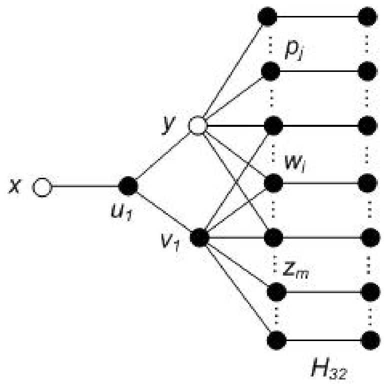


Fig. 22. The family G_{16}

Note that the graphs H_{21}, H_{22}, H_{23} and H_{27} are particular instances of H_{32}. We next describe desired graphs G based on H_{32}. We now discard the assumption concerning the induced star $K_{1,3}$ i.e. edges joining x, y, v_1 are allowable. At first we add the edge yv_1 to H_{32} and obtain graph $H_{33} = G_{17}$ which satisfies (1.1) (Fig. 23). We now consider the following exhaustive cases (Fig. 24). It easy to see that (1.1) is true for $H_{34} = G_{18}$ and $H_{35} = G_{19}$ but is false for H_i, $i = 36, \ldots, 39$.

Case 2. Each $\gamma_p(G)$-set S satisfies $\alpha(S) = 2$.

Case 2.1. There exists a set S containing vertices u,v that dominate $\{x, y\}$ such that u is paired with v in some perfect matching M of S. Without loss of generality we may assume that $u = u_1$, $v = v_1$.

Case 2.1.1. $k = 1$. Then the unique graphs $H_{40} = G_{20}$ and $H_{41} = G_{21}$ satisfying (1.1) are depicted in Fig. 25.

Now for a connected graph G with $k \geq 2$ the spanning forest $G_{sf}(S, M, S_i) = G_{sf}$ for $S_i = \{u, v\}$ is the sum of components $G^{(1)}, G^{(2)}, \ldots, G^{(k)}$, where $G^{(1)} = P_4$ and $G^{(i)} = K_2$ for $i = 2, \ldots, k$ (Fig. 26).

Case 2.1.2. $k = 2$. Now we start with the spanning forest of Fig. 27. In our construction of the desired connected graphs we add at least one edge to the graph H_{42}. Therefore, consider the following cases.

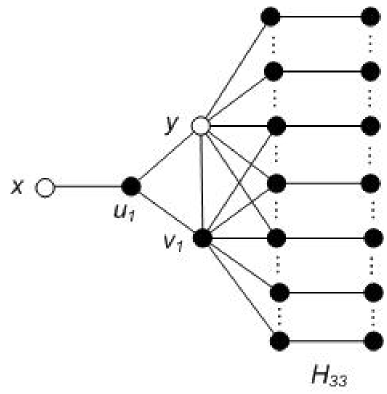


Fig. 23. The family G_{17}

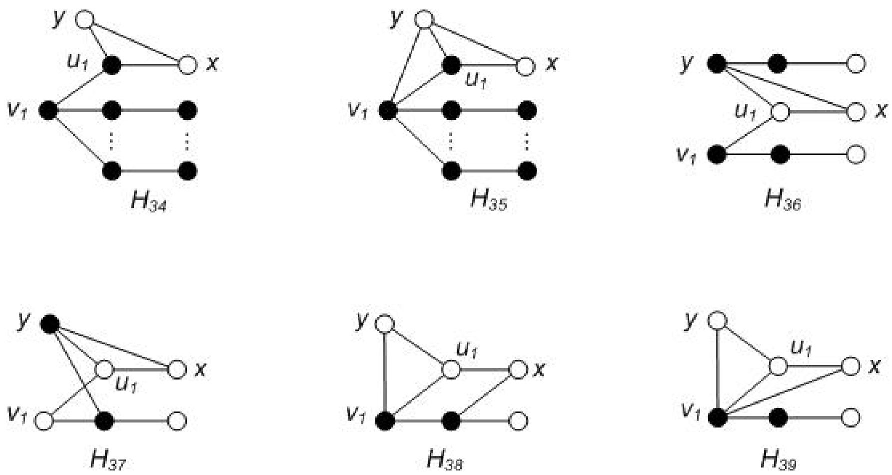


Fig. 24. The exhaustive cases

Fig. 25. The case for $k = 1$

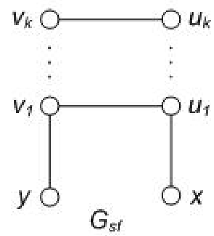


Fig. 26. The spanning forest of G

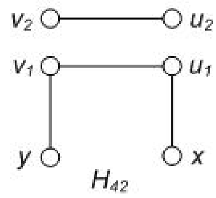


Fig. 27. The graph G_{sf} for $k = 2$

Case 2.1.2.1. One edge (Fig. 28). Then we have $H_{43} = G_{22}$ and $H_{44} = G_{23}$ satisfy (1.1).

