ALL GRAPHS
WITH PAIRED-DOMINATION NUMBER TWO
LESS THAN THEIR ORDER

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Abstract. Let \( G = (V, E) \) be a graph with no isolated vertices. A set \( S \subseteq V \) is a paired-dominating set of \( G \) if every vertex not in \( S \) is adjacent with some vertex in \( S \) and the subgraph induced by \( S \) contains a perfect matching. The paired-domination number \( \gamma_p(G) \) of \( G \) is defined to be the minimum cardinality of a paired-dominating set of \( G \). Let \( G \) be a graph of order \( n \). In [Paired-domination in graphs, Networks 32 (1998), 199–206] Haynes and Slater described graphs \( G \) with \( \gamma_p(G) = n \) and also graphs with \( \gamma_p(G) = n - 1 \). In this paper we show all graphs for which \( \gamma_p(G) = n - 2 \).

Keywords: paired-domination, paired-domination number.

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1. INTRODUCTION

All graphs considered in this paper are finite, undirected, without loops, multiple edges and isolated vertices. Let \( G = (V, E) \) be a graph with the vertex set \( V \) and the edge set \( E \). Then we use the convention \( V = V(G) \) and \( E = E(G) \). Let \( G \) and \( G' \) be two graphs. If \( V(G) \subseteq V(G') \) and \( E(G) \subseteq E(G') \) then \( G \) is a subgraph of \( G' \) (and \( G' \) is a supergraph of \( G \)), written as \( G \subseteq G' \). The number of vertices of \( G \) is called the order of \( G \) and is denoted by \( n(G) \). When there is no confusion we use the abbreviation \( n(G) = n \). Let \( C_n \) and \( P_n \) denote the cycle and the path of order \( n \), respectively. The open neighborhood of a vertex \( v \in V \) in \( G \) is denoted \( N_G(v) = N(v) \) and defined by \( N(v) = \{ u \in V : vu \in E \} \) and the closed neighborhood \( N[v] \) of \( v \) is \( N[v] = N(v) \cup \{ v \} \). For a set \( S \) of vertices the open neighborhood \( N(S) \) is defined as the union of open neighborhoods \( N(v) \) of vertices \( v \in S \), the closed neighborhood \( N[S] \) of \( S \) is \( N[S] = N(S) \cup S \). The degree \( d_G(v) = d(v) \) of a vertex \( v \) in \( G \) is the number of edges incident to \( v \) in \( G \); by our definition of a graph, this is equal to \( |N(v)| \).
A leaf in a graph is a vertex of degree one. A subdivided star $K^*_{1,t}$ is a star $K_{1,t}$, where each edge is subdivided exactly once.

In the present paper we continue the study of paired-domination. Problems related to paired-domination in graphs appear in [1–5]. A set $M$ of independent edges in a graph $G$ is called a matching in $G$. A perfect matching $M$ in $G$ is a matching in $G$ such that every vertex of $G$ is incident to an edge of $M$. A set $S \subseteq V$ is a paired-dominating set, denoted PDS, of a graph $G$ if every vertex in $V - S$ is adjacent to a vertex in $S$ and the subgraph $G[S]$ induced by $S$ contains a perfect matching $M$. Therefore, a paired-dominating set $S$ is a dominating set $S = \{u_1, v_1, u_2, v_2, \ldots, u_k, v_k\}$ with matching $M = \{e_1, e_2, \ldots, e_k\}$, where $e_i = u_i v_i$, $i = 1, \ldots, k$. Then we say that $u_i$ and $v_i$ are paired in $S$. Observe that in every graph without isolated vertices the end-vertices of any maximal matching form a PDS. The paired-domination number of $G$, denoted $\gamma_p(G)$, is the minimum cardinality of a PDS of $G$. We will call a set $S$ a $\gamma_p(G)$-set if $S$ is a paired-dominating set of cardinality $\gamma_p(G)$. The following statement is an immediate consequence of the definition of PDS.

**Observation 1.1** ([4]). If $u$ is adjacent to a leaf of $G$, then $u$ is in every PDS.

Haynes and Slater [4] show that for a connected graph $G$ of order at least six and with minimum degree $\delta(G) \geq 2$, two-thirds of its order is the bound for $\gamma_p(G)$.

**Theorem 1.2** ([4]). If a connected graph $G$ has $n \geq 6$ and $\delta(G) \geq 2$, then

$$\gamma_p(G) \leq 2n/3.$$  

Henning in [5] characterizes the graphs that achieve equality in the bound of Theorem 1.2.

