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SOBOLEV NORM ESTIMATES OF SOLUTIONS FOR THE SUBLINEAR EMDEN-FOWLER EQUATION

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Abstract. We study the sublinear Emden-Fowler equation in small domains. As the domain becomes smaller, so does any solution. We investigate the convergence rate of the Sobolev norm of solutions as the volume of the domain converges to zero. The result is obtained by estimating the first eigenvalue of the Laplacian with the help of the variational method.

Keywords: Emden-Fowler equation, sign-changing solution, positive solution, variational method, norm estimate.

Mathematics Subject Classification: 35J20, 35J25.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we study the sublinear Emden-Fowler equation

$$-\Delta u = |u|^{p-1}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where Ω is a bounded open connected domain in $\mathbb{R}^N$ with $N \geq 2$ and $0 < p < 1$. Then (1.1) has a unique positive solution, which was proved by Brezis and Oswald [3] (see also [5] without the smoothness assumption of $\partial\Omega$). Throughout the paper, we denote the unique positive solution by $U(x)$. As Ω is shrinking, the positive solution U of (1.1) becomes smaller. Indeed, we proved in [5] that

$$0 < U_1(x) < U_2(x) \quad \text{for } x \in \Omega_1, \tag{1.2}$$

provided that $\Omega_1 \subset \Omega_2$ and $\Omega_1 \neq \Omega_2$, where $U_i(x)$ denotes the positive solution of (1.1) with Ω replaced by Ω_i with $i = 1, 2$. As the volume of Ω tends to zero, the $H_0^1(\Omega)$ norm and the $L^{p+1}(\Omega)$ norm of all solutions converge to zero. The purpose of this paper is to investigate the convergence rate of these norms. The $L^\infty(\Omega)$ estimate of solutions has been studied in our paper [6] in terms of the thickness of Ω. In the present paper, we give the estimate of the $H_0^1(\Omega)$ norm of solutions by using the volume of Ω.

Multiplying (1.1) by u and integrating it over Ω, we have

$$\int_\Omega |\nabla u|^2 dx = \int_\Omega |u|^{p+1} dx, \tag{1.3}$$

or equivalently $\|\nabla u\|_2^2 = \|u\|_{p+1}^{p+1}$. Here $\|\cdot\|_q$ denotes the $L^q(\Omega)$ norm. Therefore the estimate of $\|\nabla u\|_2$ is equivalent to that of $\|u\|_{p+1}$. Hereafter we shall give the estimate of $\|\nabla u\|_2$ only. We denote the volume of Ω by $|\Omega|$. Let $\lambda_1(\Omega)$ denote the first eigenvalue of the problem

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1.4}$$

Then the first main result is as follows.

Theorem 1.1. *For any solution u of (1.1), it holds that*

$$\|\nabla u\|_2^2 \le |\Omega| \lambda_1(\Omega)^{-(1+p)/(1-p)}, \tag{1.5}$$

$$\|\nabla u\|_2^2 \le C|\Omega|^{(N+2-(N-2)p)/(N(1-p))} \tag{1.6}$$

with

$$C := \left(\lambda_1(B_1)|B_1|^{2/N}\right)^{-(1+p)/(1-p)},$$

where B_1 denotes the unit ball in $\mathbb{R}^N$.

Even if $|u|^{p-1}u$ in (1.1) is replaced by a general function $f(u)$ satisfying $|f(u)| \le C|u|^p$ with a constant $C > 0$, Theorem 1.1 is still valid with the right hand sides of (1.5) and (1.6) multiplied by positive constants. For simplicity, we deal with $|u|^{p-1}u$ only.

Theorem 1.1 gives the upper estimate of the norm. We next study the lower estimate of the norm. It was proved by Ambrosetti and Badiale [1] (see also [2] by Ambrosetti, Brezis and Cerami) that there exists a sequence of solutions u_n such that $\|\nabla u_n\|_2 \to 0$. Therefore there does not exist a uniform positive lower bound for $\|\nabla u\|_2$ of all solutions. We give a lower estimate for the positive solution only.

Theorem 1.2. *Let $R > 0$ be the radius of the maximum ball in Ω. Then there exists a $C > 0$ independent of Ω and R such that*

$$\|\nabla U\|_2^2 \ge CR^{(N+2-(N-2)p)/(1-p)} \tag{1.7}$$

for the positive solution U of (1.1). If Ω is a ball with radius R, then

$$\|\nabla U\|_2^2 = cR^{(N+2-(N-2)p)/(1-p)} \tag{1.8}$$

with a constant $c > 0$ independent of U and R.

