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ON GELFAND PAIRS ASSOCIATED TO TRANSITIVE GROUPOIDS

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Abstract. Let G be a topological locally compact, Hausdorff and second countable groupoid with a Haar system and K a compact subgroupoid of G with a Haar system too. (G, K) is a Gelfand pair if the algebra of bi-K-invariant functions is commutative under convolution. In this paper, we give a characterization of Gelfand pairs associated to transitive groupoids which generalize a well-known result in the groups case. Using this result, we prove that the study of Gelfand pairs associated to transitive groupoids is equivalent to that of Gelfand pairs associated to its isotropy groups.

Keywords: transitive groupoids, groupoid representation, Gelfand pairs.

Mathematics Subject Classification: 22A22, 46JXX.

1. INTRODUCTION

Let G be a locally compact group and K a compact subgroup of G. The convolution algebra $L^1(G\backslash\backslash K)$ of bi-K-invariant functions plays a central role in the Harmonic Analysis of pairs (G, K). In particular, if $L^1(G\backslash\backslash K)$ is commutative then (G, K) is a Gelfand pair. The notion of Gelfand pair has been sufficiently studied in [2, 3, 5, 6]. A well-known result of Gelfand in [6], asserts that (G, K) is a Gelfand pair if and only if for any irreducible unitary continuous representation π of G, the space of K-fixed vectors is at most one dimensional. Since groupoids are generalizations of groups, we would like to extend the notion of Gelfand pairs on groups to groupoids. For a groupoid G, $L^1(G)$ is abelian if and only if G is an abelian group bundle (that is a group bundle with abelian isotropy groups). Groups bundles are totally intransitive groupoids. Our purpose in this paper is to study the commutativity of $L^1(G\backslash\backslash K)$ when G is a transitive locally compact groupoid. The representation theory of groupoids is more involved. Hilbert spaces are replaced by Hilbert bundles on the unit space of the groupoid. The integrated representation on the groupoid algebra has the Hilbert space of square integrable sections as the representation space. The outline of this paper is as follows. The next section is devoted to preliminaries. In section 3, we give the results of this paper. The first result, for a suitably chosen quasi-invariant measure on the unit space of the groupoid, extends to transitive groupoids a well-known necessary condition for Gelfand pairs on groups. We extend, in the second result, to transitive locally compact groupoids the Gelfand theorem mentioned above. The space of K-fixed vectors is replaced by the space of K-fixed square integrable sections. Finally, thanks to this result, we shall prove the main result of this paper, which asserts that (G, K) is a Gelfand pair if and only if for any $m \in G^{(0)}, L^1(G_m^m \backslash\backslash K_m^m)$ is commutative. In last section, we give some examples to which our result applies.

2. PRELIMINARIES

In this section we give some notations and definitions to help with understanding this paper.

We shall use the definition of a locally compact groupoid and the definition of a Haar system on a groupoid given by J. Renault in [13]. Let G be a locally compact, Hausdorff, second countable groupoid. $G^{(0)}$ denotes the unit space of G and $G^{(2)}$ the set of composable pairs. For $x \in G$, $r(x) = xx^{-1}$ and $d(x) = x^{-1}x$ are respectively the range and the domain of x. For $u, v \in G^{(0)}$, let us put

$$G^u = r^{-1}(u), \quad G_v = d^{-1}(v), \quad G_v^u = G^u \cap G_v$$

and for each unit element u,

$$G_u^u = \{x \in G : r(x) = d(x) = u\}$$

is the isotropy group at u. The group bundle

$$G' = \{x \in G : r(x) = d(x)\}$$

is called the isotropy group bundle of G. There exists an equivalent relation on $G^{(0)}$ defined as follows: $u, v \in G^{(0)}$, $u \sim v$ iff $G_v^u \neq \emptyset$. The equivalence class of u is denoted by $[u]$ and is called the orbit of u. As a subset of $G^{(0)} \times G^{(0)}$, the graph $R = \{(r(x), d(x)) : x \in G\}$ of this equivalent relation is a groupoid on $G^{(0)}$. The anchor map (r, d) is a continuous homomorphism of G into $G^{(0)} \times G^{(0)}$ with image R. A groupoid is transitive if (r, d) is onto i.e. the image of (r, d) is equal to $G^{(0)} \times G^{(0)}$. Otherwise, the groupoid is transitive if it has a single orbit. Let $\{\lambda^u : u \in G^{(0)}\}$ be a left Haar system on G. For $u \in G^{(0)}$, λ_u will denote the image of λ^u by the inverse map and $\{\lambda_u : u \in G^{(0)}\}$ is a right Haar system on G. Let μ be a quasi-invariant measure on $G^{(0)}$ for the Haar system $\{\lambda^u : u \in G^{(0)}\}$, $\nu = \int \lambda^u d\mu(u)$ be the induced measure by μ on $G, \nu^{-1} = \int \lambda_u d\mu(u)$ be the inverse of ν, $\nu^2 = \int \lambda^u \times \lambda_u d\mu(u)$ be the induced measure by μ on $G^{(2)}$ and Δ the modular function of μ. In [7], it was proved that Δ is a homomorphism ν^2 a.e from G to $\mathbb{R}_+^*$, the group of multiplicative positive real numbers. There is a decomposition of the left Haar system λ for G

other the equivalence relation $\sim$. Firstly, there is a measure β_v^u concentrated on G_v^u for all $(u, v) \in R$ such that:

— β_u^u is a left Haar measure on G_u^u,
— β_v^u is a translate of β_v^v, i.e. $\beta_v^u = x\beta_v^v$ if $x \in G_v^u$.

