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GLOBAL EXISTENCE AND ASYMPTOTIC BEHAVIOR FOR A NONLINEAR DEGENERATE SIS MODEL

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Abstract. In this paper we investigate the global existence and asymptotic behavior of a reaction diffusion system with degenerate diffusion arising in the modeling and the spatial spread of an epidemic disease.

Keywords: reaction diffusion systems, degenerate diffusion, global existence, asymptotic behavior, population dynamics.

Mathematics Subject Classification: 35K55, 35K57, 35K65, 35B40, 92D30.

1. INTRODUCTION

In this paper we study global existence and asymptotic behavior for a degenerate parabolic system of the form

$$\begin{cases} S_t - \Delta\phi_S(S) = -\, I\left(\gamma S - \delta\right), \\ I_t - \Delta\phi_I(I) = I\left(\gamma S - \delta\right), \end{cases} \qquad (x,t) \in \Omega \times (0,T) = Q_T, \tag{1.1}$$

in $\Omega \times (0,\infty)$, subject to the initial conditions

$$S\left(x,0\right) = S_0\left(x\right), \quad I\left(x,0\right) = I_0\left(x\right), \quad x \in \Omega, \tag{1.2}$$

and to the Neumann boundary conditions

$$\frac{\partial\phi_S(S)}{\partial\eta}(x,t) = \frac{\partial\phi_I(I)}{\partial\eta}(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T). \tag{1.3}$$

Herein, Ω is an open, bounded and connected domain in $\mathbb{R}^N$, $N \geq 1$, with a smooth boundary $\partial\Omega$, Δ is the Laplace operator in $\mathbb{R}^N$, $I_0,\ S_0 \in C\left(\overline{\Omega}\right)$, S_0, $I_0 \geq 0$. Finally, for $k \in \{S, I\}$, $\phi_k \in C^2(\mathbb{R})$, $\phi_k(0) = \phi_k'(0) = 0$ and $\phi_k(s) > 0$, $\phi_k'(s) > 0$ for $s > 0$.

This problem leads to the so-called $(S - I - S)$ model: S, I represent respectively the densities of susceptibles and infectives, γSI is the force of infection or the incidence term, it represents the number of susceptible individuals S infected by contact with infective individuals I per time unit, finally δI is the number of infectives who become susceptibles after recovery.

System (1.1)–(1.3) is uniformly parabolic in the region $D = [S \neq 0] \cap [I \neq 0]$ and degenerate into first order equations on $Q_T \setminus D$. Note that degenerate diffusion is a good approach in modeling slow diffusion of individuals in the spatial spread of an epidemic disease, see Okubo [14].

In the spatially homogeneous case we found one of the models of propagation of an epidemic disease described in [6, 11]. In fact that model deals with susceptibles, infectives and removed, but if we eliminate the removed ones by adding them to susceptibles we form the model below, without demography (no new borns or deaths) and in that setting it is well known that when $t \to \infty$

$$\begin{cases} (S, I) \to (S_0 + I_0, 0) & \text{if} \quad S_0 + I_0 \leq \frac{\delta}{\gamma}, \\ (S, I) \to \left(\frac{\delta}{\gamma}, S_0 + I_0 - \frac{\delta}{\gamma}\right) & \text{otherwise.} \end{cases} \tag{1.4}$$

A comprehensive analysis of generic models with linear diffusion is initiated in Fitzgibbon and Langlais [8] and Fitzgibbon *et al.* [9]. These models include a logistic effect on the demography, yielding $L^1(\Omega)$ a priori estimates on solutions independent of the initial data for large time; this allows to use a bootstrapping argument to show global existence and exhibit a global attractor. Finally quasilinear but non degenerate systems of the form (1.1) was investigated by Fitzgibbon *et al.* [10].

For degenerate reaction-diffusion equations, and in the case where $\phi_K(s) = s^m$, $m > 1$ was studied by Aliziane and Moulay [5], Aliziane and Langlais [3,4] studied the SEIR model. Finally Hadjadj *et al.* [2] studied the case where the source term depends on the gradient of solution, they resolved the problem of existence of globally bounded weak solutions or blow-up, depending on the relations between the parameters that appear in the problem.

This paper is organized as follows. In Section 2 the notion of a weak solution is introduced and we state our main results. In Section 3 we will construct our solution as a limit of solutions of quasilinear and nondegenerate problems depending on a parameter ε, derive uniform a priori estimates on these solutions, and prove existence, uniqueness and regularity results in Section 4. Finally, in the last section we prove the large time behavior results which generalize (1.4).