In [4] the authors give the solutions of the graph-equations $\gamma_p(G) = n$ and $\gamma_p(G) = n - 1$, where $G$ is a graph of order $n$.

**Theorem 1.3** ([4]). A graph $G$ with no isolated vertices has $\gamma_p(G) = n$ if and only if $G$ is $mK_2$.

Let $\mathcal{F}$ be the collection of graphs $C_3$, $C_5$, and the subdivided stars $K^*_{1,t}$. Now, we can formulate the following statements.

**Theorem 1.4** ([4]). For a connected graph $G$ with $n \geq 3$, $\gamma_p(G) \leq n - 1$ with equality if and only if $G \in \mathcal{F}$.

**Corollary 1.5** ([4]). If $G$ is a graph with $\gamma_p(G) = n - 1$, then $G = H \cup rK_2$ for $H \in \mathcal{F}$ and $r \geq 0$.

In the present paper we consider the graph-equation

$$\gamma_p(G) = n - 2,$$  

(1.1)

where $n \geq 4$ is the order of a graph $G$.

Our aim in this paper is to find all graphs $G$ satisfying (1.1). For this purpose we need the following definition and statements.
Definition 1.6. A subgraph $G$ of a graph $G'$ is called a special subgraph of $G'$, and $G'$ is a special supergraph of $G$, if either $V(G) = V(G')$ or the subgraph $G'[V(G') - V(G)]$ has a perfect matching.

It is clear that if $V(G) = V(G')$ then the concepts “subgraph” and “special subgraph” are equivalent. Now we can formulate the following fact.

Fact 1.7. Let $G$ be a special subgraph of $G'$.

a) If $S$ is a PDS in $G$ then $S' = S \cup (V(G') - V(G))$ is a PDS in $G'$.

b) If $\gamma_p(G) = n - r$ then $\gamma_p(G') \leq n' - r$, where $n = |V(G)|$, $n' = |V(G')|$ and $0 \leq r \leq n - 2$.

Proof. a) Assume that

$$S = \{u_1, v_1, u_2, v_2, \ldots, u_t, v_t\} \quad \text{and} \quad V(G') - V(G) = \{u_{t+1}, v_{t+1}, \ldots, u_k, v_k\},$$

where $u_i$ and $v_i$ are paired in $S$ (for $i = 1, \ldots, t$) and in $V(G') - V(G)$ (for $i = t+1, \ldots, k$). Hence $M = \{e_1, e_2, \ldots, e_k\}$, where $e_i = u_i v_i$, for $i = 1, \ldots, k$, is a perfect matching in $G'[S']$. By definition of a PDS and by $V(G) - S = V(G') - S'$ we obtain the statement of a).

b) Let $S$ be a $\gamma_p$-set in $G$, thus $|V(G) - S| = r$. It follows from a) that $S' = S \cup (V(G') - V(G))$ is a PDS in $G'$. Moreover, we have the equality

$$|S'| = n' - |V(G') - S'| = n' - |V(G) - S| = n' - r.$$

Therefore we obtain $\gamma_p(G') \leq |S'| = n' - r$. \qed

Now assume that $G$ is a connected graph of order $n \geq 4$ satisfying (1.1). Let $S = \{u_1, v_1, u_2, v_2, \ldots, u_k, v_k\}$ be a $\gamma_p(G)$-set with a perfect matching $M = \{e_1, e_2, \ldots, e_k\}$, where $e_i = u_i v_i$ for $i = 1, 2, \ldots, k$, and $V - S = \{x, y\}$. By letting $\alpha(S)$ denote the minimum cardinality of a subset of $S$ which dominates $V - S$, i.e.

$$\alpha(S) = \min\{|S' : S' \subseteq S, V - S \subseteq N(S')\}.$$

Let $S_i$ be any set of size $\alpha(S)$ such that $S_i \subseteq S$ and $V - S \subseteq N(S_i)$. For $S$, $M$ and $S_i$ we define a graph $H$ as follows:

$$V(H) = V(G) \quad \text{and} \quad E(H) = M \cup \{uv : u \in S_i, v \in \{x, y\}\}.$$ 

It is clear that $H$ is a spanning forest of $G$; we denote it as $G sf(S, M, S_i)$.