Notice that β_v^u is independent of the choice of $x \in G_v^u$. Then, there is a unique Borel Haar system $\alpha = \{\alpha^u : u \in G^{(0)}\}$ for R with the property that for every $u \in G^{(0)}$, we have $\lambda^u = \int \beta_v^\omega d\alpha^u(\omega, v)$. Renault [12] proves that there exists a continuous homomorphism δ of G to $\mathbb{R}_+$ such that for all quasi-invariant measures μ on $G^{(0)}$, the modular functions Δ of G, defined by μ and λ, and $\widetilde{\Delta}$ of R, defined by μ and α, satisfy $\Delta = \delta\widetilde{\Delta} \circ (r, d)$. We also notice that for all $u \in G^{(0)}$, $\delta \mid G_u^u$ is the modular function of G_u^u relative to the left Haar measure β_u^u. It is proved in [11, 12] that there is a transitive quasi-invariant measure $\widetilde{\mu}$ (i.e. a quasi-invariant measure concentrated on an orbit) such that $\widetilde{\Delta} = 1$ and so $\Delta = \delta$. In particular, for a transitive groupoid there is a unique quasi-invariant measure on $G^{(0)}$ with full support such that the modular function Δ is a continuous homomorphism of G. $\mathcal{C}_c(G)$ will denote the space of complex-valued continuous functions on G with compact support, endowed with the inductive limit topology. In [13], Renault defines the following norm on $\mathcal{C}_c(G)$:

$$\|f\|_I = \max\{\|f\|_{I,r}, \|f\|_{I,d}\},$$

where

$$\|f\|_{I,r} = \sup\left\{ \int_{G^u} f(x)d\lambda^u(x) : u \in G^{(0)}\right\}$$

and

$$\|f\|_{I,d} = \sup\left\{ \int_{G_u} f(x)d\lambda_u(x) : u \in G^{(0)}\right\}.$$

In [9], the author defines the space $L^1(G, \nu) = L^1(G, \lambda, \mu)$ of integrable functions on G with respect to a fixed Haar system λ and a quasi-invariant probability measure μ by:

$$L^1(G, \lambda, \mu) = \left\{ f : G \to \mathbb{C} \mid f \text{ is } \lambda\text{-measurable}, \|f\|_1 = \int_G \mid f(x) \mid d\nu(x) < \infty \right\}.$$

Always, it is proved that $L^1(G, \lambda, \mu)$ is a Banach $*$-algebra under the following convolution product:

$$(f * g)(x) = \int_{G^{r(x)}} f(y)g(y^{-1}x)d\nu(y), \quad f, g \in L^1(G, \lambda, \mu),$$

The involution is defined as follows: for $f \in L^1(G, \lambda, \mu)$,

$$f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})} = \Delta(x^{-1})\overline{\check{f}(x)}.$$

Let K be a compact subgroupoid of G with unit space $G^{(0)}$. We assume that K is equipped with a normalized Haar system $\gamma = \{\gamma^u : u \in G^{(0)}\}$ that means $\gamma_u(K_u) = \gamma^u(K^u) = 1$ for each $u \in G^{(0)}$. As it is explained above, $\{\gamma^u : u \in G^{(0)}\}$ has a decomposition $\{(\gamma^u_v)_{(u,v)\in R_K} : (\rho^u)_{u\in G^{(0)}}\}$, where R_K is the graph of the equivalence relation on $G^{(0)}$ seen as a unit space of K, such that $\gamma^u = \int \gamma^\omega_v d\rho^u(\omega, v)$. We put

$$L^1(G\backslash\backslash K) = \{f \in L^1(G, \lambda, \mu) : f(kxk') = f(x) \text{ for all } x\in G,\ k\in K_{r(x)},\ k'\in K^{d(x)}\};$$

the space of bi-K-invariant integrable functions which is a Banach $*$-subalgebra of $L^1(G, \lambda, \mu)$. For any $f \in L^1(G, \lambda, \mu)$, let us denote by $f^\natural$ the bi-K-invariant function defined by

$$f^\natural(x) \quad = \quad \iint f(kxk')d\gamma_{r(x)}(k)d\gamma^{d(x)}(k'),$$