2. MAIN RESULTS

2.1. BASIC ASSUMPTIONS AND NOTATIONS

Herein, Ω is an open, bounded and connected domain of the N-dimensional Euclidean space $\mathbb{R}^N$, $N \geq 1$, with a smooth boundary $\partial\Omega$, a $(N-1)$-dimensional manifold so that locally Ω lies on one side of $\partial\Omega$, $x = (x_1, \ldots, x_N)$ is the generic element of $\mathbb{R}^N$.

The gradient with respect to x is ∇ and the Laplace operator in $\mathbb{R}^N$ is Δ, $sign_\varepsilon$ is a smooth approximation of the function signum, finally if r is a real number, then we set $r^+ = \sup(r,0)$, $r^- = \sup(-r,0)$.

Then we set $\Omega \times (0,T) = Q_T$ and for $0 \leq \tau < T$, $\Omega \times (\tau,T) = Q_{\tau,T}$. The norm in $L^p(\Omega)$ is $\|\cdot\|_{p,\Omega}$ and the norm in $L^p(Q_{\tau,T})$ is $\|\cdot\|_{p,Q_{\tau,T}}$ for $1 \leq p \leq \infty$. Finally $H^1(Q_{\tau,T}) = H^1(\Omega) \times (\tau,T)$.

2.2. MAIN RESULTS

It is well known that the general problem (1.1)–(1.3) has no classical solutions. A suitable notion of a generalized solution is required. We adopt the notion of a weak solution introduced by Oleinik *et al.* [15].

Definition 2.1. A couple of nonnegative and continuous functions (S,I) is a solution of system (1.1)–(1.3) in Q_T, $T>0$, for each $\varphi,\psi \in C^1(\bar{Q}_T)$, such that $\frac{\partial\varphi}{\partial\eta} = \frac{\partial\psi}{\partial\eta} = 0$ on $\partial\Omega \times (0,T)$.

1. $\nabla\phi_S(S)$, $\nabla\phi_I(I)$ exist in a distributional sense and $\nabla\phi_S(S)$, $\nabla\phi_I(I) \in L^2(Q_T)$;
2. S and I verify the identities:

a) $\int\limits_\Omega S(x,T)\varphi(x,T)\,dx - \int\limits_0^T\int\limits_\Omega [S\varphi_t - \nabla\phi_S(S)\nabla\varphi - I(\gamma S-\delta)\varphi]\,dxdt =$
$= \int\limits_\Omega S(x,0)\varphi(x,0)\,dx,$

b) $\int\limits_\Omega I(x,T)\psi(x,T)\,dx - \int\limits_0^T\int\limits_\Omega [I\psi_t - \nabla\phi_I(I)\nabla\psi + I(\gamma S-\delta)\psi]\,dxdt =$
$= \int\limits_\Omega I(x,0)\psi(x,0)\,dx.$

We are now ready to state our results.

Theorem 2.2. *For each initial non negative data (S_0,I_0) in $C(\overline{\Omega}) \times C(\overline{\Omega})$ there exists a unique weak solution (S,I) of problem (1.1)–(1.3) on Q_∞. Furthermore,*

(i) $S \in C(Q_\infty) \cap L^\infty(Q_\infty)$,
(ii) $I \in C(Q_\infty)$; *and if* $S_0 \leq \frac{\delta}{\gamma}$, *then* $I \in C(Q_\infty) \cap L^\infty(Q_\infty)$.

Remark 2.3. These results can be extended to the case $S_0, I_0 \in L^\infty(\Omega)$ with $S,I \in C([\tau,T];L^\infty(\Omega))$ in the definition of the weak solution and using results of Di Benedetto [7] to get:

(i) $S \in C(Q_{\tau,\infty}) \cap L^\infty(Q_\infty)$ for all $\tau > 0$,
(ii) $I \in C(Q_{\tau,T})$; and if $S_0 \leq \frac{\delta}{\gamma}$, then $I \in C(Q_{\tau,\infty}) \cap L^\infty(Q_\infty)$ for all $\tau > 0$.

For the large time behavior of the weak solution, we obtain the following result.

