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UNIVERSAL THIRD PARTS OF ANY COMPLETE 2-GRAPH AND NONE OF DK_5

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Abstract. It is shown that there is no digraph F which could decompose the complete digraph on 5 vertices minus any 2-arc remainder into three parts isomorphic to F for each choice of the remainder. On the other hand, for each $n \geq 3$ there is a universal third part F of the complete 2-graph ${}^2K_n$ on n vertices, i.e., for each edge subset R of size $|R| = \|{}^2K_n\| \bmod 3$, there is an F-decomposition of ${}^2K_n - R$. Using an exhaustive computer-aided search, we find all, exactly six, mutually nonisomorphic universal third parts of the 5-vertex 2-graph. Nevertheless, none of their orientations is a universal third part of the corresponding complete digraph.

Keywords: decomposition, remainder, universal parts, isomorphic parts.

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1. INTRODUCTION

By a 2-graph we mean a multigraph with edge multiplicity at most two. The problem we deal with is a specification ($t = 3$) of the edge (arc) t-decomposition of the complete 2-graph ${}^2K_n$ (complete digraph DK_n) on n vertices – hence of size $n(n-1)$ – into t isomorphic parts with an edge (arc) *t-remainder R, where the name t-remainder means that the size of the remainder is $|R| = n(n-1) \bmod t$, which is as small as possible for fixed n and t.* If those parts are isomorphic to F then the isomorphism class of F is called a t-th *part*, with remainder R if $|R| \neq 0$. The symbol $\langle R \rangle$ stands for the 2-graph (digraph) induced by R. Moreover, the isomorphism class of the $\langle R \rangle$ is called a *shape* of R. In the case of t-packings of a t-th part F realises all t-remainders (on at most n vertices), then F is called a *universal* t-th *part.* A decomposition (packing) with parts isomorphic to F is called an *F-decomposition* (*F-packing*). We use the notation and terminology of graph theory as in [1, 3, 18].

Decompositions of the complete graphs into specified parts (matchings, triangles or Hamiltonian cycles) started in the 19th century, see e.g. Kirkman, Steiner and Lucas

([7] about Walecki) in Bosák [1]. Decompositions of the complete graphs into a fixed number of parts, say t, and with $|R| = 0$, were originated by Sachs and Ringel who, for $t{=}2$, independently introduced the notion of the self-complementary graphs in early 1960s. The self-complementary digraphs were described by Read [12]. The existence of t-th parts with $|R| = 0$ was proved independently in [4,13] for the complete graphs and in [5] for the complete digraphs. In the case of any $|R|$, the number-theoretical floor and ceiling were introduced to the graphical decompositions in [14] and the notion of the universal parts in [15]. Decompositions of the complete digraphs into three parts (possibly self-converse or without 2-cycles) or into t parts, with a remainder in general, are presented in papers [8–11]. The existence of a universal t-th part of the complete graph K_n is proved in the manuscripts [6, 16, 17] for $t \leq 6$ and any n, for $n \leq 10$ and any t and, moreover, for any n and $t \geq n - 3$ with exceptions $t = n - 3$ and odd $n \geq 11$.

The following conjecture is a motivation of our study.

Conjecture 1.1 ([15])**.** *A universal t-th part of the complete graph exists.*

We state the following conjecture and our main results.

Conjecture 1.2. *A universal t-th part of any complete 2-graph exists.*

Theorem 1.3. *A universal third part of any complete 2-graph exists.*

Using an exhaustive computer-aided search, we find all, exactly six, mutually nonisomorphic universal third parts of ${}^2K_5$, see Theorem 3.2 for the listing of the parts. This result helps with proving the following chief result of our article.

Theorem 1.4. *There is no universal third part of the complete digraph on 5 vertices.*

In the proofs which follow, a decomposition of the multigraph ${}^2K_n$ into three parts isomorphic to F, with a 3-remainder R (which is nonempty if $n \bmod 3 = 2$), is represented by an $n \times n$ matrix, in which the entry k in row i and column j (with $i \neq j$) means that the edge ij belongs to R if $k = 0$ and to part k otherwise, $k = 1, 2, 3$. This matrix with zeroes on the main diagonal replaced by dots is a modification of the adjacency matrix of ${}^2K_n$ and is called (a 3- or an F-) *decomposition matrix.*

Let $\lfloor \frac{{}^2K_n}{3} \rfloor$ denote the set of universal third parts of the complete 2-graph ${}^2K_n$.