Fig. 28. $H_{42} + e$

Case 2.1.2.2. Two edges. Now by adding one edge to H_{43} and H_{44} we obtain some graphs by exhaustion (Fig. 29). Observe (1.1) fails for H_{45}, H_{46} and holds for $H_{47} = G_{24}$, $H_{48} = G_{25}$ and $H_{49} = G_{26}$. Moreover graphs H_i for $i = 50, 51, 52$ are discussed in Case 1.
Case 2.1.2.3. Three edges. Then it suffices to add one edge to H_i, $i = 47, 48, 49$. One resulting graph is the graph H_{53} depicted in Fig. 30, which does not satisfy (1.1). One can verify that the remaining graphs in this case are supergraphs of H_{45}, H_{46} or are graphs discussed in Case 1.

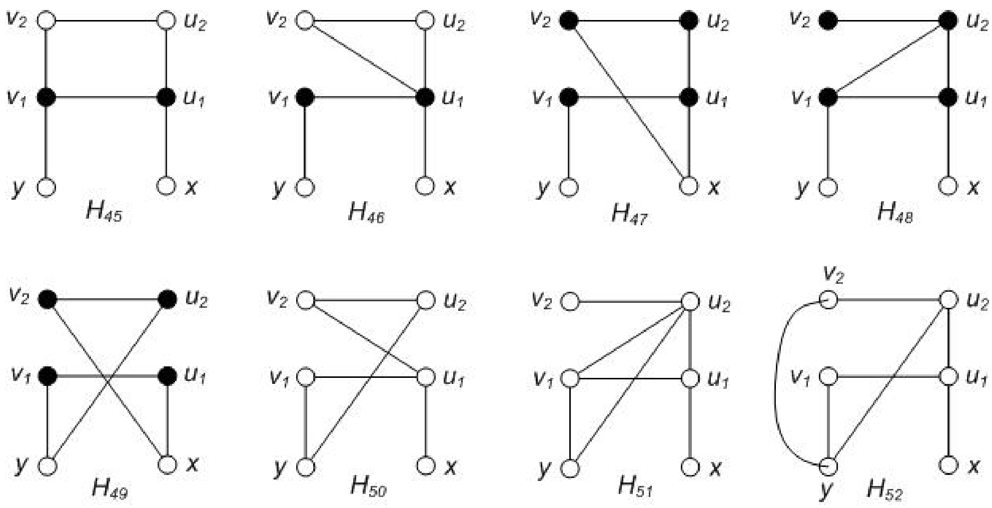


Fig. 29. $H_{43} + e$ and $H_{44} + e$

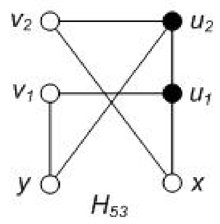


Fig. 30. $H_i + e$ for $i = 47, 48, 49$

Case 2.1.3. $k \geq 3$. At first we show some graphs for which (1.1) does not hold (Fig. 31). For H_i, $i = 54, \ldots, 57$, (1.1) is false; in H_{54} the vertex u_1 is paired with u_2 and v_1 with u_3.

Now we start with the spanning forest depicted in Fig. 32.

Taking account of the forbidden special subgraphs H_i, $i = 54, \ldots, 57$, we can reconstruct G based on G_{sf}. By the connectedness of G it is necessary to join vertices of both the edges u_iv_i, u_jv_j with at least one vertex among u_1, v_1, x, y. Thus we consider the following cases (without loss of generality we take the vertices u_i and u_j of the above edges). If $u_iu_1 \in E(G)$ then we have two options: $u_ju_1 \in E(G)$ or $u_jx \in E(G)$. Instead, if $u_ix \in E(G)$ then we have the following options: $u_jx \in E(G)$ or $u_ju_1 \in E(G)$. Replace u_1 by v_1 and x by y we obtain analogous results. This way we construct the desired graph $G = H_{58}$ for which (1.1) holds (Fig. 33). Note that $H_{58} = G_{27}$. We end this case with adding new edges in H_{58}. At first, if $u_iz \in E(G)$ and $v_iz \in E(G)$, where $2 \leq i \leq k$, $z = u_1, v_1, x, y$, then we return to Case 1. Therefore, let us consider all possible cases, which are depicted in Fig. 34. Then we obtain that (1.1) is true for $H_{60} = G_{28}$ but is false for H_{59} and H_{61}.