Theorem 2.4. *Assume* $0 \leq S_0 \leq \frac{\delta}{\gamma}$ *and let* $M = \frac{1}{|\Omega|}\int\limits_\Omega (S_0+I_0)(x)dx$. *Then:*

1. *if* $M \leq \frac{\delta}{\gamma}$, *then*
$$\lim_{t\to\infty} S(t,\cdot) = M \quad \text{and} \quad \lim_{t\to\infty} I(t,\cdot) = 0 \quad \text{in } C(\bar{\Omega});$$
2. *if* $M > \frac{\delta}{\gamma}$, *then*
$$\lim_{t\to\infty} S(t,\cdot) = \frac{\delta}{\gamma} \quad \text{and} \quad \lim_{t\to\infty} I(t,\cdot) = M - \frac{\delta}{\gamma} \quad \text{in } C(\bar{\Omega}).$$

3. AUXILIARY PROBLEM AND A PRIORI ESTIMATES

In this section we consider in $\Omega \times (0, \infty)$ the auxiliary quasilinear non-degenerate system

$$\begin{cases} S_t - \Delta d_1(S) = -(I - \varepsilon)\left(\gamma(S - \varepsilon) - \delta\right), \\ I_t - \Delta d_2(I) = (I - \varepsilon)\left(\gamma(S - \varepsilon) - \delta\right), \end{cases} \quad (x, t) \in \Omega \times (0, T) = Q_T, \tag{3.1}$$

subject to the initial and boundary conditions

$$\begin{cases} S(x, 0) = S_{0,\varepsilon}(x), \;\; I(x, 0) = I_{0,\varepsilon}(x), \quad x \in \Omega, \\ \dfrac{\partial d_1(S)}{\partial n}(x, t) = \dfrac{\partial d_2(I)}{\partial n}(x, t) = 0, \qquad (x, t) \in \partial\Omega \times (0, T). \end{cases} \tag{3.2}$$

d_1, $d_2 : \mathbb{R}^N \to (\frac{\varepsilon}{2}, \infty)$ are smooth and increasing functions with

$$d_1(S) = \phi_S(S), \varepsilon \leq S, \quad \text{and} \quad d_2(I) = \phi_I(I), \varepsilon \leq I. \tag{3.3}$$

If $U_{0,\varepsilon}$ represents one of the smooth functions $S_{0,\varepsilon}$ or $I_{0,\varepsilon}$ over $\overline{\Omega}$, then we require

$$\begin{cases} U_{0,\varepsilon}(x) \geq \varepsilon, \; x \in \Omega, \; 0 < \varepsilon \leq 1, \\ \displaystyle\int_\Omega (U_{0,\varepsilon}(x) - \varepsilon) dx = \int_\Omega U_0(x) dx, \\ U_{0,\varepsilon} \to U_0 \text{ in } C(\Omega) \text{ as } \varepsilon \to 0. \end{cases} \tag{3.4}$$

We refer to [1] for a construction of such a set of initial data. From standard results, [12, Theorem 7.4], local existence and uniqueness of a classical solution $(S_\varepsilon, I_\varepsilon)$ of (3.1)–(3.2) in some maximal interval $[0, T_{max,\varepsilon})$ are guaranteed.

It is easy to check that $[\varepsilon, \infty)^2$ is an invariant region (see [16]), thus

$$0 < \varepsilon \leq S_\varepsilon(x, t), 0 < \varepsilon \leq I_\varepsilon(x, t), \quad x \in \Omega, \; 0 < t < T_{max,\varepsilon}. \tag{3.5}$$

Then one can apply results in [10] to show global existence, i.e. $T_{max,\varepsilon} = \infty$, of a classical solution for (3.1)–(3.2). Using (3.3) and (3.5) we obtain global existence for

$$\begin{cases} S_t - \Delta\phi_S(S) = -\left(I - \varepsilon\right)\left(\gamma(S - \varepsilon) - \delta\right), \\ I_t - \Delta\phi_I(I) = (I - \varepsilon)\left(\gamma(S - \varepsilon) - \delta\right), \end{cases} \quad (x, t) \in \Omega \times (0, T) = Q_T, \tag{3.6}$$

in $\Omega \times (0, \infty)$, together with (3.2).

We derive a priori estimates. First, adding the two equations in (3.1) and using a straightforward integration one can derive the conservation of the total mass:

$$\int_\Omega S_\varepsilon(x,T)\,dx + \int_\Omega I_\varepsilon(x,T)\,dx = \int_\Omega S_{0,\varepsilon}(x) + I_{0,\varepsilon}(x)\;dx, \quad T \geq 0. \tag{3.7}$$

In what follows, T is a positive number, $M_1, \ldots, M_n$ are positive constants independent of $T, \varepsilon, 0 < \varepsilon \leq 1$, and $F_1, \ldots, F_n$ are non decreasing functions of T independent of ε.