If Ω is a ball with radius R, then $|\Omega| = |B_1|R^N$, where B_1 denotes the unit ball. Substituting this value into (1.6) and comparing (1.6) with (1.8), we see that the exponent of $|\Omega|$ in (1.6) is the best possible.

Remark 1.3. Theorem 1.1 gives the upper estimate of $\|\nabla u\|_2$ in terms of the volume of Ω. However, Theorem 1.2 provides the lower estimate of the positive solution by using R. We remark that there does not exist the lower estimate in terms of the volume of Ω. Suppose on the contrary that there is an estimate

$$\|\nabla U\|_2^2 \geq C|\Omega|^\theta \tag{1.9}$$

for the positive solution U. As $|\Omega| \to 0$ in (1.6), $\|\nabla U\|_2$ converges to zero. Hence θ must be positive. However, there exists a sequence of domains Ω_n such that $\|\nabla U_n\|_2 \to 0$ and $|\Omega_n| \to \infty$ as $n \to \infty$, where U_n is the positive solution in Ω_n. We shall give such a sequence Ω_n in Example 1.4 or in Example 1.12. Thus estimate (1.9) does not hold.

Example 1.4. Let $N = 2$ and $\Omega := (0, a) \times (0, b)$ with $b = a^{-\delta}$. Then

$$|\Omega| = ab = a^{1-\delta} \to \infty \tag{1.10}$$

as $a \to \infty$ provided that $\delta < 1$. The first eigenvalue of (1.4) for $\Omega = (0, a) \times (0, b)$ is computed as $\lambda_1(\Omega) = \pi^2(a^{-2} + b^{-2})$. Then (1.5) leads to

$$\begin{aligned} \|\nabla u\|_2^2 &\leq ab\pi^{-2(1+p)/(1-p)}(a^{-2} + b^{-2})^{-(1+p)/(1-p)} \leq \\ &\leq \pi^{-2(1+p)/(1-p)} a^{1-\delta(3+p)/(1-p)} \to 0 \end{aligned} \tag{1.11}$$

as $a \to \infty$ provided that $(1 - p)/(3 + p) < \delta$. When $(1 - p)/(3 + p) < \delta < 1$, we have both (1.10) and (1.11). This is an example stated in Remark 1.3.

We shall give an estimate of $\lambda_1(\Omega)$ for some special domains Ω. We first consider the case where Ω lies between two parallel hyperplanes a distance d apart. After rotating and translating Ω, we assume

$$\Omega \subset \{(x_1, \ldots, x_N) : 0 < x_1 < d\}.$$

Theorem 1.5. *Under the assumption above, we have*

$$\lambda_1(\Omega) \geq (\pi/d)^2, \quad \|\nabla u\|_2^2 \leq |\Omega|(d/\pi)^{2(1+p)/(1-p)} \tag{1.12}$$

for any solution u *of* (1.1).

The theorem above is not effective for an annulus. We shall investigate the estimate of solutions in an annulus like domain Ω.

Definition 1.6. Suppose that $0 \notin \overline{\Omega}$. Let S^{N-1} be the unit sphere in $\mathbb{R}^N$. We define

$$\Sigma := \{\sigma \in S^{N-1} : r\sigma \in \overline{\Omega} \text{ at some } r > 0\},$$

$$a(\sigma) := \min\{r > 0 : r\sigma \in \overline{\Omega}\}, \quad b(\sigma) := \max\{r > 0 : r\sigma \in \overline{\Omega}\}$$

for $\sigma \in \Sigma$. Moreover, we put

$$\alpha := \inf_{\sigma\in\Sigma} a(\sigma), \quad \beta := \sup_{\sigma\in\Sigma} a(\sigma), \quad \gamma := \sup\{b(\sigma) - a(\sigma) : \sigma \in \Sigma\}.$$

Theorem 1.7. *Suppose that* $0 \notin \overline{\Omega}$ *and* u *be any solution of* (1.1)*. Then*

$$\lambda_1(\Omega) \geq (\alpha/\beta)^{N-1}(\pi/\gamma)^2, \tag{1.13}$$

$$\|\nabla u\|_2^2 \leq |\Omega| \left\{(\beta/\alpha)^{(N-1)}(\gamma/\pi)^2\right\}^{(1+p)/(1-p)}. \tag{1.14}$$

