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A NOTE ON BOUNDED HARMONIC FUNCTIONS OVER HOMOGENEOUS TREES

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Abstract. Let $\mathcal{T}_k$ be the homogeneous tree of degree $k \geq 3$. J. M. Cohen and F. Colonna have proved that if f is a bounded harmonic function on $\mathcal{T}_k$, then $|f(x)-f(y)| \leq \|f\|_\infty \cdot 2(k-2)/k$ for any adjacent vertices x and y in $\mathcal{T}_k$. We give here a new and very simple proof of this inequality.

Keywords: bounded harmonic function, homogeneous tree.

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1. INTRODUCTION

The homogeneous tree of degree k ($k \in \mathbb{N}$), written $\mathcal{T}_k$, is a connected graph without a circuit where every vertex has exactly k neighbours. The distance $d(x,y)$ between two vertices x and y of $\mathcal{T}_k$ is the number of edges in the unique path from x to y (using the terminology of [2]). If x is a vertex of $\mathcal{T}_k$ and $r \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we write $S(x,r)$ the sphere of centre x and radius r (i.e. the set of vertices at distance r from x) and $B(x,r)$ the ball of centre x and radius r. By definition $|S(x,1)| = k$ and it is easily shown that, for $r \geq 1$, $|S(x,r)| = k(k-1)^{r-1}$ and, for $r \geq 0$, $|B(x,r)| = (k(k-1)^r-2)/(k-2)$ if $k \geq 3$, $|B(x,r)| = 1 + 2r$ if $k = 2$.

A complex valued function f defined on the vertices of $\mathcal{T}_k$ is *harmonic at a vertex* x if its value at x is the mean of its values at the neighbours of x:

$$f(x) = \frac{1}{|S(x,1)|} \sum_{z \in S(x,1)} f(z).$$

The function f is *harmonic on* $\mathcal{T}_k$ if it is harmonic at every vertex of $\mathcal{T}_k$.

In many aspects $\mathcal{T}_k$ can be seen as a discrete analogue of the euclidean plane or, better, the hyperbolic plane, in particular with respect to properties of harmonic functions ([3] is at the origin of the topic, [1] is a recent presentation). For example, every harmonic function f on $\mathcal{T}_k$ satisfies the mean value property on balls:

$$f(x) = \frac{1}{|B(x,r)|} \sum_{z \in B(x,r)} f(z)$$

for any vertex x and radius r [5, p.186]. But, in contrast to $\mathbb{R}^n$, when $k \geq 3$ there are bounded harmonic functions on $\mathcal{T}_k$ which are not constant; and about those functions J. M. Cohen and F. Colonna have proved the following.

Proposition. *Let $k \geq 3$. If f is a bounded harmonic function on $\mathcal{T}_k$ and x, y are two adjacent vertices in $\mathcal{T}_k$, then*

$$|f(x) - f(y)| \leq \frac{2(k-2)}{k} \|f\|_\infty .$$

Their first proof [4, Theorem 1, p.65] uses elementary tools but is somewhat involved. They later sketch another proof [4, p.71] which they term more elegant, but it is cetainly not elementary, since it uses the Poisson integral representation of bounded harmonic functions and harmonic measures on the boundary of the tree. Now, we can get this inequality in a way which is both elementary and elegant.

2. PROOF

The key idea is to imitate Nelson's proof [6] of Liouville's theorem: every bounded harmonic function on $\mathbb{R}^n$ is constant. So, take f a bounded harmonic function on $\mathcal{T}_k$, x and y two adjacent vertices in $\mathcal{T}_k$, and $r \in \mathbb{N}$. We calculate

$$\begin{aligned} |f(x) - f(y)| &= \left| \frac{1}{|B(x,r)|} \sum_{z \in B(x,r)} f(z) - \frac{1}{|B(y,r)|} \sum_{z \in B(y,r)} f(z) \right| \leq \\ &\leq \frac{1}{|B(x,r)|} \sum_{z \in B(x,r) \triangle B(y,r)} |f(z)| \leq \\ &\leq \|f\|_\infty \frac{|B(x,r) \triangle B(y,r)|}{|B(x,r)|} = \\ &= \|f\|_\infty \frac{2(k-1)^r}{(k(k-1)^r - 2)/(k-2)} \end{aligned}$$

(where $B(x,r) \triangle B(y,r)$ is the symmetric difference of the two balls). We have here a decreasing sequence, but which does not converge to zero (in contrast to $\mathbb{R}^n$): we arrive, when $r \to +\infty$, at

$$|f(x) - f(y)| \leq \|f\|_\infty \frac{2(k-2)}{k} .$$

3. REMARK

Fix a pair x_0, x_1 of adjacent vertices in $\mathcal{T}_k$ $(k \geq 3)$; Cohen and Colonna exhibit a non-zero bounded harmonic function F on $\mathcal{T}_k$ such that $|F(x_0) - F(x_1)| = \|F\|_\infty \cdot 2(k-2)/k$ [4, p.65]. Therefore, when f runs through all non-zero bounded harmonic functions on $\mathcal{T}_k$, the supremum of

$$\sup_{d(x,y)=1} \frac{|f(x) - f(y)|}{\|f\|_\infty}.$$

is in fact a maximum (equal to $2(k-2)/k$.) Its minimum is clearly 0: take f a constant function. But what about non-constant functions? Here is an answer.

Choose $\varepsilon > 0$. Take $\ell \in \mathbb{N}$ with $\ell \geq 1/\varepsilon$ and a path $p = (x_{-\ell}, \ldots, x_0, \ldots, x_\ell)$ in $\mathcal{T}_k$. Given a vertex z in $\mathcal{T}_k$, assume x_j is the closest vertex of p to z and let $n = d(z, x_j)$. We define a function f on the vertices of $\mathcal{T}_k$ by

$$f(z) = \begin{cases} -\ell - \dfrac{1}{k-2}\left(1 - \dfrac{1}{(k-1)^n}\right) & \text{if } j = -\ell, \\ j & \text{if } -\ell < j < \ell, \\ \ell + \dfrac{1}{k-2}\left(1 - \dfrac{1}{(k-1)^n}\right) & \text{if } j = \ell. \end{cases}$$

Straightforward calculations show that f is harmonic on $\mathcal{T}_k$ with

$$\inf_z f(z) = -\frac{\ell(k-2)+1}{k-2}$$

and

$$\sup_z f(z) = \frac{\ell(k-2)+1}{k-2}.$$

Moreover, for any adjacent vertices x and y, $|f(x) - f(y)| \leq 1$; hence

$$\frac{|f(x) - f(y)|}{\|f\|_\infty} \leq \frac{k-2}{\ell(k-2)+1} < \frac{k-2}{\ell(k-2)} = \frac{1}{\ell} \leq \varepsilon.$$

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