Lemma 3.1. *There exist a constant M_1 and a nondecreasing function F_1 independent of $\varepsilon \in (0,1]$ such that*

$$0 < \varepsilon \leq S_\varepsilon(x,t) \leq M_1, \qquad x \in \Omega,\; t \geq 0, \tag{3.8}$$

$$0 < \varepsilon \leq I_\varepsilon(x,t) \leq F_1(T), \qquad x \in \Omega,\;\; 0 \leq t \leq T. \tag{3.9}$$

If $0 \leq S_0(x) \leq \frac{\delta}{\gamma}$, *then F_1 is a constant.*

Proof. As $\|S_{0,\varepsilon}\|_{\infty,\Omega} + \frac{\delta}{\gamma}$ is a supersolution of equation for S_ε in (3.1)–(3.2), estimation (3.8) follows.

Multiplying the equation for I_ε by $p(I_\varepsilon - \varepsilon)^{p-1}$, $p \geq 1$ and integrating over Ω, we obtain

$$\frac{d}{dt}\int_\Omega (I_\varepsilon - \varepsilon)^p(x,t)dx \leq p\gamma \int_\Omega (I_\varepsilon - \varepsilon)^p(S_\varepsilon - \varepsilon)(x,t)dx, \qquad t \geq 0.$$

Estimation (3.8) and Gronwall's inequality lead to estimate (3.9).

Now if $0 \leq S_0(x) \leq \frac{\delta}{\gamma}$, we can construct $S_{0,\varepsilon}$ such that $0 \leq S_{0,\varepsilon}(x) \leq \frac{\delta}{\gamma} + \varepsilon$, then by the maximum principle applied to the equation for S_ε we obtain

$$0 \leq S_\varepsilon(x,t) - \varepsilon \leq \frac{\delta}{\gamma}, \qquad x \in \Omega,\; t \geq 0. \tag{3.10}$$

A second application of the maximum principle to the equation for I_ε gives

$$\varepsilon \leq I_\varepsilon(x,t) \leq \|I_0\|_{\infty,\Omega} + 1, \qquad x \in \Omega,\;\; 0 \leq t \leq T.$$

□

Lemma 3.2. *There exists a constant M_2 independent of $\varepsilon \in (0,1]$ such that*

$$\int_0^\infty \int_\Omega (I_\varepsilon - \varepsilon)\left(\delta - \gamma(S_\varepsilon - \varepsilon)\right)^2 dx dt \leq M_2. \tag{3.11}$$

Proof. Let $U_\varepsilon = (\gamma(S_\varepsilon - \varepsilon) - \delta)$. Then the equation for S_ε can be written as

$$\begin{cases} U_t - div(\phi_S'(S_\varepsilon)\nabla U) + \gamma(I-\varepsilon)U = 0, & (x,t) \in \Omega \times (0,T), \\ \frac{\partial U}{\partial n}(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ U(x,0) = \gamma(S_{0,\varepsilon} - \varepsilon) - \delta, & x \in \Omega. \end{cases} \tag{3.12}$$

We multiply (3.12) by U_ε and integrate over Q_T to find

$$\frac{1}{2}\int_\Omega U_\varepsilon^2(x,T)dx + \int_0^T\int_\Omega \phi_S'(S_\varepsilon)\left|\nabla U_\varepsilon\right|^2(x,t)dxdt \;+\gamma\int_0^T\int_\Omega (I_\varepsilon-\varepsilon)U_\varepsilon^2 dxdt =$$
$$= \frac{1}{2}\int_\Omega \left(\gamma(S_{0,\varepsilon}-\varepsilon)-\delta\right)^2(x)dx.$$

Then the estimate (3.11) follows by (3.4). □

Lemma 3.3. *There exists a nondecreasing function F_2 independent of $\varepsilon\in(0,1]$ such that*

$$\int_0^T\int_\Omega \left|\nabla\phi_S(S_\varepsilon)\right|^2(x,t)dxdt + \int_0^T\int_\Omega \left|\nabla\phi_I(I_\varepsilon)\right|^2(x,t)dxdt \le F_2(T). \tag{3.13}$$

If $0\le S_0(x)\le\frac{\delta}{\gamma}$ for each $x\in\Omega$, then F_2 is a constant.