for all $x \in G$. Let $\mathcal{H} = (H_u)_{u\in G^{(0)}}$ be a Hilbert bundle over $G^{(0)}$ and $\mathcal{U}(\mathcal{H})$ the unitary groupoid of the bundle $\mathcal{H}$. $(\pi, \mathcal{H})$ is a unitary continuous representation of G if π is a groupoid morphism of G into $\mathcal{U}(\mathcal{H})$ such that for all square integrable sections ξ and η of $\mathcal{H}$, the map $x \mapsto \langle\pi(x)\xi(d(x)), \eta(r(x))\rangle$ is continuous. A closed nonzero subbundle $\mathcal{M}$ of $\mathcal{H}$ (i.e. M_u is a nonzero closed subspace of H_u for each $u \in G^{(0)}$) is invariant under π if $\pi(x)M_{d(x)} \subset M_{r(x)}$, for each $x \in G$. If π admits a non trivial closed invariant subbundle $\mathcal{M}$, it is called reducible. Otherwise it is called irreducible. If ξ is a section of $\mathcal{H}$, the subbundle $\mathcal{M}_\xi$ whose leaf at $u \in G^{(0)}$ is the closed linear span of the set $\{\pi(x)\xi(d(x)) : x \in G^u\}$ is called the cyclic subbundle generated by ξ. We say that ξ is cyclic if $(\mathcal{M}_\xi)_u$ is dense in H_u, for each $u \in G^{(0)}$. We denote by $\Gamma_\mu(\mathcal{H})$ the Hilbert space of a square integrable section of $\mathcal{H}$. In [13], J. Renault associates to any unitary representation $(\pi, \mathcal{H})$ a representation L of $\mathcal{C}_c(G)$ on $\Gamma_\mu(\mathcal{H})$ defined by

$$(L(f)\xi, \eta) = \int f(x)\langle\pi(x)\xi(d(x)), \eta(r(x))\rangle d\nu_0(x),$$

for all $f \in \mathcal{C}_c(G)$, ξ,$\eta \in \Gamma_\mu(\mathcal{H})$, where $\nu_0 = \Delta^{\frac{-1}{2}}\nu$. L is a bounded non degenerate $*$-representation of $\mathcal{C}_c(G)$ where $\mathcal{C}_c(G)$ is equipped with the norm $\|\cdot\|_I$. We may also define L for μ a.a. $u \in G^{(0)}$ by

$$L(f)\xi(u) = \int_{G^u} f(x)\pi(x)\xi(d(x))\Delta^{\frac{-1}{2}}(x)d\lambda^u(x).$$

Let us put

$$H_1 = \{\xi \in \Gamma_\mu(\mathcal{H})\colon \text{the map } f \mapsto (L(f)\xi, \xi) \text{ is continuous in } L^1\text{-norm}\}.$$

H_1 is a nontrivial invariant subspace of $\Gamma_\mu(\mathcal{H})$ (see [9]). Let us set $\tilde{L} = L \mid H_1$. In [9], Massoud *et al.* show that $\tilde{L}$ extends to a continuous representation of $L^1(G, \lambda, \mu)$ on H_1, still denoted $\tilde{L}$. They also show that if π is irreducible then the representation $\tilde{L}$ of $L^1(G, \lambda, \mu)$ on H_1 is irreducible.

3. GELFAND PAIRS

Throughout this section G is a Hausdorff, second countable, transitive locally compact groupoid with a left Haar system $\lambda = \{\lambda^u \colon u \in G^{(0)}\}$. K is a compact subgroupoid of G containing $G^{(0)}$. It follows that $G^{(0)}$ is the unit space of K. Let μ be the quasi-invariant measure on $G^{(0)}$ such that $supp(\mu) = G^{(0)}$ and the modular function Δ associated to (λ, μ) is a continuous homomorphism. Since K contains $G^{(0)}$, it follows that $G^{(0)}$ is compact. So we assume that μ is normalized. For each $u \in G^{(0)}$, the measure α^u is concentrated on $u \times [u]$, and $\alpha^u = \varepsilon_u \times \mu$, where ε_u is the unit point mass at u. So $\lambda^u = \int \beta^u_\omega d\mu(\omega)$. K is equipped with a normalized Haar system $\gamma = \{\gamma^u \colon u \in G^{(0)}\}$.