2. THE MAIN RESULT

The main purpose of this paper is to construct all graphs $G$ of order $n$ for which $\gamma_p(G) = n - 2$. At first consider the family $\mathcal{G}$ of graphs in Fig. 1. We shall show that only the graphs in family $\mathcal{G}$ are connected and satisfy condition (1.1).
Theorem 2.1. Let $G$ be a connected graph of order $n \geq 4$. Then $\gamma_p(G) = n - 2$ if and only if $G \in \mathcal{G}$.

Proof. Our aim is to construct all connected graphs $G$ for which (1.1) holds. Let $G$ be a connected graph of order $n \geq 4$ satisfying (1.1). We shall prove that $G \in \mathcal{G}$.

Let us consider the following cases.

Case 1. There exists a $\gamma_p(G)$-set $S$ such that $\alpha(S) = 1$.

Case 1.1. $k = 1$. Then we have the graphs shown in Fig. 2. It is clear that the graphs $H_i$ satisfy (1.1) and $H_i = G_i$ for $i = 1, 2, 3, 4$.

Figure 2 illustrates the graphs $H_i$, where the shaded vertices form a $\gamma_p$-set. We shall continue to use this convention in our proof.

At present for $k \geq 2$ we shall find all connected graphs $G$ satisfying (1.1) and having a $\gamma_p(G)$-set $S$ with $\alpha(S) = 1$. It is clear that in Case 1 any graph $G_{sf}(S, M, S_i)$ is independent of the choice of $S, M$ and $S_i$, so we can write $G_{sf}(S, M, S_i) = G_{sf}$. The spanning forest $G_{sf}$ consists of $k$ components $G^{(1)}, G^{(2)}, \ldots, G^{(k)}$, where $G^{(1)} = K_{1,3}$.
with \( V(K_{1,3}) = \{x,y,u_1,v_1\} \), where \( u_1 \) is the central vertex, while \( G^{(i)} = K_2 \) for \( i = 2,\ldots,k \) (see Fig. 3). Now by adding suitable edges to \( G_{sf} \) we are able to reconstruct \( G \).

\[
\begin{align*}
&v_k \quad \circ \quad u_k \\
&v_1 \quad \circ \quad u_1 \\
&y \quad \circ \quad G_{sf} \quad x
\end{align*}
\]

**Fig. 3.** The spanning forest of \( G \)

**Case 1.2.** \( k = 2 \). Now we start with the graph \( H_5 \) (Fig. 4). In our construction of the desired connected graphs we add one or more edges to \( H_5 \). Thus, let us consider the following cases regarding the number of these edges.

**Case 1.2.1.** One edge (Fig. 5). One can see that \( H_6 = G_5 \) satisfies (1.1) but \( H_7 \) does not.

\[
\begin{align*}
&v_2 \quad \circ \quad u_2 \\
&v_1 \quad \circ \quad u_1 \\
&y \quad \circ \quad H_6 \quad x
\end{align*}
\]

**Fig. 4.** The spanning forest \( H_5 \)

\[
\begin{align*}
&v_2 \quad \bullet \quad u_2 \\
&v_1 \quad \bullet \quad u_1 \\
&y \quad \circ \quad H_6 \quad x
\end{align*}
\]

\[
\begin{align*}
&v_2 \quad \circ \quad u_2 \\
&v_1 \quad \circ \quad u_1 \\
&y \quad \circ \quad H_7 \quad x
\end{align*}
\]

**Fig. 5.** The graphs obtained by adding one edge to \( H_5 \)

**Case 1.2.2.** Two edges. For \( H_7 \) we have \( \gamma_p(H_7) = 6 - 4 = |V(H_7)| - 4 \). Thus, by Fact 1.7 b) for any special supergraph \( G' \) of \( H_7 \) we obtain \( \gamma_p(G') \leq |V(G')| - 4 \). Hence, we deduce that it suffices to add one edge to \( H_6 \). Since adding the edges \( u_1u_2 \) or \( u_1v_2 \)
leads to $H_7$, we shall omit these edges in our construction. Now consider the graphs of Fig. 6.

Certainly, $\gamma_p(H_8) = n - 4$, $\gamma_p(H_i) = n - 2$ and $H_i = G_{i-3}$ for $i = 9, \ldots, 12$. Using the above argument for $H_8$ we do not take $v_1u_2$. Let us consider the following cases.

Case 1.2.3. Three edges. It follows from Fact 1.7 b) that it suffices to add one edge to $H_i$ for $i = 9, \ldots, 12$.