2. PROOF OF THEOREM 1.3

Lemma 2.1. *For $n = 3$, $\lfloor \frac{{}^2K_3}{3} \rfloor = \{P_3, {}^2K_2\}$.*
For $n = 4$, $\lfloor \frac{{}^2K_4}{3} \rfloor$ is the set of all 2-graphs with 4 edges on 3 or 4 vertices with the exception of the 2-graph, say U, obtained from the path P_4 by doubling the middle edge.

Proof. For $n = 3, 4$, the 3-remainder is empty and an F-decomposition of ${}^2K_n$ into three parts exists. Namely, $F = P_3$ or $F = {}^2K_2$ if $n = 3$. If $n = 4$, F can be any of seven 2-graphs with 4 edges on 3 or 4 vertices under the assumption that F is different from U. This exclusion of U is easy to see. Suppose a U-decomposition exists. Then the doubled edges in any U-decomposition necessarily induce a doubled path. Hence remaining edges of ${}^2K_4$ induce another doubled path, which is not decomposable into three 2-matchings, whence $F \neq U$. □

For seven F-decomposition matrices, see Table 1.

Table 1. 3-decomposition matrices for $n = 4$

. 1 2 3	. 1 1 1	. 1 1 3	. 1 1 2	. 1 1 2	. 1 2 1	. 1 1 1
1 . 3 2	2 . 2 2	1 . 2 2	1 . 2 3	1 . 1 3	3 . 1 2	2 . 1 2
2 3 . 1	3 3 . 3	1 2 . 3	3 3 . 1	2 3 . 2	3 2 . 1	3 2 . 2
3 2 1 .	1 2 3 .	3 2 3 .	2 3 2 .	2 3 3 .	2 3 3 .	3 3 3 .

Lemma 2.2. *The 2-graph F^1 on five vertices, uniquely determined by its degree sequence* $(5, 3, 2, 2, 0)$ (*see first item in Fig.* 1) *is the universal third part of* ${}^2K_5$.

Proof. Note that a 3-remainder, R, of ${}^2K_5$ has two edges and therefore three shapes $\langle R \rangle = 2K_2,\ C_2,\ P_3$. The following decomposition matrices (see Table 2) show that 3-packings of F^1 in ${}^2K_5$ realize those shapes of R.

Table 2. F^1-decomposition matrices with remainder R

$\langle R \rangle = 2K_2$	$\langle R \rangle = C_2$	$\langle R \rangle = P_3$
. 1 1 1 2	. 1 1 1 2	. 1 1 1 2
1 . 1 3 2	1 . 3 1 3	1 . 3 1 3
2 0 . 3 2	1 3 . 0 2	1 3 . 3 2
1 3 3 . 3	2 3 0 . 2	2 3 0 . 2
2 2 3 0 .	2 3 2 3 .	2 3 2 0 .

We now describe a recursive step of the proof. Assume that for $n \geq 3$ there is an F-decomposition $F_1,\ F_2,\ F_3$ with 3-remainder R of $G := {}^2K_n$. Then $|R| = \|{}^2K_n\| \bmod 3 = 2$ if $n \bmod 3 = 2$ and $|R| = 0$ otherwise. Consider $\tilde{G} = {}^2K_{n+3}$ which includes G and three new vertices x_1, x_2, x_3. Note that $\{\tilde{F_1}, \tilde{F_2}, \tilde{F_3}\}$ is a required 3-decomposition of $\tilde{G}$ if we assume that each $\tilde{F_j}$ includes F_j, all double edges joining x_j to G and the double edge which joins together the two remaining new vertices, $j = 1, 2, 3$. Another possibility is that $\tilde{F_j}$ includes both F_j and the single edges joining all of G to two new vertices different from x_j together with the two single edges joining those two vertices to x_j.