Theorem 3.1. *If (G, K) is a Gelfand pair then μ is invariant, that is $\Delta = 1$.*

Proof. For any $f \in \mathcal{C}_c(G)$, we have $\nu(f^\natural) = \nu(f)$. In fact,

$$
\begin{aligned}
\nu(f^\natural) &= \int f^\natural(x) d\nu(x) = \iiiint f(kxk') d\gamma_{r(x)}(k) d\gamma^{d(x)}(k') d\lambda^u(x) d\mu(u) = \\
&= \iiiint f(xk') d\gamma_u(k) d\gamma^{d(x)}(k') d\lambda^{r(k)}(x) d\mu(u) = \\
&= \iiiint f(xk') d\gamma^u(k) d\gamma^{d(x)}(k') d\lambda^{r(k)}(x) d\mu(u) = \\
&= \iiint f(xk') d\gamma^{d(x)}(k') d\lambda^u(x) d\mu(u) = \\
&= \iiint f(xk') \Delta(x) d\gamma^u(k') d\lambda_{r(k')}(x) d\mu(u) = \\
&= \iiint f(x) \Delta(x) d\gamma_u(k') d\lambda_{d(k')}(x) d\mu(u) = \\
&= \iint f(x) d\lambda^u(x) d\mu(u) = \nu(f).
\end{aligned}
$$

Let $f, g \in \mathcal{C}_c(G \backslash\backslash K)$ such that $g(x) = 1$ for each $x \in supp(f) \cup supp(f)^{-1}$.

Since $\mathcal{C}_c(G \backslash\backslash K)$ is abelian, we obtain

$$
\begin{aligned}
\int f(x) d\nu(x) &= \int f(x) g(x^{-1}) d\nu(x) = \int_{G^{(0)}} \int_{G^u} f(x) g(x^{-1}u) d\nu(x) d\mu(u) = \\
&= \int_{G^{(0)}} (g * f)(u) d\mu(u) = \int_{G^{(0)}} \int_{G^u} g(x) f(x^{-1}) d\nu(x) = \int f(x^{-1}) d\nu(x).
\end{aligned}
$$

Thus for all $h \in \mathcal{C}_c(G)$, we have $\int h^\natural(x) d\nu(x) = \int h^\natural(x^{-1}) d\nu(x)$ and using the first equality we have $\int h(x) d\nu(x) = \int h(x^{-1}) d\nu(x)$, i.e. $\nu = \nu^{-1}$. So the modular function of (λ, μ) is identically equal to one. □

Let $(\pi, \mathcal{H})$ be a unitary continuous representation of G. Since G is transitive, then the Hilbert bundle $\mathcal{H}$ is trivial i.e. there exists a Hilbert space H such that

$\mathcal{H} = G^{(0)} \times H$. Let $m \in G^{(0)}$. Since G is transitive, then the map $d : G^m \to G^{(0)}$ is a continuous surjection and there exists a section $\sigma : G^{(0)} \to G^m$ such that $d \circ \sigma = id_{G^{(0)}}$ and $\sigma(d(K^m)) \subset K^m$. Let us consider the map:

$$\theta : G \to G_m^m, \quad x \mapsto \theta(x) = \sigma(r(x))x\sigma(d(x))^{-1}.$$

θ is a morphism of groupoids which is surjective. In [1], the author establishes a connection between the unitary representation of G and unitary representation of G_m^m. In fact, if π is a unitary representation of G_m^m on H, then $\tilde{\pi}$ defined by

$$\tilde{\pi}(x)(d(x), h) = (r(x), \pi(\theta(x))h)$$

is a unitary representation of G on $G^{(0)} \times H$. Conversely if $\tilde{\pi}$ is a unitary representation of G on $G^{(0)} \times H$, $\pi = \tilde{\pi} \mid G_m^m$ is a unitary representation of G_m^m. In the next result, we prove that if one is irreducible then the other one is irreducible.

Lemma 3.2. *If π is an irreducible unitary representation of G_m^m on H, then $\tilde{\pi}$ is irreducible. Also if $\tilde{\pi}$ is an irreducible unitary representation of G then $\tilde{\pi} \mid G_m^m$ is irreducible.*

Proof. First suppose π is an irreducible unitary representation of G_m^m on H. Let us show that $\tilde{\pi}$ is irreducible. Let $G^{(0)} \times M$ be a closed subbundle of $G^{(0)} \times H$ invariant under $\tilde{\pi}$. We have

$$\tilde{\pi}(x)(d(x), M) = (r(x), \pi(\theta(x))M).$$

Hence $\pi(\theta(x))M \subset M$ for all $x \in G$. Since θ is surjective, then $\pi(y)M \subset M$ for all $y \in G_m^m$. So $M = \{0\}$ or $M = H$, since π is irreducible. Consequently, $G^{(0)} \times M$, is trivial and $\tilde{\pi}$ is irreducible.

Now, suppose $\tilde{\pi}$ is an irreducible unitary representation of G on $G^{(0)} \times H$ and let us show that $\pi = \tilde{\pi} \mid G_m^m$ is irreducible. Let M be a nonzero closed subspace of H invariant under π. For $v \in G^{(0)}$, let us consider N_v the closed linear span of the set $\{\tilde{\pi}(x)(d(x), h) : x \in G_m^v, h \in M\}$. In particular $N_m = M$. $\{N_v\}_{v \in G^{(0)}}$ is a closed subbundle of $G^{(0)} \times H$. For all $y \in G$, $x \in G_m^{d(y)}$, $h \in M$, we have

$$\tilde{\pi}(y)(\tilde{\pi}(x)(d(x), h)) = \tilde{\pi}(yx)(d(x), h).$$

Since $yx \in G_m^{r(y)}$, then $\tilde{\pi}(y)(\tilde{\pi}(x)(d(x), h)) \in N_{r(y)}$. Hence $\tilde{\pi}(y)N_{d(y)} \subset N_{r(y)}$. So $\{N_v\}_{v \in G^{(0)}}$ is invariant under $\tilde{\pi}$. But $\tilde{\pi}$ is irreducible, so it comes that $N_v = \{0\}$ or $N_v = H$ for any $v \in G^{(0)}$. In particular $N_m = M = H$. So π is irreducible. $\square$