Remark 1.8. By the definition of α and β, the domain Ω is included in an annulus $A(\alpha, \beta)$, which is defined by

$$A(\alpha, \beta) := \{x \in \mathbb{R}^N : \alpha < |x| < \beta\}. \tag{1.15}$$

This inclusion implies that

$$\lambda_1(\Omega) \geq \lambda_1(A(\alpha, \beta)). \tag{1.16}$$

However (1.13) is a sharper estimate than (1.16). We shall explain this claim. Consider the square in $\mathbb{R}^2$,

$$D := \{(x_1, x_2) : |x_1| < 1, \ |x_2| < 1\}.$$

For $\varepsilon > 0$, we define

$$(1+\varepsilon)D := \{(1+\varepsilon)x : x \in D\}, \quad \Omega := (1+\varepsilon)D \setminus \overline{D}.$$

Then $(1+\varepsilon)D$ is a square slightly larger than D and Ω is a domain enclosed by the two boundaries $\partial((1+\varepsilon)D)$ and ∂D. By definition, α is the radius of the inscribed circle of D, β is the radius of the circumscribed circle of $(1+\varepsilon)D$ and γ is the length of the segment connecting corner points $(1,1)$ and $(1+\varepsilon, 1+\varepsilon)$ in the coordinate plane. Thus $\alpha = 1$, $\beta = \sqrt{2}(1+\varepsilon)$ and $\gamma = \sqrt{2}\varepsilon$. Then (1.16) means that

$$\lambda_1(\Omega) \geq \lambda_1(A(1, \sqrt{2}(1+\varepsilon))).$$

The right hand side does not diverge as $\varepsilon \to 0$. However, our estimate (1.13) says

$$\lambda_1(\Omega) \geq (\sqrt{2}(1+\varepsilon))^{-(N-1)}(\pi/(\sqrt{2}\varepsilon))^2,$$

which means that $\lambda_1(\Omega) \to \infty$ as $\varepsilon \to 0$. Consequently, (1.13) gives a sharper estimate than (1.16).

We show the maximality of the positive solution in the next proposition.

Proposition 1.9. *Let* U *be the positive solution of* (1.1)*. Then* U *is the maximum solution of all solutions, that is, if* v *is a sign-changing solution, then*

$$|v(x)| < U(x) \quad \text{for } x \in \Omega. \tag{1.17}$$

Moreover, for any $q \in [1, \infty]$*, the solution that attains the maximum of* $\|v\|_q$ *among all solutions* v *is either the positive solution or the negative solution. This assertion remains valid for* $\|\nabla v\|_2$ *instead of* $\|v\|_q$*.*

We give the optimal estimate of the positive solution in a rectangle and an annulus.

Theorem 1.10. *Let $\Omega = \prod_{i=1}^{N}(0, a_i)$ be an N-dimensional rectangle. Then there exist constants $c_1, c_2 > 0$ independent of Ω such that*

$$c_1 AB^{-(1+p)/(1-p)} \le \|\nabla U\|_2^2 \le c_2 AB^{-(1+p)/(1-p)} \tag{1.18}$$

for the positive solution U of (1.1), where $A := \prod_{i=1}^{N} a_i$ and $B := \sum_{i=1}^{N} a_i^{-2}$. In particular, if Ω is a cube, we put $a_i = a$ for all i to get

$$c_1 a^{(N+2-(N-2)p)/(1-p)} \le \|\nabla U\|_2^2 \le c_2 a^{(N+2-(N-2)p)/(1-p)}.$$

Theorem 1.11. *Let Ω be an annulus $A(a, b)$ defined by (1.15) with $\alpha = a$ and $\beta = b$. Then for any $K > 1$ there exist constants $c_1, c_2 > 0$ independent of a and b such that*

$$c_1 a^{N-1}(b-a)^{(3+p)/(1-p)} \le \|\nabla U\|_2^2 \le c_2 a^{N-1}(b-a)^{(3+p)/(1-p)} \tag{1.19}$$

for the positive solution U of (1.1) if $1 < b/a < K$.