Proof. The estimate on $\nabla\phi_I(I_\varepsilon)$ is obtained upon multiplying the equation for I_ε by $\phi_I(I_\varepsilon)$ and integrating over $Q_T=\Omega\times(0,T)$:

$$\begin{aligned}&\int_\Omega \Phi_I(I_\varepsilon(x,T))dx + \int_0^T\int_\Omega \left|\nabla\phi_I(I_\varepsilon)\right|^2 dxdt =\\ &= \int_\Omega \Phi_I(I_{0,\varepsilon}(x))dx + \int_0^T\int_\Omega \phi_I(I_\varepsilon)\left(I_\varepsilon-\varepsilon\right)\left(\gamma(S_\varepsilon-\varepsilon)-\delta\right)(x,t)\,dxdt,\end{aligned} \tag{3.14}$$

where

$$\Phi_I(I)=\int_0^I \phi_I(s)ds.$$

By the Cauchy inequality and (3.9), one has

$$\begin{aligned}&\int_\Omega \Phi_I(I_\varepsilon(x,T))dx + \int_0^T\int_\Omega \left|\nabla\phi_I(I_\varepsilon)\right|^2 dxdt \;\le \int_\Omega \Phi_I(I_{0,\varepsilon}(x))dx+\\ &+T^{\frac{1}{2}}mes^{\frac{1}{2}}(\Omega)\;F_1\phi_I(F_1(T))\left(\int_0^T\int_\Omega (I_\varepsilon-\varepsilon)\left(\delta-\gamma(S_\varepsilon-\varepsilon)\right)^2(x,t)dxdt\right)^{\frac{1}{2}}.\end{aligned} \tag{3.15}$$

Using Lemma 3.2 one obtains the desired estimate. If $0\le S_0(x)\le\dfrac{\delta}{\gamma}$ for each $x\in\Omega$, putting (3.10) in (3.14) we obtain

$$\int_0^\infty\int_\Omega \left|\nabla\phi_I(I_\varepsilon)\right|^2 dxdt \;= \int_\Omega \Phi_I(I_{0,\varepsilon}(x))dx. \tag{3.16}$$

Now to obtain an estimate on ∇S_ε we multiply the equation for S_ε by S_ε^m, integrate over Q_T and use the Cauchy inequality and Lemma 3.1. We have

$$\begin{aligned} &\int_\Omega \Phi_S(S_\varepsilon(x,T))dx + \int_0^T\int_\Omega |\nabla\phi_S(S_\varepsilon)|^2\,dxdt = \\ &= \int_\Omega \Phi_S(S_{0,\varepsilon}(x))dx + \int_0^T\int_\Omega \phi_S(S_\varepsilon)\,(I_\varepsilon-\varepsilon)\,(\delta-\gamma(S_\varepsilon-\varepsilon))\,(x,t)\,dxdt \le \end{aligned} \tag{3.17}$$

$$\begin{aligned} &\le \int_\Omega \Phi_S(S_{0,\varepsilon}(x))dx + \\ &\quad + \phi_S(M_1)F_1 mes^{\frac{1}{2}}(\Omega)\left(\int_0^T\int_\Omega (I_\varepsilon-\varepsilon)\,(\delta-\gamma(S_\varepsilon-\varepsilon))^2\,(x,t)dxdt\right)^{\frac{1}{2}}. \end{aligned} \tag{3.18}$$

Then the estimate on ∇S_ε in (3.13) follows from (3.18) and (3.11).

If $0 \le S_0(x) \le \dfrac{\delta}{\gamma}$, $x \in \Omega$, by integrating equation in I_ε one gets

$$\int_0^\infty\int_\Omega (I_\varepsilon-\varepsilon)\,(\delta-\gamma(S_\varepsilon-\varepsilon))\,dxdt \le \int_\Omega I_{0,\varepsilon}(x)dx. \tag{3.19}$$

Then by (3.19) and (3.17), we have

$$\int_0^T\int_\Omega |\nabla\phi_S(S_\varepsilon)|^2\,dxdt \le \int_\Omega \Phi_S(S_{0,\varepsilon}(x))dx + \phi_S\left(\frac{\delta}{\gamma}\right)\int_\Omega I_{0,\varepsilon}(x)dx. \qquad \square$$

Lemma 3.4. *There exist nondecreasing functions F_3, F_4 independent of ε, $0<\varepsilon\le 1$, such that for all $t>t_0>0$,*

$$\int_\Omega |\nabla\phi_S(S_\varepsilon)|^2\,(x,t)\,dx + \int_\Omega |\nabla\phi_I(I_\varepsilon)|^2\,(x,t)\,dx \le F_3(t),$$

$$\int_{t_0}^t\int_\Omega |(\phi_S(S_\varepsilon))_t|^2\,dxds + \int_{t_0}^t\int_\Omega |(\phi_I(I_\varepsilon))_t|^2\,dxds \le F_4(t).$$

If $0 \le S_0(x) \le \frac{\delta}{\gamma}$ for each $x \in \Omega$, then F_3 and F_4 are constants.