Let us notice that any section ξ of $\mathcal{H}$ is bounded since $G^{(0)}$ is compact. It follows that any section ξ of $\mathcal{H}$ is square integrable and it also belongs to H_1 defined above. We set

$$\Gamma_\mu^K(\mathcal{H}) = \{\xi \in \Gamma_\mu(\mathcal{H}) : \pi(k)\xi(d(k)) = \xi(r(k)) \text{ for each } k \in K\},$$

the space of K-invariant square integrable section of $\mathcal{H}$. If $\xi \in \Gamma_\mu(\mathcal{H})$, then the section $P_K\xi$ defined by

$$P_K\xi(u) = \int \pi(k)\xi(d(k))d\gamma^u(k)$$

is K-invariant and square integrable. Also $\Gamma_\mu^K(\mathcal{H})$ is a closed subspace of $\Gamma_\mu(\mathcal{H})$. We have the following result.

Theorem 3.3. *(G, K) is a Gelfand pair if and only if for any irreducible unitary representation $(\pi, \mathcal{H})$, the dimension of $\Gamma_\mu^K(\mathcal{H})$ is less than one.*

Proof. Suppose (G, K) is a Gelfand pair and let $\tilde{L}$ be the extension of π to $L^1(G, \lambda, \mu)$ restricted to $H_1 = \Gamma_\mu(\mathcal{H})$. For $f \in L^1(G\backslash\backslash K)$, $\xi \in \Gamma_\mu^K(\mathcal{H})$ and $k \in K$,

$$\begin{aligned}\pi(k)((\tilde{L}(f)\xi)(d(k))) &= \int\limits_{G^{d(k)}} f(x)\pi(kx)\xi(d(x))d\lambda^{d(k)}(x) = \\ &= \int\limits_{G^{r(k)}} f(x)\pi(x)\xi(d(x))d\lambda^{r(k)}(x) = \tilde{L}(f)\xi(r(k)).\end{aligned}$$

Thus $\tilde{L}(L^1(G\backslash\backslash K))\Gamma_\mu^K(\mathcal{H}) \subset \Gamma_\mu^K(\mathcal{H})$. So let us consider the representation $\tilde{L}^\natural$ of $L^1(G\backslash\backslash K)$ on $\Gamma_\mu^K(\mathcal{H})$. We suppose that $\Gamma_\mu^K(\mathcal{H}) \neq \{0\}$. Let U be a closed subspace of $\Gamma_\mu^K(\mathcal{H})$ invariant under $\tilde{L}^\natural$. Suppose $U \neq \{0\}$ and let us put $W = U^\perp$. For $\xi \in U$, set

$$V^\xi = \{\tilde{L}(f)\xi : f \in L^1(G, \lambda, \mu)\}.$$

V^ξ is invariant under $\tilde{L}$ and is dense in $\Gamma_\mu(\mathcal{H})$. For all $\eta \in W$ and $f \in L^1(G, \lambda, \mu)$, we have

$$\begin{aligned}0 &= \langle \tilde{L}^\natural(f^\natural)\xi, \eta\rangle = \\ &= \int f(kxk')\langle \pi(x)\xi(d(x)), \eta(r(x))\rangle d\gamma_{r(x)}(k)d\gamma^{d(x)}(k')d\lambda^u(x)d\mu(u) = \\ &= \int f(xk')\langle \pi(k^{-1}x)\xi(d(x)), \eta(d(k))\rangle d\gamma_u(k)d\gamma^{d(x)}(k')d\lambda^{r(k)}(x)d\mu(u) = \\ &= \int f(xk')\langle \pi(x)\xi(d(x)), \eta(r(k))\rangle d\gamma_u(k)d\gamma^{d(x)}(k')d\lambda^{r(k)}(x)d\mu(u) = \\ &= \int f(xk')\langle \pi(x)\xi(d(x)), \eta(r(x))\rangle d\gamma^u(k)d\gamma^{d(x)}(k')d\lambda^{r(k)}(x)d\mu(u) = \\ &= \int f(xk')\langle \pi(x)\xi(d(x)), \eta(r(x))\rangle d\gamma^{d(x)}(k')d\lambda^u(x)d\mu(u) = \\ &= \int f(xk')\langle \pi(x)\xi(d(x)), \eta(r(x))\rangle d\gamma^u(k')d\lambda_{r(k')}(x)d\mu(u) = \\ &= \int f(x)\langle \pi(x)\xi(d(k')), \eta(r(x))\rangle d\gamma^u(k')d\lambda_{d(k')}(x)d\mu(u) = \\ &= \int f(x)\langle \pi(x)\xi(d(x)), \eta(r(x))\rangle d\gamma_u(k')d\lambda_{d(k')}(x)d\mu(u) = \\ &= \int f(x)\langle \pi(x)\xi(d(x)), \eta(r(x))\rangle d\nu(x) = \langle \tilde{L}(f)\xi, \eta\rangle.\end{aligned}$$