Example 1.12. We consider the expanding annulus $\Omega = A(a, b)$ with $a \to \infty$. Then the convergence or divergence of $\|\nabla U\|_2$ for the positive solution U depends on the divergence rate of b. If $b - a \ge c_0$ with a constant $c_0 > 0$, the norm $\|\nabla U\|_2$ diverges to infinity as $a \to \infty$ because of (1.19). So, we consider the case where $b - a \to 0$. Put $b = a + a^{-\delta}$. Then we have

$$a^{N-1}(b-a)^{(3+p)/(1-p)} = a^{N-1-\delta(3+p)/(1-p)}.$$

Substitute this identity into (1.19) and put

$$\theta(N, p) := (N-1)(1-p)/(3+p).$$

Then the norm $\|\nabla U\|_2$ with $\Omega = A(a, a + a^{-\delta})$ behaves as $a \to \infty$,

$$\begin{aligned} \|\nabla U\|_2 \to \infty \quad &\text{if} \quad 0 < \delta < \theta(N,p), \\ C_1 \le \|\nabla U\|_2 \le C_2 \quad &\text{if} \quad \delta = \theta(N,p), \\ \|\nabla U\|_2 \to 0 \quad &\text{if} \quad \delta > \theta(N,p), \end{aligned} \tag{1.20}$$

where $C_1, C_2 > 0$ are some constants.

We now construct another example of domains Ω_n stated in Remark 1.3. In (1.20), we choose δ slightly larger than $\theta(N, p)$. Using the binomial expansion of $b^N = (a + a^{-\delta})^N$, we have a constant $c > 0$ such that

$$|\Omega| = |B_1|(b^N - a^N) \ge ca^{N-1-\delta} \to \infty \quad \text{as } a \to \infty. \tag{1.21}$$

Thus (1.20) and (1.21) give the example stated in Remark 1.3.

2. PROOF OF THE MAIN RESULTS

In this section, we prove the main results. Let B_R denote a ball centered at origin with radius R.

Lemma 2.1. $\lambda_1(\Omega) \geq \lambda_1(B_1)(|B_1|/|\Omega|)^{2/N}$.

Proof. We use the isoperimetric inequality (see [4, p.87, Theorem 2])

$$\lambda_1(\Omega) \geq \lambda_1(B) \quad \text{if } |\Omega| = |B| \text{ with a ball } B. \tag{2.1}$$

By an easy scaling computation, we see that $\lambda_1(B_R) = R^{-2}\lambda_1(B_1)$. Choose $R > 0$ such that $|\Omega| = |B_R|$. Then $|\Omega| = |B_R| = |B_1|R^N$, which is rewritten as $R = (|\Omega|/|B_1|)^{1/N}$. By (2.1), we have

$$\lambda_1(\Omega) \geq \lambda_1(B_R) = R^{-2}\lambda_1(B_1) = \lambda_1(B_1)(|B_1|/|\Omega|)^{2/N}.$$

□

Recall that the first eigenvalue is characterized by

$$\lambda_1(\Omega) = \inf_{H_0^1(\Omega)\setminus\{0\}} \left(\int_\Omega |\nabla v|^2 dx \right) \Big/ \left(\int_\Omega |v|^2 dx \right),$$

which implies that

$$\|v\|_2 \leq \lambda_1(\Omega)^{-1/2} \|\nabla v\|_2 \quad \text{for } v \in H_0^1(\Omega). \tag{2.2}$$

Proof of Theorem 1.1. Let u be any solution of (1.1). Since $0 < p < 1$, we use the Hölder inequality to get

$$\int_\Omega |u|^{p+1} dx \leq \left(\int_\Omega |u|^2 dx \right)^{(p+1)/2} |\Omega|^{(1-p)/2} = |\Omega|^{(1-p)/2} \|u\|_2^{p+1}. \tag{2.3}$$

By (2.2) and (2.3), we obtain

$$\int_\Omega |u|^{p+1} dx \leq |\Omega|^{(1-p)/2} \lambda_1(\Omega)^{-(p+1)/2} \|\nabla u\|_2^{p+1}.$$

Substituting this inequality into (1.3), we get

$$\|\nabla u\|_2^2 \leq |\Omega|^{(1-p)/2} \lambda_1(\Omega)^{-(p+1)/2} \|\nabla u\|_2^{p+1}.$$

Dividing both sides by $\|\nabla u\|_2^{p+1}$, we get (1.5). Combining Lemma 2.1 with (1.5), we obtain (1.6). The proof is complete. □

By (1.2), we have

$$\|U_1\|_{L^{p+1}(\Omega_1)} < \|U_2\|_{L^{p+1}(\Omega_2)},$$

where U_i is the unique positive solution of Ω_i with $i = 1, 2$. This inequality with (1.3) implies

$$\|\nabla U_1\|_{L^2(\Omega_1)} < \|\nabla U_2\|_{L^2(\Omega_2)}, \tag{2.4}$$

provided that $\Omega_1 \subset \Omega_2$ and $\Omega_1 \neq \Omega_2$.