Case 1.2.3.1. $H_9$. Observe that $H_i = G_{i-3}, i = 13, 14, 15$, satisfy (1.1). Moreover, the graphs depicted in Fig. 7 are the unique graphs for which (1.1) holds in this case. Indeed, the edge $v_2y$ leads to a supergraph of $H_8$, and joining $u_2$ to $x$ we have $H_{15}$.

Case 1.2.3.2. $H_{10}$. Then we obtain a supergraph of $H_7$ by means of edge $v_2y$, a supergraph of $H_8$ by means of $xy$, $u_2x$, instead by adding $u_2y$ we return to $H_{15}$.

Therefore, it remains to research the graph of Fig. 8. It obvious that (1.1) holds for $H_{16} = G_{13}$. 
Case 1.2.3.3. $H_{11}$. Then it suffices to consider the graph of Fig. 9. Really, edges $v_2x$, $v_2y$ lead to a supergraph of $H_8$ and $u_2x$, $u_2y$ lead to $H_{13}$. Observe that for $H_{17} = G_{14}$ equality (1.1) is true.

Case 1.2.4. Four edges.

Case 1.2.4.1. $H_{13}$. Let $G$ be a graph obtained by adding a new edge $e$ to $H_{13}$. If $e = v_1y$ then $H_7 \subseteq G$; if $e = v_2y, v_2x$, then $H_8 \subseteq G$ and for $e = v_1x, u_2x$ we have the graph $G_{15} \in \mathcal{G}$ (Fig. 10).

Case 1.2.4.2. $H_{14}$. Keeping the above convention we note: if $e = xy$ then $H_7 \subseteq G$; if $e = v_2y, v_2x, u_2x$ then $H_8 \subseteq G$. 

Fig. 8. The graph obtained from $H_{10}$ by adding an edge

Fig. 9. $H_{11} + e$

Fig. 10. $H_{15} + e$
Case 1.2.4.3. $H_{15}$. If $e = v_2y$ then $H_7 \subseteq G$; if $e = xy, v_1y, u_2x$ then $H_8 \subseteq G$; if $e = v_1x$ then $G = G_{15}$. It is easy to see that (1.1) is true for $G_{15}$.

Case 1.2.4.4. $H_{16}$. In this case we conclude: if $e = xy$ then $H_7 \subseteq G$; if $e = v_2y, u_2x$ then $H_8 \subseteq G$; if $e = u_2y$ then $G = G_{15}$.

Case 1.2.4.5. $H_{17}$. Then we obtain the following results: if $e = v_2y, v_1y, u_2y$ then $H_7 \subseteq G$; if $e = v_2x$ then $H_8 \subseteq G$; if $e = u_2x$ then we have the graph $H_{18}$ depicted in Fig. 11. It is clear that $H_{18} = G_{15}$.

Case 1.2.5. Five edges.

Case 1.2.5.1. $G_{15}$. Then it suffices to consider the following: if $e = v_1y$ then $H_7 \subseteq G$; if $e = v_1x$ then $H_8 \subseteq G$. Therefore, Case 1.2 is complete.

For case $k \geq 3$ we only consider graphs satisfying the condition $G[S'] = G_{sf}[S'] = K_{1,3}$ for $S' = \{x, y, u_1, v_1\}$. In other words, $G$ contains the induced star $K_{1,3}$, where $V(K_{1,3}) = \{x, y, u_1, v_1\}$ and $u_1$ is the central vertex.

Case 1.3. $k = 3$. Then we start with the basic graph of Fig. 12. To obtain connected graphs we add two or more edges to $H_{19}$ and investigate whether (1.1) holds for the resulting graphs. At first we find a forbidden subgraph $H \subseteq G$ i.e. such that $\gamma_p(H) = n - 4$. We have already shown two forbidden special subgraphs $H_7, H_{20}$, and we now present the other one in Fig. 13. For a while we return to the general case $k \geq 3$. The forbidden special subgraphs $H_7$ and $H_{20}$ determine a means of construction of graphs $G$ from $G_{sf}$.

Fig. 12. The spanning forest $G_{sf} = H_{19}$

Fig. 11. $H_{17} + e$, where $e = u_2x$
Claim 1. Let $G$ be a connected graph satisfying (1.1) and obtained from $G_{sf} = H_{19}$. Then vertex $u_i$ or $v_i$, $i = 2, \ldots, k$, can be adjacent to the vertices $v_1, x, y$, only.