Proof. We multiply the equation for S_ε by $(\phi_S(S_\varepsilon))_t$ and integrate over $\Omega\times(\tau,t)$, $\frac{t}{2}\le\tau\le t\le T$, to find

$$\int_\tau^t\int_\Omega \phi'_S(S_\varepsilon)\,(S_\varepsilon)_t^2\,(x,s)dxds+\frac{1}{2}\int_\Omega\left|\nabla\phi_S(S_\varepsilon)\right|^2(x,t)\,dx=$$
$$=\int_\tau^t\int_\Omega (\phi_S(S_\varepsilon))_t\,(I_\varepsilon-\varepsilon)(\delta-\gamma(S_\varepsilon-\varepsilon))(x,s)dxds+\frac{1}{2}\int_\Omega\left|\nabla\phi_S(S_\varepsilon)\right|^2(x,\tau)\,dx, \tag{3.20}$$

but

$$\int_\tau^t\int_\Omega (\phi_S(S_\varepsilon))_t\,(I_\varepsilon-\varepsilon)(\delta-\gamma(S_\varepsilon-\varepsilon))(x,s)dxds\le$$
$$\le\frac{1}{2}\int_\tau^t\int_\Omega \phi'_S(S_\varepsilon)\,(S_\varepsilon)_t^2\,(x,s)dxds+ \tag{3.21}$$
$$+\frac{1}{2}\left\|\phi'_S(S_\varepsilon)\right\|_{\infty,\Omega}\left\|I_\varepsilon-\varepsilon\right\|_{\infty,\Omega}\int_\tau^t\int_\Omega(I_\varepsilon-\varepsilon)(\delta-\gamma(S_\varepsilon-\varepsilon))^2(x,s)dxds.$$

Putting this estimate in (3.20) one obtains

$$\int_\Omega\left|\nabla\phi_S(S_\varepsilon)\right|^2(x,t)\,dx\le$$
$$\le\int_\Omega\left|\nabla\phi_S(S_\varepsilon)\right|^2(x,\tau)\,dx+$$
$$+\left\|\phi'_S(S_\varepsilon)\right\|_{\infty,Q_T}\left\|I_\varepsilon\right\|_{\infty,Q_T}\int_\tau^t\int_\Omega(I_\varepsilon-\varepsilon)(\delta-\gamma(S_\varepsilon-\varepsilon))^2(x,s)dxds.$$

Integrating this inequality in τ over $(\frac{t}{2},t)$ one finds

$$\int_\Omega\left|\nabla\phi_S(S_\varepsilon)\right|^2(x,t)\,dx\le$$
$$\le\frac{2}{t}\int_{\frac{t}{2}}^t\int_\Omega\left|\nabla\phi_S(S_\varepsilon)\right|^2(x,\tau)\,dxd\tau+$$
$$+\left\|\phi'_S(S_\varepsilon)\right\|_{\infty,Q_T}\left\|I_\varepsilon\right\|_{\infty,Q_T}\int_{\frac{t}{2}}^t\int_\Omega(I_\varepsilon-\varepsilon)(\delta-\gamma(S_\varepsilon-\varepsilon))^2(x,s)dxds.$$

The estimate for $\nabla\phi_S(S_\varepsilon)$ follows by Lemmas 3.1, 3.2 and 3.3. In the same way one can obtain the estimate for $\nabla\phi_I(I_\varepsilon)$.

The estimate for $(\phi_S(S_\varepsilon))_t$ is immediately deduced from (3.20), (3.21) and Lemma 3.3 keeping in mind that

$$|(\phi_S(S_\varepsilon))_t|^2 = (\phi_S'(S_\varepsilon))^2 (S_\varepsilon)_t^2 \le \Big(\sup_{0<s\le\|S_\varepsilon\|_{\infty,Q_T}} \phi_S'(s)\Big)\phi_S'(S_\varepsilon)\,(S_\varepsilon)_t^2\,.$$

The estimate for $(\phi_I(I_\varepsilon))_t$ follows immediately. □

4. PROOFS FOR THE EXISTENCE AND UNIQUENESS

In this section we supply a quick proof of Theorem 2.2.