Proof of Theorem 1.2. Let $R > 0$ be as in Theorem 1.2. By a parallel translation, we assume that $B_R \subset \Omega$, where B_R is the ball centered at the origin with radius R. Let v_R be a radial positive solution of (1.1) with Ω replaced by B_R. Then $v_R = v_R(r)$ with $r = |x|$ satisfies

$$v_R'' + \frac{N-1}{r} v_R' + v_R^p = 0, \quad v > 0 \quad \text{in } (0, R), \tag{2.5}$$

$$v_R'(0) = v_R(R) = 0, \tag{2.6}$$

where $v_R' = dv_R/dr$. Observe that the transformation $\lambda^{-2/(1-p)} v(\lambda x)$ with $\lambda > 0$ leaves (2.5) invariant. Therefore we have the relation

$$v_R(r) = R^{2/(1-p)} v_1(r/R),$$

where $v_1(r)$ is a solution of (2.5), (2.6) with $R = 1$. Then we compute

$$\begin{aligned} \|v_R\|_{L^{p+1}(B_R)}^{p+1} &= R^{2(1+p)/(1-p)} \omega_N \int_0^R v_1(r/R)^{p+1} r^{N-1} dr = \\ &= C R^{(N+2-(N-2)p)/(1-p)}, \end{aligned} \tag{2.7}$$

where ω_N denotes the surface area of the unit sphere of $\mathbb{R}^N$ and we have set

$$C := \omega_N \int_0^1 v_1(t)^{p+1} t^{N-1} dt.$$

By (2.4) and (2.7), we have

$$\|\nabla U\|_{L^2(\Omega)}^2 \geq \|\nabla v_R\|_{L^2(B_R)}^2 = \|v_R\|_{L^{p+1}(B_R)}^{p+1} = C R^{N+2-(N-2)p)/(1-p)}.$$

If Ω is a ball with radius R, then (2.7) ensures (1.8). □

Proof of Theorem 1.5. The lower estimate of $\lambda_1(\Omega)$ in (1.12) is well known, however we prove it for the sake of completeness. Let $u \in H_0^1(\Omega)$. Put $u = 0$ in $\mathbb{R}^N \setminus \Omega$. Since $(\pi/d)^2$ is the first eigenvalue of (1.4) with $N = 1$ and $\Omega = (0, d)$, we have

$$\int_0^d u(x_1, \ldots, x_N)^2 dx_1 \leq (d/\pi)^2 \int_0^d u_{x_1}(x_1, \ldots, x_N)^2 dx_1.$$

Integrating both sides with respect to $x_2, \ldots, x_N$, we have

$$\int_\Omega u(x)^2 dx \leq (d/\pi)^2 \int_\Omega u_{x_1}(x)^2 dx \leq (d/\pi)^2 \int_\Omega |\nabla u(x)|^2 dx,$$

which implies that $\lambda_1(\Omega) \geq (\pi/d)^2$. Combining this inequality with (1.5), we get the second inequality of (1.12). □

To prove Theorem 1.7, we prepare a brief lemma.

Lemma 2.2. *Let $0 < a < b$. Then for $u \in H_0^1(a,b)$, it holds that*

$$\int_a^b u(r)^2 r^{N-1} dr \le (b/a)^{N-1}((b-a)/\pi)^2 \int_a^b u'(r)^2 r^{N-1} dr.$$