Now we add at least two edges to $H_{19}$. We consider the following cases.

Case 1.3.1. Two edges. Then we obtain the graphs $H_{21}$ and $H_{22}$ for which (1.1) holds (Fig. 14).

Case 1.3.2. Three edges. At present it suffices to add one edge in $H_{21}, H_{22}$. This way we obtain the graphs depicted in Figure 15. Observe that (1.1) fails for $H_{24}$ since $H_{20} \subseteq H_{24}$. Thus, $H_{23}$ satisfies (1.1) but $H_i$, $i = 24, 25, 26$, do not.

Case 1.3.3. Four edges. By adding one edge to $H_{23}$ we obtain the unique graph for which (1.1) holds (see Fig. 16). One can verify that in the remaining options we have special supergraphs of $H_7, H_8, H_{20}, H_{25}$ or $H_{26}$.

Case 1.3.4. Five edges. Each new edge in $H_{27}$ leads to a special supergraph of $H_7, H_8, H_{20}, H_{25}$ or $H_{26}$. But the following statement is obvious.

Claim 2. The graphs $H_7, H_8, H_{20}, H_{25}$ and $H_{26}$ are forbidden special subgraphs for (1.1).
All graphs with paired-domination number two less than their order

We now study a generalization of the case $k = 3$. We keep our earlier assumption regarding the induced star $K_{1,3}$ with vertex set $\{u_1, v_1, x, y\}$.

**Case 1.4.** $k \geq 3$. Then we give one property of graphs satisfying (1.1).

**Claim 3.** Let $G$ be a connected graph for which (1.1) holds and $k \geq 3$. If $G$ contains the induced star $K_{1,3}$ with $V(K_{1,3}) = \{x, y, u_1, v_1\}$ then at least one vertex of $K_{1,3}$ is a leaf in $G$.

**Proof.** Consider some cases.

**Case A.** $k = 3$. It follows from our earlier investigations that $H_{21}, H_{22}, H_{23}$ and $H_{27}$ are the unique connected graphs satisfying (1.1) in this case. Thus, we have the desired result.

**Case B.** $k \geq 4$. Claim 1 and Fact 1.7 b) imply that a special subgraph $G[S]$ induced by $S = \{x, y, u_1, v_1, u_2, v_2, u_3, v_3\}$ is connected and satisfies (1.1), i.e. it must be one of the graphs $H_{21}, H_{22}, H_{23}, H_{27}$.

**Case B.1.** $G[S] = H_{21}$. We show that $x$ is a leaf in $G$. Suppose not and let $x$ be adjacent to $v_i$, where $i \geq 4$. Then we obtain the graph $H_{28}$ in Fig. 17, for which (1.1) does not hold.
Fig. 17. \( x \) is adjacent to \( v_i \) for \( i \geq 4 \)

Case B.2. \( G[S] = H_{22} \).

Case B.2.1. Suppose that in \( G \) vertex \( v_i, i \geq 4 \), is adjacent to \( x \) and \( y \). Then for graph \( H_{29} \) depicted in Fig. 18 equality (1.1) is false since \( H_{20} \subseteq H_{29} \).

Fig. 18. \( v_i, \) for \( i \geq 4, \) is adjacent to \( x \) and \( y \)

Case B.2.2. Assume that in \( G \) vertices \( v_i \) and \( u_i, i \geq 4 \), are adjacent to \( x \) and \( y \), respectively (see Fig. 19). In this way we obtain graph \( H_{30} \) which does not satisfy (1.1) since \( H_{26} \subseteq H_{30} \).

Case B.2.3. Now, in \( G \) let vertices \( v_i \) and \( u_j, 4 \leq i < j \), be adjacent to \( x \) and \( y \), respectively (Fig. 20). As can be seen, (1.1) fails for \( H_{31} \), furthermore \( u_j \) is paired with \( y \), \( u_i \) with \( v_i \), \( u_3 \) with \( v_3 \) and \( v_4 \) with \( v_2 \). It follows from the above consideration that we omit the cases: \( G[S] = H_{23} \) and \( G[S] = H_{27} \), since \( H_{21}, H_{22} \) are subgraphs of \( H_{23}, H_{27} \). In all cases we obtain special subgraphs of \( G \) for which (1.1) fails, therefore \( G \) does not satisfy (1.1), a contradiction.
All graphs with paired-domination number two less than their order

Fig. 19. $v_i$ is adjacent to $x$ and $u_i$ to $y$, where $i \geq 4$

Fig. 20. $v_i$ is adjacent to $x$ and $u_j$ to $y$ for $4 \leq i < j$

We are now in a position to construct the desired graphs for $k \geq 3$. Let $G$ be a connected graph satisfying the following conditions:

a) (1.1) holds,

b) $k \geq 3$,

c) $G$ contains the induced $K_{1,3}$ with $V(K_{1,3}) = \{x, y, u_1, v_1\}$.