Starting decompositions for the three initial orders $n = 3, 4, 5$ are presented in Lemmas 2.1 and 2.2 above. □

3. ALL UNIVERSAL THIRD PARTS OF ${}^2K_5$

Let R be a 3-remainder in ${}^2K_5$. Since R includes two edges, there are three shapes of R. Let $A(R)$ be the degree sequence of ${}^2K_5 - R$. Then

$$A(R) = \begin{cases} (8, 8, 8, 6, 6) & \text{for } \langle R \rangle = C_2, \\ (8, 8, 7, 7, 6) & \text{for } \langle R \rangle = P_3, \\ (8, 7, 7, 7, 7) & \text{for } \langle R \rangle = 2K_2. \end{cases}$$

We are going to find all degree sequences of would-be third parts of ${}^2K_5$. The order and the size of those parts are 5 and 6 respectively, multiplicity of edges being at most 2. Therefore we find all partitions of 12 into 5 or less parts, each of which is at most 6. There are 29 of such partitions. We note that if $\Delta = 6$ then remaining parts are to be 2 or 1, and if $\Delta = 5$, at most 3. This observation eliminates 12 of the partitions without any 2-graphic realization. The remaining 17 partitions can be proved to be 2-graphic. A few mutually equivalent characterization of r-graphic partitions are presented by Chungphaisan [2]. One of those characterizations, which is a generalized Erdős-Gallai theorem, is as follows.

Theorem 3.1 ([2])**.** *A nonincreasing sequence $d = (d_1, \ldots, d_n)$ of nonnegative integers is r-graphic if and only if $\sum_{i=1}^{n} d_i$ is even, and for every positive integer $k \le n$,*

$$\sum_{i=1}^{k} d_i \le rk(k-1) + \sum_{i=k+1}^{n} \min\{rk, d_i\}.$$

Let F be a third part of ${}^2K_5 - R$. Then F is of size 6. Therefore if $\alpha = (a_1, \ldots, a_5)$ is a (nonincreasing) degree sequence of F and $A(R) = (A_1, \ldots, A_5)$ then the following condition is satisfied.

(a) There exist three permutations $\sigma_1, \sigma_2, \sigma_3$ such that $a_i + a_{\sigma_1(i)} + a_{\sigma_2(i)} = A_{\sigma_3(i)}$, for $i = 1, \ldots, 5$.

An F-decomposition of the multigraph ${}^2K_5 - R$ into three parts is represented by a 3×5 matrix, called a *degree-decomposition matrix*, in which the first row is a degree sequence of F and the remaining two are permutations of it. Moreover, column sums make up a permutation of $A(R)$. Two degree-decomposition matrices are called *equivalent matrices* if interchanging columns and/or rows in one of the matrices gives the other. A degree-decomposition matrix M is called a *standard degree-decomposition matrix* if the concatenation of the consecutive columns of M is a sequence which is a lexicographical maximum among all matrices equivalent to M.

The following Table 3 summarizes results of computer calculations. The symbol + therein means that the condition (a) is satisfied for the corresponding A, $A = A(R)$.

Table 3. Partitions of 12 which are 2-graphic

$\alpha\backslash A$	88866	88776	87777		$\alpha\backslash A$	88866	88776	87777
62220	+	-	-		44211	-	-	-
62211	+	-	-		43320	+	+	+
53310	-	+	-		43311	-	+	-
53220	+	+	+		43221	+	+	+
53211	-	+	+		42222	+	-	-
52221	+	-	-		33330	-	-	-
44400	-	-	-		33321	-	+	+
44310	+	+	-		33222	+	+	+
44220	+	-	-					

Each partition α listed in Table 3 is a degree sequences of a 2-graph F. Every α which is accompanied by three symbols $+$ therein is called an *acceptable F-sequence.* Thus only the following four partitions are acceptable F-sequences:

$$(5,3,2,2,0),\ (4,3,3,2,0),\ (4,3,2,2,1),\ (3,3,2,2,2).$$

All 2-graphic realizations of those sequences are presented in Fig. 1 and are a result of our exhaustive search for drawings. Note that F^1, F^2–F^4, F^5–F^{11}, and F^{12}–F^{17} make up the corresponding four lists of realizations.