Proof. The number $(\pi/(b-a))^2$ is the first eigenvalue of (1.4) with $N = 1$ and $\Omega = (a,b)$. This implies that

$$\int_a^b u(r)^2 dr \le ((b-a)/\pi)^2 \int_a^b u'(r)^2 dr.$$

Then we have

$$\int_a^b u(r)^2 r^{N-1} dr \le b^{N-1}((b-a)/\pi)^2 \int_a^b u'(r)^2 dr \le$$

$$\le (b/a)^{N-1}((b-a)/\pi)^2 \int_a^b u'(r)^2 r^{N-1} dr. \qquad \square$$

Proof of Theorem 1.7. Let $u \in H_0^1(\Omega)$ and put $u = 0$ outside of Ω. We define $a(\sigma) = b(\sigma) = 0$ for $\sigma \in S^{N-1} \setminus \Sigma$. We write $u(x) = u(r,\sigma)$ with the polar coordinate $x = r\sigma$, $r > 0$ and $\sigma \in S^{N-1}$. Observe that

$$\int_\Omega u^2 dx = \int_{S^{N-1}} \left(\int_{a(\sigma)}^{b(\sigma)} u(r,\sigma)^2 r^{N-1} dr \right) d\sigma.$$

Using Lemma 2.2 with $a = a(\sigma)$ and $b = b(\sigma)$, we have

$$\int_\Omega u^2 dx \le (\beta/\alpha)^{N-1}(\gamma/\pi)^2 \int_\Omega |\nabla u|^2 dx,$$

or equivalently,

$$\left(\int_\Omega |\nabla u|^2 dx \right) \Big/ \left(\int_\Omega |u|^2 dx \right) \ge (\alpha/\beta)^{N-1}(\pi/\gamma)^2,$$

for any $u \in H_0^1(\Omega)$. Taking the infimum on $u \in H_0^1(\Omega) \setminus \{0\}$, we obtain (1.13). Substituting (1.13) into (1.5), we get (1.14). $\square$

Proof of Proposition 1.9. Let v be any sign-changing solution. Let D be any connected component of the set of x satisfying $v(x) \ne 0$. If $v(x) < 0$ in D, we replace v

by $-v$. Then $v(x)$ is a positive solution of (1.1) with Ω replaced by D. By (1.2), we have $0 < v(x) < U(x)$ in D. This proves (1.17). From (1.17) it follows readily that $\|v\|_q < \|U\|_q$ for any $q \in [1, \infty]$. Putting $q = p+1$ and using (1.3), we conclude that $\|\nabla v\|_2 < \|\nabla U\|_2$. This completes the proof. □

To prove Theorem 1.10, we use a variational method. We define a Rayleigh quotient

$$R(u) := \|\nabla u\|_2^2 / \|u\|_{p+1}^2.$$

Moreover, we define the Nehari manifold

$$\mathcal{N} := \Big\{ u \in H_0^1(\Omega) \setminus \{0\} \colon \int_\Omega (|\nabla u|^2 - |u|^{p+1}) dx = 0 \Big\}.$$

By the Sobolev embedding, R has a positive lower bound in $H_0^1(\Omega) \setminus \{0\}$. Denote the infimum of R by R_0. Observe that for any $u \in H_0^1(\Omega) \setminus \{0\}$, there is a $\lambda > 0$ such that $\lambda u \in \mathcal{N}$. Furthermore, $R(\lambda u) = R(u)$ for any $\lambda > 0$. Hence we have

$$R_0 := \inf \big\{ R(u) \colon u \in H_0^1(\Omega) \setminus \{0\} \big\} = \inf \{ R(u) \colon u \in \mathcal{N} \}.$$

The infimum is achieved at a certain point $u \in \mathcal{N}$. Then u satisfies (1.1). For the proof, we refer the readers to [7]. We call u a *least energy solution* if $u \in \mathcal{N}$ and $R(u) = R_0$. Since $R(|u|) = R(u)$, $|u|$ is also a least energy solution. Then the strong maximum principle ensures that $|u(x)| > 0$ in Ω. Thus u is a positive or negative solution. We choose a positive solution as a least energy solution. Since a positive solution is unique, a least energy solution coincides with it. Accordingly, we have

$$R(U) = \inf_{v \in H_0^1(\Omega) \setminus \{0\}} R(v) = \inf_{v \in \mathcal{N}} R(v),$$

for the positive solution U. We use (1.3) to get

$$R(U) = \|\nabla U\|_2^2 / \|U\|_{1+p}^2 = \|\nabla U\|_2^{-2(1-p)/(1+p)}.$$

Combining the two inequalities above, we have the next proposition.