According to Claims 1–3 we can reconstruct $G$ based on $G_{sf}$. By Claim 3, at least one vertex of $K_{1,3}$, say $x$, is a leaf in $G$. Hence, by Claim 1, a vertex $u_i$ or $v_i$, $i = 2, \ldots, k$, can be adjacent to $v_1$, $y$, only. Observe that one vertex among $u_i$, $v_i$, for $i = 2, \ldots, k$ is a leaf. Indeed, if $v_jy$ and $u_iy$ ($v_1v_1$ and $u_1v_1$) are edges of $G$ then $H_8$ is a special subgraph of $G$, but if $v_jy$, $u_iv_1 \in E(G)$ then $H_{25}$ is a special subgraph of $G$ (Fig. 21). From the above investigations we obtain the desired graph in Fig. 22. One can see that (1.1) holds for $H_{32} = G_{16}$. We emphasize that the numbers of edges $yw_i$ or $v_1w_1$, $yp_j$, $v_1z_m$ can be zero here.
Note that the graphs $H_{21}, H_{22}, H_{23}$ and $H_{27}$ are particular instances of $H_{32}$. We next describe desired graphs $G$ based on $H_{32}$. We now discard the assumption concerning the induced star $K_{1,3}$ i.e. edges joining $x$, $y$, $v_1$ are allowable. At first we add the edge $yv_1$ to $H_{32}$ and obtain graph $H_{33} = G_{17}$ which satisfies (1.1) (Fig. 23).

We now consider the following exhaustive cases (Fig. 24). It easy to see that (1.1) is true for $H_{34} = G_{18}$ and $H_{35} = G_{19}$ but is false for $H_i$, $i = 36, \ldots, 39$.

Case 2. Each $\gamma_p(G)$-set $S$ satisfies $\alpha(S) = 2$.

Case 2.1. There exists a set $S$ containing vertices $u, v$ that dominate $\{x, y\}$ such that $u$ is paired with $v$ in some perfect matching $M$ of $S$. Without loss of generality we may assume that $u = u_1$, $v = v_1$.

Case 2.1.1. $k = 1$. Then the unique graphs $H_{30} = G_{20}$ and $H_{41} = G_{21}$ satisfying (1.1) are depicted in Fig. 25.

Now for a connected graph $G$ with $k \geq 2$ the spanning forest $G_{sf}(S, M, S_i) = G_{sf}$ for $S_i = \{u, v\}$ is the sum of components $G^{(1)}, G^{(2)}, \ldots, G^{(k)}$, where $G^{(1)} = P_4$ and $G^{(i)} = K_2$ for $i = 2, \ldots, k$ (Fig. 26).

Case 2.1.2. $k = 2$. Now we start with the spanning forest of Fig. 27. In our construction of the desired connected graphs we add at least one edge to the graph $H_{42}$. Therefore, consider the following cases.
All graphs with paired-domination number two less than their order

Fig. 23. The family $G_{17}$

Fig. 24. The exhaustive cases

Fig. 25. The case for $k = 1$
Case 2.1.2.1. One edge (Fig. 28). Then we have $H_{43} = G_{22}$ and $H_{44} = G_{23}$ satisfy (1.1).

Case 2.1.2.2. Two edges. Now by adding one edge to $H_{43}$ and $H_{44}$ we obtain some graphs by exhaustion (Fig. 29). Observe (1.1) fails for $H_{45}$, $H_{46}$ and holds for $H_{47} = G_{24}$, $H_{48} = G_{25}$ and $H_{49} = G_{26}$. Moreover graphs $H_i$ for $i = 50, 51, 52$ are discussed in Case 1.

Case 2.1.2.3. Three edges. Then it suffices to add one edge to $H_i$, $i = 47, 48, 49$. One resulting graph is the graph $H_{53}$ depicted in Fig. 30, which does not satisfy (1.1). One can verify that the remaining graphs in this case are supergraphs of $H_{45}$, $H_{46}$ or are graphs discussed in Case 1.
Case 2.1.3. \( k \geq 3 \). At first we show some graphs for which (1.1) does not hold (Fig. 31).