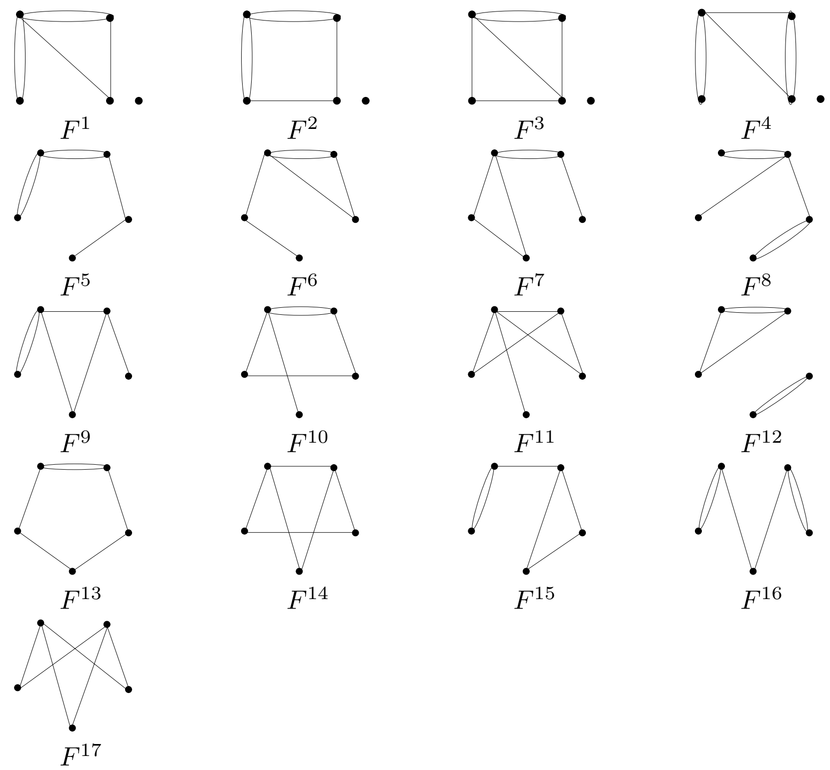

Fig. 1. All 2-graphic realizations of acceptable F-sequences

For each of the four acceptable F-sequences, all standard degree-decomposition matrices M^j have been generated by the above-mentioned computer program. Matrices are listed in Tables 4, 5 and 6. The superscript j increases if we move from left to right along any row of the list as well as if we go down to a new row of matrices. The name M^j is put on a matrix only in case the matrix is referred to later on.

Table 4. Standard degree-decomposition matrices M^j for the remainder $\langle R\rangle = P_3$, $j = 1, 2, \ldots, 29$

53220 53220
32205 32205
03252 02253

43320 43320 43320 43320 43320 43320
43023 42033 42033 32043 33204 33024
02433 03423 03324 03324 02343 02343

43221 43221 43221 43221 43221 43221 43221 43221 43221 43221
32421 23421 23421 22431 32412 32412 23412 23412 23412 31422
12234 21234 22134 22134 12243 12234 22134 21243 22143 13224

43221 43221 43221 43221 43221 43221 43221 43221 43221
31422 32214 32214 32214 23214 23214 22314 31224 31224
12234 12432 13242 12243 21432 22341 21342 13422 12432

33222 33222
33222 32322
22332 23232

Table 5. Standard degree-decomposition matrices M^j for the remainder $\langle R\rangle = C_2$, $j = 30, \ldots, 38$

53220
30225
05223

43320 43320
43203 43023 (M^{31}, M^{32})
02343 02343

43221 43221 43221 43221 43221
32421 31422 31422 23214 31224
13224 14223 12243 22431 14223

33222
32322 (M^{38})
23322

Table 6. Standard degree-decomposition matrices M^j for the remainder $\langle R\rangle = 2K_2$, $j = 39, 40, \ldots, 46$

53220 53220
32205 23205 (M^{39}, M^{40})
02352 02352

43320
33024 (M^{41})
02433

43221 43221 43221 43221
22413 32214 23214 23214 ($M^{42}, \ldots, M^{45}$)
22143 12342 12342 21342

33222
32322 (M^{46})
22233

Theorem 3.2. $\left\lfloor \frac{^2K_5}{3} \right\rfloor = \{F^1, F^6, F^9, F^{11}, F^{14}, F^{15}\}$.

Proof. We first prove that each of the six listed multigraphs is a universal part for $n = 5$. To this end, the decomposition matrices are presented in Table 7.