4.1. EXISTENCE

From estimates established in the previous section one has $(S_\varepsilon - \varepsilon)_{0<\varepsilon\le1}$ and $(\nabla\phi_S(S_\varepsilon))_{0<\varepsilon\le1}$ are respectively bounded in $L^2(Q_T)$ and $\left(L^2(Q_T)\right)^N$ for a fixed $T > 0$. Then there exists two sequences which are still denoted $(S_\varepsilon - \varepsilon)_{0<\varepsilon\le1}$ and $(\nabla\phi_S(S_\varepsilon))_{0<\varepsilon\le1}$ such that $(S_\varepsilon - \varepsilon)_{0<\varepsilon\le1}$ converges weakly to some function S in $L^2(Q_T)$ and $(\nabla\phi_S(S_\varepsilon))_{0<\varepsilon\le1}$ converges weakly to V in $\left(L^2(Q_T)\right)^N$. On the other hand $(S_\varepsilon)_{0<\varepsilon\le1}$ is bounded in $L^\infty(Q_T)$; using the weak formulation of the equation for S_ε one can invoke the results in Di Benedetto [7] to get $(S_\varepsilon)_{0<\varepsilon\le1}$ relatively compact in $C\left(\overline{\Omega}\times[0,T]\right)$. It follows that $(S_\varepsilon - \varepsilon)_{0<\varepsilon\le1}$converges to S in $C\left(\overline{\Omega}\times[0,T]\right)$ and $(\phi_S(S_\varepsilon))_{0<\varepsilon\le1}$ converges to $\phi_S(S)$ in $C\left(\overline{\Omega}\times[0,T]\right)$. As a first consequence of this, $V = \nabla\phi_S(S)$. By the same way one can prove that there is a function I such that $(I_\varepsilon)_{0<\varepsilon\le1}$ converges to I in $C\left(\overline{\Omega}\times[0,T]\right)$ and $(\nabla\phi_I(I))_{0<\varepsilon\le1}$ converges weakly to $\nabla\phi_I(I)$ in $\left(L^2(Q_T)\right)^N$. Now let us multiply equation for S_ε in (3.6) by φ, equation for I_ε by ψ, integrate by parts over $\Omega\times(0,T)$ and let ε goes to zero, to conclude that (S, I) is the desired solution.

The regularity results for $\nabla\phi_S(S)$, $\nabla\phi_I(I)$, $(\phi_S(S))_t$ and $(\phi_I(I))_t$ follow from the a priori estimates in Lemmas 3.1, 3.2 and 3.4.

4.2. UNIQUENESS

The uniqueness is obtained by choosing an adequate test function in the definition of the weak solution as in [13].

Let (S_1, I_1) and (S_2, I_2) be two weak solutions of problem (1.1)–(1.3). They verify the integral identity

$$\begin{aligned}&\int_\Omega (S_1 - S_2)(x,T)\varphi_1(x,T)dx - \int_{Q_T} (\phi_S(S_1) - \phi_S(S_2))\Delta\varphi_1(x,t)dxdt = \\ &= \int_{Q_T} [\partial_t\varphi_1(S_1 - S_2) - (f(S_1, I_1) - f(S_2, I_2))\varphi_1](x,t)dxdt\end{aligned} \tag{4.1}$$

and

$$\int\limits_{\Omega}(I_1-I_2)(x,T)\varphi_2(x,T)dx-\int\limits_{Q_T}(\phi_I(I_1)-\phi_I(I_2))\Delta\varphi_2(x,t)dxdt=$$
$$=\int\limits_{Q_T}[\partial_t\varphi_2(I_1-I_2)+(f(S_1,I_1)-f(S_2,I_2))\varphi_2](x,t)dxdt \tag{4.2}$$

for every $\varphi_i \in C^1(\bar{Q}_T)$, $i=1,2$, such that $\dfrac{\partial\varphi_i}{\partial\eta}=0$ on $\partial\Omega\times(0,T)$ and $\varphi_i>0$, where $f(S,I)=I\,(\gamma S-\delta)$. Let us introduce two functions ψ_1,ψ_2 as follows:

$$\psi_1(x,t)=\begin{cases}\dfrac{\phi_S(S_1)-\phi_S(S_2)}{S_1-S_2}(x,t), & \text{if } S_1\neq S_2,\\ 0, & \text{otherwise,}\end{cases}$$

$$\psi_2(x,t)=\begin{cases}\dfrac{\phi_I(I_1)-\phi_I(I_2)}{I_1-I_2}(x,t), & \text{if } I_1\neq I_2,\\ 0, & \text{otherwise.}\end{cases}$$