For \( H_i, i = 54, \ldots, 57 \), (1.1) is false; in \( H_{54} \) the vertex \( u_1 \) is paired with \( u_2 \) and \( v_1 \) with \( u_3 \).

Now we start with the spanning forest depicted in Fig. 32.

Taking account of the forbidden special subgraphs \( H_i, i = 54, \ldots, 57 \), we can reconstruct \( G \) based on \( G_{sf} \). By the connectedness of \( G \) it is necessary to join vertices of both the edges \( u_i v_i \), \( u_j v_j \) with at least one vertex among \( u_1, v_1, x, y \). Thus we consider the following cases (without loss of generality we take the vertices \( u_i \) and \( u_j \) of the above edges). If \( u_i u_1 \in E(G) \) then we have two options: \( u_j v_1 \in E(G) \) or \( u_j x \in E(G) \). Instead, if \( u_i x \in E(G) \) then we have the following options: \( u_j x \in E(G) \) or \( u_j u_1 \in E(G) \). Replace \( u_1 \) by \( v_1 \) and \( x \) by \( y \) we obtain analogous results. This way we construct the desired graph \( G = H_{58} \) for which (1.1) holds (Fig. 33). Note that \( H_{58} = G_{27} \). We end this case with adding new edges in \( H_{58} \). At first, if \( u_i z \in E(G) \) and \( v_i z \in E(G) \), where \( 2 \leq i \leq k \), \( z = u_1, v_1, x, y \), then we return to Case 1. Therefore,
let us consider all possible cases, which are depicted in Fig. 34. Then we obtain that (1.1) is true for $H_{60} = G_{28}$ but is false for $H_{59}$ and $H_{61}$.

**Fig. 31.** The forbidden graphs

![Forbidden Graphs](image)

**Fig. 32.** The spanning forest for $k \geq 3$, where $2 \leq i < j \leq k$

![Spanning Forest](image)

**Fig. 33.** $H_{58} = G_{27}$

Case 2.2. For each $S$ and for all vertices $u, v \in S$ that dominate $\{x, y\}$ the vertex $u$ is not paired with $v$ in any perfect matching of $S$. In this case the spanning forest $G_{sf}(S, M, S_i) = G_{sf}$, for each $M$ and $S_i = \{u, v\}$, is depicted in Fig. 35.
All graphs with paired-domination number two less than their order

Fig. 34. $H_{58} + e$

Fig. 35. The spanning forest $G_{sf}$ of a connected graph $G$

Now we search for connected graphs based on $G_{sf}$ and consider the following cases.

Case 2.2.1. $k = 2$. Then by adding one edge we obtain the three options of Fig. 36: $H_{62}$ does not satisfy (1.1) while $H_{63} = G_5$ and $H_{64} = G_{23}$.

Fig. 36. The case $k = 2$

Case 2.2.2. $k = 3$. Now consider the spanning forest depicted in Fig. 37. By joining the vertices $u_1, v_1, x$ to $u_2, v_2, y$ we could obtain $H_i, i = 62, 63, 64$, or their supergraphs. Hence the obtained graphs do not satisfy (1.1) or belong to Case 1 or Case 2.1. Therefore, it suffices to consider edges joining the above vertices to $u_3$ or $v_3$ (Fig. 38). Then $H_i, i = 65, \ldots, 69$, do not satisfy (1.1) but $H_{70}$ belongs to the family $G_{16}$. 
Case 2.2.3. $k > 3$. Then we obtain graphs for which (1.1) fails or graphs belonging to Case 1.

Conversely, let $G$ be any graph of the family $\mathcal{G}$. It follows from the former investigations that (1.1) holds for $G$. \qed
We end this paper with the following statement obtained by Theorems 1.3, 1.4, 2.1 and Corollary 1.5.

**Corollary 2.2.** If $G$ is a graph of order $n \geq 4$, then $\gamma_p(G) = n - 2$ if and only if

1) exactly two of the components of $G$ are isomorphic to graphs of the family $F$ given in Theorem 1.4 and every other component is $K_2$ or

2) exactly one of the components of $G$ is isomorphic to a graph of the family $G$ given in Theorem 2.1 and every other component is $K_2$.

**REFERENCES**


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