We next show that if F^* is any of the remaining eleven multigraphs in Fig. 1 (and listed in Table 8), then a 3-remainder exists which is not realized by 3-packings of F^* in $^2K_5$, see Lemma 3.3. □

Table 7. Decomposition matrices for $n = 5$

F^1	F^6	F^9	F^{11}	F^{14}	F^{15}
		$\langle R\rangle = 2K_2$			
. 1 1 1 2	. 1 1 1 2	. 1 1 1 2	. 1 1 1 1	. 1 1 2 1	. 1 2 2 1
1 . 1 3 2	1 . 2 1 2	2 . 1 2 1	2 . 1 1 2	3 . 1 1 3	2 . 1 1 3
2 0 . 3 2	0 3 . 3 1	3 3 . 3 2	2 3 . 3 2	2 2 . 3 2	3 0 . 1 2
1 3 3 . 3	2 3 3 . 3	1 3 0 . 3	3 3 0 . 2	3 2 0 . 1	3 2 3 . 3
2 2 3 0 .	3 2 2 0 .	0 2 2 3 .	2 0 3 3 .	2 0 3 3 .	1 3 2 0 .
		$\langle R\rangle = C_2$			
. 1 1 1 2	. 1 1 1 2	. 1 1 1 2	. 1 1 1 1	. 1 1 2 1	. 1 1 1 2
1 . 3 1 3	1 . 1 2 3	2 . 1 3 1	3 . 1 1 2	3 . 2 1 1	3 . 2 2 1
1 3 . 0 2	2 3 . 0 2	2 3 . 2 2	2 3 . 0 2	2 3 . 1 2	2 3 . 1 3
2 3 0 . 2	3 3 0 . 1	1 3 2 . 0	2 3 0 . 2	3 2 3 . 0	3 3 2 . 0
2 3 2 3 .	2 3 3 2 .	3 3 3 0 .	2 3 3 3 .	2 3 3 0 .	2 1 3 0 .
		$\langle R\rangle = P_3$			
. 1 1 1 2	. 1 1 1 2	. 1 1 1 3	. 1 1 1 1	. 1 2 1 1	. 1 3 1 2
1 . 3 1 3	1 . 2 1 3	2 . 1 2 1	2 . 1 1 3	2 . 1 2 1	2 . 1 2 1
1 3 . 3 2	2 3 . 2 1	2 2 . 3 2	2 2 . 2 2	3 2 . 2 1	3 2 . 2 1
2 3 0 . 2	3 2 2 . 3	1 3 0 . 3	2 3 3 . 3	2 3 3 . 3	1 3 3 . 3
2 3 2 0 .	3 0 0 3 .	3 3 2 0 .	3 0 3 0 .	3 3 0 0 .	2 3 0 0 .

Lemma 3.3. *The multigraph F^3 is not a third part of ${}^2K_5$ with remainder C_2. Moreover, none of the remaining ten multigraphs in Fig.* 1 *is a third part with the remainder $2K_2$.*

Proof. There are 11 cases to deal with, see Table 8. For example, we consider the relatively difficult case of the multigraph F^5 and the matrix M^{44}. We first note that degree-3 and degree-4 vertices in each row of M^{44} are mutually doubly adjacent, see F^5 in Fig. 1. Therefore the degree-1 vertex in row 3 is adjacent to column 5 and that in row 2 to column 1. Consequently, since the degree-3 vertex in any row is adjacent to the neighbor of the degree-1 vertex, there is an excessive 1-2 edge in row 2, as stated in Table 8. In a similar way we deal with the multigraph F^{12} and the matrix M^{46}. Remaining cases are rather simple and detailed proofs can be derived from Table 8. □

Table 8. Obstructions for being universal

F^i	j in M^j	edges		j in M^j	edges	
F^2	41	4-5	lack			
F^3	31	4-5	excess	32	3-5	excess
F^4	41	1-2	excess			
F^5	42	3-4	lack	43	1-2 or 1-5	excess
	44	1-2	excess	45	1-2 or 2-5	excess
F^7, F^{10}	42	1-2	excess	43	3-4	excess
	44	2-5	excess	45	4-5	excess
F^8	42	1-5	lack or exc.	43	1-2	lack or exc.
	44	1-2	lack or exc.	45	1-2	lack or exc.
F^{12}	46	2-4 or 3-5	excess			
F^{13}	46	1-2 or 1-3	excess			
F^{16}	46	4-5	lack			
F^{17}	46	4-5	lack			

Corollary 3.4. *The following three sequences* $(5,3,2,2,0)$, $(4,3,2,2,1)$ *and* $(3,3,2,2,2)$ *are the only degree sequences among universal third parts of the complete* 2*-graph ${}^2K_5$.*

This implies the following observation to be used in what follows.