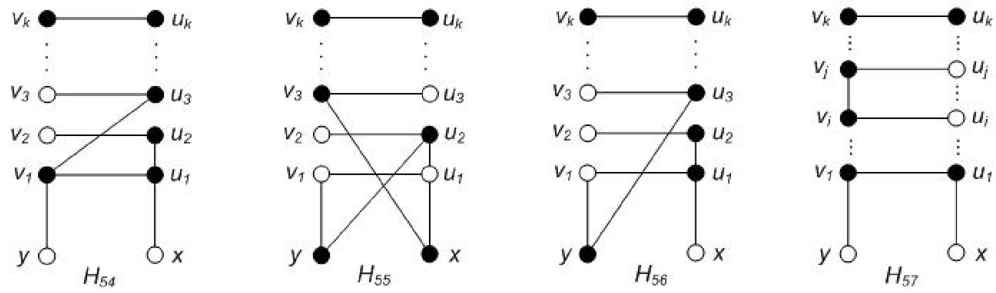


Fig. 31. The forbidden graphs

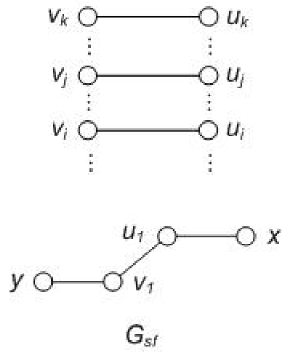


Fig. 32. The spanning forest for $k \geq 3$, where $2 \leq i < j \leq k$

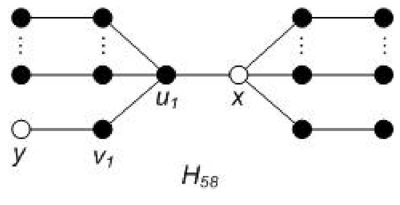


Fig. 33. $H_{58} = G_{27}$

Case 2.2. For each S and for all vertices $u, v \in S$ that dominate $\{x, y\}$ the vertex u is not paired with v in any perfect matching of S. In this case the spanning forest $G_{sf}(S, M, S_i) = G_{sf}$, for each M and $S_i = \{u, v\}$, is depicted in Fig. 35.

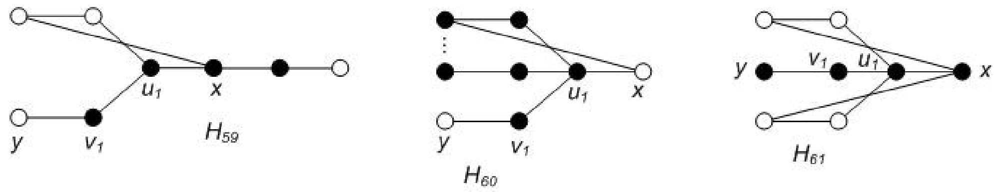


Fig. 34. $H_{58} + e$

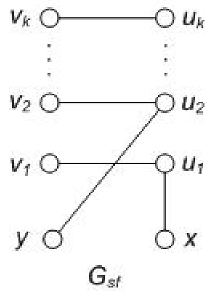


Fig. 35. The spanning forest G_{sf} of a connected graph G

Now we search for connected graphs based on G_{sf} and consider the following cases.
Case 2.2.1. $k = 2$. Then by adding one edge we obtain the three options of Fig. 36: H_{62} does not satisfy (1.1) while $H_{63} = G_5$ and $H_{64} = G_{23}$.