Thus $\eta \in (V^\xi)^\perp = \{0\}$ and we conclude that $W \subset \{0\}$ and $U = \Gamma_\mu^K(\mathcal{H})$. So $\tilde{L}^\natural$ is irreducible. Since $L^1(G\backslash\backslash K)$ is a commutative Banach $*$-algebra, then $\dim \Gamma_\mu^K(\mathcal{H}) = 1$.

For the converse, let us assume that $\dim \Gamma_\mu^K(\mathcal{H}) \leq 1$ for any irreducible unitary representation of G on a Hilbert bundle $\mathcal{H}$. Let us show first that $\mathcal{C}_c(G\backslash\backslash K)$ possesses sufficiently many one-dimensional $*$-representations. Take $f \in \mathcal{C}_c(G\backslash\backslash K)$ such that $f \neq 0$. Let us put $g = f \mid G_m^m$ for some $m \in G^{(0)}$. Then $g \in \mathcal{C}_c(G_m^m\backslash\backslash K_m^m)$ and $g \neq 0$. By well-known results due to Gelfand and Raikov [4], we can find a continuous elementary positive-definite function φ_m on G_m^m satisfying $\int g(x)\varphi_m(x)d\beta_m^m(x) \neq 0$. Denote the irreducible unitary representation of G_m^m on H associated with φ_m by π_m. Then $\varphi_m(x) = \langle \pi_m(x)h, h\rangle$, where h is a cyclic vector. Let π be the unitary representation of G on $G^{(0)} \times H = \mathcal{H}$ associated to π_m. It is irreducible according to Lemma 3.2. Put $\xi(u) = (u, h)$ for any $u \in G^{(0)}$. ξ is a section of $G^{(0)} \times H$. Since h is cyclic, then ξ is a cyclic square integrable section. We have $\langle \tilde{L}(f)\xi, \xi\rangle \neq 0$, because

$$\begin{aligned}\langle \tilde{L}(f)\xi, \xi\rangle &= \int f(x)\langle \pi(x)\xi(d(x)), \xi(r(x))\rangle d\nu(x) = \int f(x)\langle \pi_m(\theta(x))h, h\rangle d\lambda^u(x)d\mu(u) = \\ &= \int f(\sigma(u)^{-1}x\sigma(v))\langle \pi_m(x)h, h\rangle d\beta_m^m(x)d\mu(u)d\mu(v) = \\ &= \int g(x)\langle \pi_m(x)h, h\rangle d\beta_m^m(x) = \int g(x)\varphi_m(x)d\beta_m^m(x).\end{aligned}$$

We also obtain

$$\begin{aligned}\langle \tilde{L}(f)\xi, \xi\rangle &= \int f(x)\langle \pi(x)\xi(d(x)), \xi(r(x))\rangle d\nu(x) = \\ &= \int\int f(kx)\langle \pi(x)\xi(d(x)), \xi(r(x))\rangle d\gamma_{r(x)}(k)d\lambda^u(x)d\mu(u) = \\ &= \int\int f(x)\langle \pi(k^{-1}x)\xi(d(x)), \xi(d(k))\rangle d\gamma_u(k)d\lambda^{r(k)}(x)d\mu(u) = \langle \tilde{L}(f)\xi, P_K\xi\rangle.\end{aligned}$$

Therefore $P_K\xi \neq 0$ and hence $\Gamma_\mu^K(\mathcal{H}) \neq \{0\}$. So, by assumption, $\dim \Gamma_\mu^K(\mathcal{H}) = 1$. Thus $\tilde{L}^\natural$ is a one-dimensional $*$-representation of $\mathcal{C}_c(G\backslash\backslash K)$ on $\Gamma_\mu^K(\mathcal{H})$ and $\tilde{L}^\natural(f) \neq 0$. To complete the proof, let ρ be a one-dimensional $*$-representation of $\mathcal{C}_c(G\backslash\backslash K)$ and $f, g \in \mathcal{C}_c(G\backslash\backslash K)$, we have $\rho(f * g - g * f) = 0$. Hence by the above considerations $f * g = g * f$. Consequently $\mathcal{C}_c(G\backslash\backslash K)$ is commutative and by density $L^1(G\backslash\backslash K)$ is also commutative. □

We obtain a characterization of Gelfand pairs for transitive groupoids as a corollary of Theorem 3.3.