Corollary 3.5. *Standard degree-decomposition matrices of the universal third parts for the remainder C_2, see Table* 5*, are among matrices M^{30} and M^{33}–M^{38}.*

4. PROOF OF THEOREM 1.4

Lemma 4.1. *There are at most two half-degree sequences, namely,*

$$s^1 := (3,2,1,0,0), \quad s^2 := (2,2,1,1,0),$$

among universal third parts of DK_5.

Proof. Because $|R| = 2$, there are 5 shapes of a 3-remainder R in DK_5. The digraph $DK_5 - R$ has one of the following five sequences of degree pairs (outdegree, indegree):

$$\begin{array}{ll} ((4,4),(4,4),(4,4),(3,3),(3,3)) & \text{for } \langle R\rangle = \vec{C}_2, \\ ((4,4),(4,4),(4,3),(3,4),(3,3)) & \text{for } \langle R\rangle = \vec{P}_3, \\ ((4,4),(4,4),(4,3),(4,3),(2,4)) & \text{for } \langle R\rangle = \vec{P}_{3out}, \\ ((4,4),(4,4),(4,2),(3,4),(3,4)) & \text{for } \langle R\rangle = \vec{P}_{3in}, \\ ((4,4),(4,3),(4,3),(3,4),(3,4)) & \text{for } \langle R\rangle = 2\vec{P}_2. \end{array}$$

Therefore any corresponding half-degree sequence (indegree or outdegree alike) of $DK_5 - R$ is one of the two sequences $B = (4,4,4,4,2)$ or $B = (4,4,4,3,3)$. Suppose that F is a universal third part of DK_5. Then F is of size 6. Assume that $\beta = (b_1, \ldots, b_5)$ is a half-degree sequence of F. Then the corresponding half-degree-decomposition matrices for universal F impose the following condition to be satisfied for either choice of B.

(b) Fix any $B =: (B_1, \ldots, B_5)$. Then there exist three permutations $\sigma_1, \sigma_2, \sigma_3$ such that $b_i + b_{\sigma_1(i)} + b_{\sigma_2(i)} = B_{\sigma_3(i)}$ for $i = 1, \ldots, 5$ and, moreover, for either choice of B.

There are 8 partitions of 6 into at most five summands of which the largest is at most 4. Each of the partitions gives rise to a half-degree sequence, say $\tilde{\beta}$, of a digraph on 5 vertices, see Table 9 wherein the symbol + indicates that the condition (b) is satisfied. It is easy to see that Table 9 is correct. Hence it follows that β, if exists, has the symbol + twice and therefore is as stated. □

Table 9. Partitions of 6 versus the condition (b)

$\tilde{\beta}\backslash B$	(4,4,4,3,3)	(4,4,4,4,2)
(4,2,0,0,0)	-	+
(4,1,1,0,0)	+	-
(3,3,0,0,0)	-	-
(3,2,1,0,0)	+	+
(3,1,1,1,0)	+	-
(2,2,2,0,0)	-	+
(2,2,1,1,0)	+	+
(2,1,1,1,1)	+	-

In what follows we use the abbreviation DP for *degree pair* in the names DP-*sequence* and DP-*decomposition matrix*, the counterparts of degree sequence and degree-decomposition matrix, respectively.

Using results Theorem 3.2 and Corollary 3.4 on all universal third parts of ${}^2K_5$ we show the following result, which completes the proof of Theorem 1.4.

Lemma 4.2. *No universal third part of ${}^2K_5$ has an orientation which could be a universal third part of DK_5.*

Proof. Assume that $((c_1, d_1), \ldots, (c_5, d_5))$ is a DP-sequence of a universal third part of DK_5. Then the following two conditions are satisfied.

(i) Both $(c_1, \ldots c_5)$ and $(d_1, \ldots d_5)$ are permutations of either sequence s^1 or s^2, or both (by Lemma 4.1).

(ii) The sequence $(c_1 + d_1, \ldots, c_5 + d_5)$ is a permutation of one of the sequences $(5, 3, 2, 2, 0)$, $(4, 3, 2, 2, 1)$ and $(3, 3, 2, 2, 2)$.