Proposition 2.3. *The positive solution* U *of* (1.1) *satisfies*

$$\|\nabla U\|_2^{-2(1-p)/(1+p)} = R(U) \le R(v) \quad \text{for any } v \in H_0^1(\Omega) \setminus \{0\}. \tag{2.8}$$

Proof of Theorem 1.10. Let $\Omega = \prod_{i=1}^N (0, a_i)$. Let A and B be as in Theorem 1.10. Then $A = |\Omega|$. The first eigenvalue and the eigenfunction are computed as

$$\lambda_1(\Omega) = \pi^2 \sum_{i=1}^N a_i^{-2} = \pi^2 B, \quad \phi(x) = \prod_{i=1}^N \sin(\pi x_i / a_i).$$

Hence (1.5) implies the second inequality of (1.18). We shall prove the first inequality. To this end, we estimate the Rayleigh quotient $R(\phi)$. From an easy calculation, it follows that

$$\phi_{x_i} = (\pi / a_i) \cos(\pi x_i / a_i) \prod_{j \neq i} \sin(\pi x_j / a_j),$$

$$\|\phi_{x_i}\|_2^2 = (\pi/a_i)^2 \prod_{j=1}^{N} (a_j/2) = 2^{-N}(\pi/a_i)^2 A,$$

which shows $\|\nabla\phi\|_2^2 = 2^{-N}\pi^2 AB$. On the other hand, we have

$$\int_0^{a_i} \sin^{p+1}(\pi x_i/a_i)dx_i \geq \int_{a_i/3}^{2a_i/3} \sin^{p+1}(\pi x_i/a_i)dx_i \geq (\sqrt{3}/2)^{p+1}(a_i/3),$$

which implies

$$\|\phi\|_{p+1}^{p+1} \geq 3^{-N}(\sqrt{3}/2)^{N(p+1)}A.$$

Therefore the Rayleigh quotient is estimated as

$$R(\phi) = \|\nabla\phi\|_2^2/\|\phi\|_{p+1}^2 \leq CA^{-(1-p)/(1+p)}B,$$

with some constant $C > 0$ independent of A and B. Then (2.8) with $v = \phi$ implies that

$$\|\nabla U\|_2^2 \geq cAB^{-(1+p)/(1-p)}. \qquad \square$$

Proof of Theorem 1.11. Let U be the positive solution of (1.1). Then it is radially symmetric. We use a test function

$$v(r) := \sin(\pi(r-a)/(b-a)) \quad \text{with } r = |x|.$$

We shall estimate $R(v)$. By the change of variables $t = \pi(r-a)/(b-a)$, we compute

$$\|\nabla v\|_2^2 = \omega_N \int_a^b |v'|^2 r^{N-1}dr \leq \omega_N b^{N-1} \int_a^b |v'|^2 dr = \frac{\omega_N \pi^2 b^{N-1}}{2(b-a)}. \tag{2.9}$$

Put

$$c := \frac{2a+b}{3}, \quad d := \frac{a+2b}{3}.$$

Then we see

$$\begin{aligned} \|v\|_{1+p}^{1+p} &= \omega_N \int_a^b v(r)^{1+p} r^{N-1}dr \geq \\ &\geq \omega_N a^{N-1} \int_c^d \sin^{1+p}(\pi(r-a)/(b-a))dr \geq (\sqrt{3}/2)^{1+p}\omega_N a^{N-1}(b-a)/3. \end{aligned} \tag{2.10}$$

Combining (2.9) with (2.10), we have

$$R(v) = \|\nabla v\|_2^2/\|v\|_{1+p}^2 \leq Ca^{-2(N-1)/(1+p)} b^{N-1}(b-a)^{-(3+p)/(1+p)},$$

with a certain constant $C > 0$ independent of a and b. Then (2.8) implies that

$$\|\nabla U\|_2^2 \geq Ca^{2(N-1)/(1-p)} b^{-(N-1)(1+p)/(1-p)} (b-a)^{(3+p)/(1-p)}.$$

If $1 < b/a < K$, the inequality above implies the first inequality in (1.19). Substituting $|\Omega| = |B_1|(b^N - a^N)$, $\alpha = a$, $\beta = b$ and $\gamma = b - a$ into (1.14), we get

$$\|\nabla U\|_2^2 \leq |B_1|(b^N - a^N)(b/a)^{(N-1)(1+p)/(1-p)}((b-a)/\pi)^{2(1+p)/(1-p)}.$$

This inequality shows the second inequality in (1.19) when $1 < b/a < K$. The proof is complete. □

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