Theorem 3.4. *If we assume in addition that K is transitive, then (G, K) is a Gelfand pair if and only if for some $m \in G^{(0)}$ (hence for all $m \in G^{(0)}$) (G_m^m, K_m^m) is a Gelfand pair.*

Proof. Let us first show that if $\tilde{\pi}$ is a unitary representation of G on $G^{(0)} \times H$ and π the restriction of $\tilde{\pi}$ to G_m^m then $\Gamma_\mu^K(G^{(0)} \times H)$ and $H_{K_m^m}$ are isomorphics as vector spaces, where $H_{K_m^m}$ designates the space of K_m^m-invariant vectors under π. In fact, let us consider the map

$$\varphi : \Gamma_\mu^K(G^{(0)} \times H) \to H_{K_m^m}, \quad \rho \mapsto \rho(m).$$

For all $k \in K_m^m$,

$$\pi(k)\rho(m) = \tilde{\pi}(k)\rho(d(k)) = \rho(r(k)) = \rho(m).$$

So $\rho(m) \in H_{K_m^m}$. Clearly φ is linear. Suppose now that ρ, $\rho' \in \Gamma_\mu^K(G^{(0)} \times H)$ such that $\rho(m) = \rho'(m)$. Let $u \in G^{(0)}$. Since K is transitive, then there exists $k \in K$ such that $r(k) = u$ and $d(k) = m$. Thus,

$$\rho(u) = \rho(r(k)) = \tilde{\pi}(k)\rho(m) = \tilde{\pi}(k)\rho'(m) = \tilde{\pi}(k)\rho'(d(k)) = \rho'(r(k)) = \rho'(u),$$

so $\rho = \rho'$ and φ is injective. Let $h \in H_{K_m^m}$. We set for all $u \in G^{(0)}$, $s(u) = \tilde{\pi}(k)(m, h)$ for some $k \in K_m^u$. Notice that $s(u)$ is independent of the choice of k. In fact, if $k, k' \in K_m^u$, we have $k^{-1}k' \in K_m^m$ and

$$\tilde{\pi}(k^{-1}k')(m, h) = \pi(k^{-1}k')(m, h) = (m, h).$$

So, $\tilde{\pi}(k')(m, h) = \tilde{\pi}(k)(m, h)$.

We have, for all $k \in K$ and for some $k' \in K_m^{d(k)}$

$$\tilde{\pi}(k)s(d(k)) = \tilde{\pi}(k)\tilde{\pi}(k')(m, h) = \tilde{\pi}(kk')(m, h) = s(r(k)),$$

so s is K-invariant. s is square integrable since $G^{(0)}$ is compact. Moreover, for some $k \in K_m^m$

$$s(m) = \tilde{\pi}(k)(m, h) = \pi(k)h = h.$$

Thus $\varphi(s) = h$ and φ is surjective.

In the same way, if π is a unitary representation of G_m^m on H and $\tilde{\pi}$ the unitary representation of G on $G^{(0)} \times H$ associated to π, we show that $\Gamma_\mu^K(G^{(0)} \times H)$ and $H_{K_m^m}$ are isomorphics as vector spaces. But only for proof that $\rho(m) \in H_{K_m^m}$ we use the fact that $\theta(K_m^m) = K_m^m$.

So suppose (G, K) is a Gelfand pair and let π be an irreducible unitary representation of G_m^m on a Hilbert space H. The unitary representation $\tilde{\pi}$ of G on $G^{(0)} \times H$ defined by π is irreducible according to Lemma 3.2. Since (G, K) is a Gelfand pair then, thanks to Theorem 3.3, $\dim \Gamma_\mu^K(G^{(0)} \times H) \le 1$. So according to the isomorphism established above, $\dim H_{K_m^m} \le 1$. Hence (G_m^m, K_m^m) is a Gelfand pair thanks to Gelfand theorem mentioned in the introduction. Conversely, suppose (G_m^m, K_m^m) is a Gelfand pair and let $\tilde{\pi}$ be an irreducible unitary representation of G on $G^{(0)} \times H$. The unitary representation π of G_m^m on H defined by $\tilde{\pi}$ is irreducible according to Lemma 3.2. Since (G_m^m, K_m^m) is a Gelfand pair then, thanks to Gelfand theorem mentioned in the introduction, $\dim H_{K_m^m} \le 1$. So according to the isomorphism established above, $\dim \Gamma_\mu^K(G^{(0)} \times H) \le 1$. Hence (G, K) is a Gelfand pair thanks to Theorem 3.3. $\square$

4. EXAMPLES

Example 4.1. Let M be a compact manifold, $GL(n, \mathbb{R})$ be the group of invertible real matrices and $\mathbf{0}(n, \mathbb{R})$ be the group of orthogonal matrices.