The condition (ii) follows from Corollary 3.4.

For each of the three degree sequences in Corollary 3.4, we use three decision trees (namely, s^1-s^1 tree, s^2-s^2 tree and s^1-s^2 tree) in order to split the degree sequence into all possible DP-sequences $((c_1, d_1), \ldots, (c_5, d_5))$ such that both conditions (i) and (ii) are satisfied. The converse s^2-s^1 splits are omitted wlog. Results of this procedure are shown in Table 10.

Table 10. DP-sequences satisfying the conditions (i) and (ii)

(5,3,2,2,0)	s^1-s^1	((3,2),(2,0),(1,1),(0,3),(0,0))
		((3,0),(2,3),(1,1),(0,2),(0,0))
	s^1-s^2	((3,2),(2,1),(1,1),(0,2),(0,0))
(4,3,2,2,1)	s^1-s^1	((3,1),(2,0),(1,0),(0,3),(0,2))
		((3,0),(2,0),(1,3),(0,2),(0,1))
	s^1-s^2	((3,1),(2,1),(1,0),(0,2),(0,2))
		((3,1),(2,0),(1,2),(0,2),(0,1))
		((3,0),(2,2),(1,1),(0,2),(0,1))
	s^2-s^2	((2,2),(2,1),(1,1),(1,0),(0,2))
		((2,2),(2,0),(1,2),(1,1),(0,1))
(3,3,2,2,2)	s^1-s^2	((3,0),(2,1),(1,1),(0,2),(0,2))
M^{38}	s^1-s^1	((3,0),(2,0),(1,1),(0,2),(0,3))
F^{14}, F^{15}	s^2-s^2	((2,1),(2,0),(1,2),(1,1),(0,2))

In order to complete the proof, we show that in each of degree-decomposition matrices for the remainder C_2, namely, in matrices M^{30} and M^{33}–M^{38}, see Corollary 3.5, there is a pair of columns which cannot be split into two columns of degree pairs (taken from Table 10) which could represent an orientation in any of underlying multigraphs. In this way we show that no orientation in question can realize the remainder $\vec{C_2}$. Those columns follow:

$$\begin{matrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \end{matrix} \ (M^{30}, M^{37}, M^{38}), \quad \begin{matrix} 2 & 2 \\ 4 & 2 \\ 2 & 2 \end{matrix} \ (M^{33}, M^{34}), \quad \begin{matrix} 2 & 2 \\ 4 & 2 \\ 2 & 4 \end{matrix} \ (M^{35}), \quad \begin{matrix} 4 & 2 \\ 2 & 2 \\ 2 & 4 \end{matrix} \ (M^{36}).$$

Consider the case of a pair of all-2 columns in an M^j, $j = 30, 37, 38$. Each corresponding DP column has the sum $(3, 3)$ in order to ensure that the remainder is $\vec{C_2}$. However, the degree pair $(1, 1)$ appears at most once among DP-sequences in Table 10. Consequently, both all-2 columns in M^j split into columns with distinct degree pairs $(1, 1), (2, 0), (0, 2)$. Such degree pairs, all without any repetition, are available in Table 10 in the last two DP-sequences only. These are to be DP-sequences in a required orientation of F^{14} and/or F^{15}. Moreover, it follows that rows in the two DP columns are to represent distinct pairs of degree-2 vertices. Since degree-2 vertices are not independent in those multigraphs, no required orientation exists in this case.

It remains to deal with the next three pairs of columns listed above, with degree sequence $(4, 3, 2, 2, 1)$, and with corresponding DP-sequences in Table 10. Then each degree column with sum 8 comprises degree 4 and twice degree 2. Moreover, the column should split into DP column with sum $(4, 4)$. On the other hand, 4 as a degree splits into DP $(3, 1)$, $(1, 3)$ or $(2, 2)$, see Table 10. It is easy to see that the required splitting comprising DP's $(2, 2)$ and twice $(1, 1)$ is the only possible. Then no splitting exists for the accompanying column because $(1, 1)$ appears only once in the related DP-sequences. □

Open problem. Decide the existence of a universal third part of DK_n with $n \geq 8$ and $n \bmod 3 = 2$. The existence for $n = 8$ would solve the problem.

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