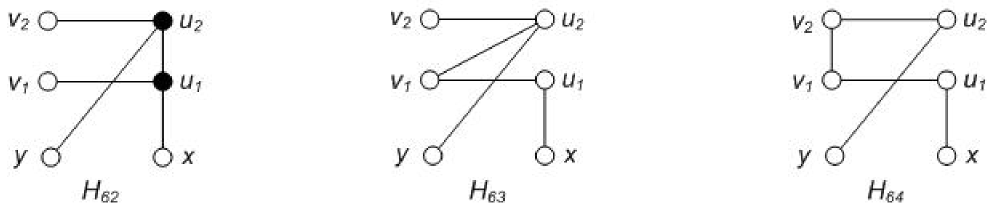


Fig. 36. The case $k = 2$

Case 2.2.2. $k = 3$. Now consider the spanning forest depicted in Fig. 37.
By joining the vertices u_1, v_1, x to u_2, v_2, y we could obtain H_i, $i = 62, 63, 64$, or their supergraphs. Hence the obtained graphs do not satisfy (1.1) or belong to Case 1 or Case 2.1. Therefore, it suffices to consider edges joining the above vertices to u_3 or v_3 (Fig. 38). Then H_i, $i = 65, \ldots, 69$, do not satisfy (1.1) but H_{70} belongs to the family G_{16}.

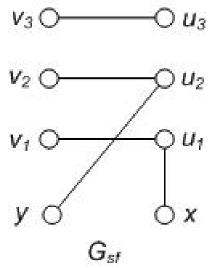


Fig. 37. The spanning forest for $k = 3$

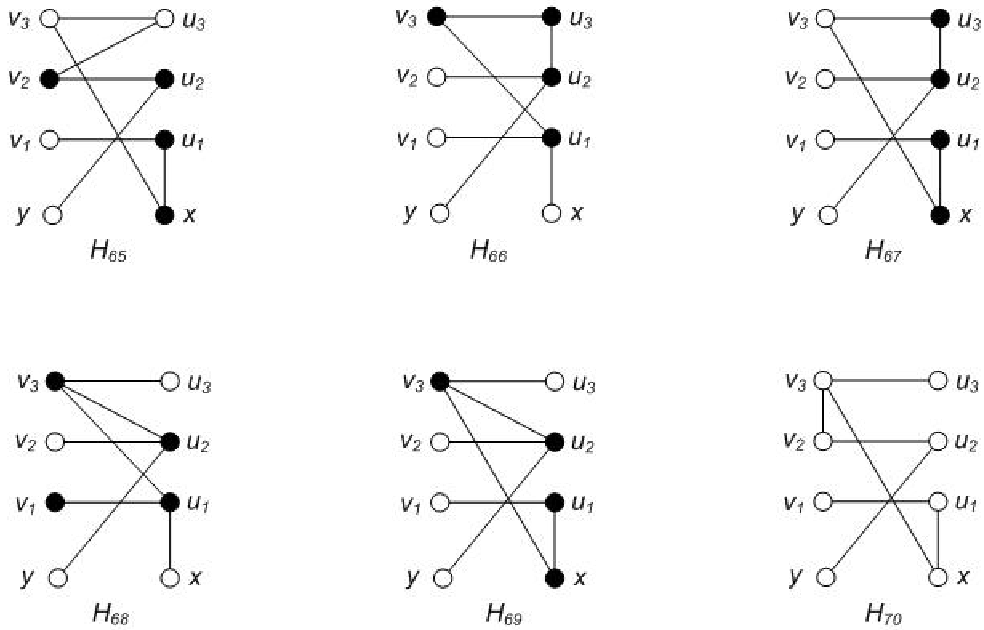


Fig. 38. The case $k = 3$

Case 2.2.3. $k > 3$. Then we obtain graphs for which (1.1) fails or graphs belonging to Case 1.

Conversely, let G be any graph of the family $\mathcal{G}$. It follows from the former investigations that (1.1) holds for G. □

We end this paper with the following statement obtained by Theorems 1.3, 1.4, 2.1 and Corollary 1.5.

Corollary 2.2. *If G is a graph of order $n \geq 4$, then $\gamma_p(G) = n - 2$ if and only if*
1) *exactly two of the components of G are isomorphic to graphs of the family $\mathcal{F}$ given in Theorem 1.4 and every other component is K_2 or*
2) *exactly one of the components of G is isomorphic to a graph of the family $\mathcal{G}$ given in Theorem 2.1 and every other component is K_2.*

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Włodzimierz Ulatowski
twoulat@mif.pg.gda.pl

Gdańsk University of Technology
Department of Technical Physics and Applied Mathematics
Narutowicza 11/12, 80–952 Gdańsk, Poland

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