Consider the transitive trivial Lie groupoid $G = M \times GL(n, \mathbb{R}) \times M$ with groupoid structure which is defined in the following way:

$$\begin{aligned} d(m, g, n) &= (n, I, n), \\ r(m, g, n) &= (m, I, m), \\ (m, g, n)(n, h, p) &= (m, gh, p) \end{aligned}$$

and

$$(m, g, n)^{-1} = (n, g^{-1}, m),$$

where $m, n, p \in M$, $g, h \in GL(n, \mathbb{R})$ and I the identity matrix. The set $K = M \times \mathbf{0}(n, \mathbb{R}) \times M$ equipped with the above groupoid structure, is a compact Lie subgroupoid of G. For any $m \in M$, we have $G_m^m = GL(n, \mathbb{R})$ and $K_m^m = \mathbf{0}(n, \mathbb{R})$. So since $(GL(n, \mathbb{R}), \mathbf{0}(n, \mathbb{R}))$ is a Gelfand pair, then thanks to Theorem 3.4, $(M \times GL(n, \mathbb{R}) \times M, M \times \mathbf{O}(n, \mathbb{R}) \times M)$ is a Gelfand pair.

In general, if G is a locally compact group and K a compact subgroup of G such that (G, K) is a Gelfand pair then $(M \times G \times M, M \times K \times M)$ is a Gelfand pair, where M is a topological compact space.

Example 4.2. Let A be an abelian locally compact group and let M be a topological compact space. We suppose that A acts transitively and continuously on M. Let us consider the groupoid action $G = A \propto M$ defined by:

$$\begin{aligned} d(a, m) &= m, \\ r(a, m) &= a.m, \\ (a, a'.m')(a', m') &= (aa', m') \end{aligned}$$

and

$$(a, m)^{-1} = (a^{-1}, a.m),$$

where $a, a' \in A, m, m' \in M$. Let us put $K = M$ the base groupoid with $d = r = id_M$. For any $m \in M$, G_m^m is a subgroup of A and $K_m^m = \{m\}$. Since G_m^m is abelian then $(G_m^m, \{m\})$ is a Gelfand pair. So, thanks to Theorem 3.4 $(A \propto M, M)$ is a Gelfand pair.

We can take $G = \mathbb{R}$, $M = S^1$ the unit circle and consider the action of $\mathbb{R}$ on S^1 defined by $t.z = e^{2\pi i t} z$ with $t \in \mathbb{R}$, $z \in S^1$. The pair $(\mathbb{R} \propto S^1, S^1)$ is a Gelfand pair as explained just above.

Example 4.3. Let H be a compact Lie group and L a closed Lie subgroup of H such that (H, L) is a Gelfand pair. Let us consider the transitive trivial Lie groupoid $G = H/L \times H \times H/L$ and $K = \frac{H \times H}{L}$ the gauge groupoid (see for example [8] for a definition) associated to the principal bundle $H(H/L, L)$. K is a closed Lie subgroupoid of G ([8, p. 14]). But G is compact, so K is compact. Note that a gauge groupoid is transitive. For any $m \in H/L$, we have $G_m^m = H$ and $K_m^m = L$, so (G_m^m, K_m^m) is a Gelfand pair and thanks to Theorem 3.4 $(H/L \times H \times H/L, \frac{H \times H}{L})$ is a Gelfand pair.

We can apply it to the Gelfand pair $(SO(n), SO(n-1))$, $n \geq 2$, where $SO(n)$ is the real special orthogonal group of order n.

Example 4.4. Let M be a compact $\mathcal{C}^\infty$-manifold and (E,p,M) be a complex rank n vector bundle with hermitan metric. Let $\Phi(E)$ denote the set of all linear isomorphisms between the various fibres of the vector bundle (E,p,M). Then $\Phi(E)$ is a Lie groupoid on M with respect to the following structure: for each isomorphism $\varphi_{m,n}\colon E_n \to E_m$, $d(\varphi_{m,n}) = n$, and $r(\varphi_{m,n}) = m$; the partial multiplication is the composition of maps. The inverse of $\varphi_{m,n}$ is its inverse as a map. With this structure, $\Phi(E)$ is called the linear frame groupoid of (E,p,M). Consider now $\Phi_U(E)$ the unitary frame groupoid of (E,p,M) consisting of all unitary isomorphisms between the various fibres of the vector bundle. It is a closed Lie subgroupoid of $\Phi(E)$ (see [8]). Note that $\Phi(E)$ and $\Phi_U(E)$ are locally trivial Lie groupoids (see [8]) so transitive. For any $m \in M$,

$$(\Phi(E))^m_m = GL(E_m) \simeq GL(n,\mathbb{C}) \text{ and } (\Phi_U(E))^m_m = U(E_m) \simeq U(n,\mathbb{C}).$$

Since $(GL(n,\mathbb{C}), U(n,\mathbb{C}))$ is a Gelfand pair, then thanks to Theorem 3.4, $(\Phi(E), \Phi_{\mathbf{U}}(E))$ is a Gelfand pair.

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