Let us consider a sequence of smooth functions $(\psi_{i,\varepsilon})_{\varepsilon\geq 0}$ such that $\psi_{i,\varepsilon}\geq\varepsilon$, $\psi_{i,\varepsilon}$ is uniformly bounded in $L^\infty(Q_T)$ and

$$\lim_{\varepsilon\to 0}\|(\psi_{i,\varepsilon}-\psi_i)/\sqrt{\psi_{i,\varepsilon}}\|_{L^2(Q_T)}=0.$$

For any $0<\varepsilon\leq 1$, let us introduce the adjoint nondegenerate boundary value problem

$$\begin{cases}\partial_t\varphi_i+\psi_{i,\varepsilon}\Delta\varphi_i=0 & \text{in } \Omega\times(0,T),\\ \dfrac{\partial\varphi_i}{\partial\eta}(x,t)=0 & \text{on } \partial\Omega\times(0,T),\\ \varphi_i(x,T)=\chi_i & \text{in } \Omega.\end{cases} \tag{4.3}$$

For any smooth χ_i with $0\leq\chi_i(x,t)\leq 1$, $i=1,\ldots,4$, and any $0<\varepsilon\leq 1$, this problem has a unique classical solution $\varphi_{i,\varepsilon}$ such that (see [13])

$$0\leq\varphi_{i,\varepsilon}(x,t)\leq 1,$$
$$\int\limits_{Q_T}\psi_{i,\varepsilon}(\Delta\varphi_{i,\varepsilon})^2dxdt\leq K_1.$$

If in (4.1)–(4.2) we replace φ_i by $\varphi_{i,\varepsilon}$, where $\varphi_{i,\varepsilon}$ is the solution of problem (4.3) with $\chi_1(x)=\chi_{1,\varepsilon}(x)=sign_\varepsilon^+(S_1-S_2)(x,T)$ and $\chi_2(x)=\chi_{2,\varepsilon}(x)=sign_\varepsilon^+(I_1-I_2)(x,T)$, then we obtain

$$\int\limits_{\Omega}\chi_{1,\varepsilon}(x)(S_1-S_2)(x,T)dx-\int\limits_{Q_T}(\psi_1-\psi_{1,\varepsilon})(S_1-S_2)\Delta\varphi_{1,\varepsilon}(x,t)dxdt=$$
$$=-\int\limits_{Q_T}(f(S_1,I_1)-f(S_2,I_2))\varphi_{1,\varepsilon}(x,t)dxdt$$

and

$$\int_\Omega \chi_{2,\varepsilon}(x)(I_1 - I_2)(x,T)dx - \int_{Q_T} (\psi_2 - \psi_{2,\varepsilon})(I_1 - I_2)\Delta\varphi_{2,\varepsilon}(x,t)dxdt =$$

$$= \int_{Q_T} (f(S_2, I_2) - f(S_1, I_1))\varphi_{2,\varepsilon}(x,t)dxdt.$$

Using the local Lipschitz continuity of f and the properties of $\psi_{i,\varepsilon}$ and $\varphi_{i,\varepsilon}$, we deduce, by letting $\varepsilon \to 0$, that

$$\int_\Omega \left((S_1 - S_2)^+ + (I_1 - I_2)^+\right)(x,T)dx \le K \int_{Q_T} \left(|S_1 - S_2| + |I_1 - I_2|\right) dxdt,$$

where K is the Lipschitz constant of the vector field f. In a similar fashion we establish an analogous inequality for $(S_1 - S_2)^-$ and $(I_1 - I_2)^-$ and deduce that

$$\int_\Omega \left(|S_1 - S_2| + |I_1 - I_2|\right)(x,T)dx \le K \int_{Q_T} \left(|S_1 - S_2| + |I_1 - I_2|\right)(x,t)dxdt.$$

We conclude by using Gronwall's Lemma.

5. LARGE TIME BEHAVIOR: PROOFS

5.1. THE ω-LIMIT SET

In this section we assume that $0 \le S_0 \le \frac{\delta}{\gamma}$ and set $\psi_k = \phi_k^{-1}$ for $k \in \{S, I\}$.

By Lemma 3.3, the set $\{(\phi_S(S)(\cdot, t), \phi_I(I)(\cdot, t))\}_{t \ge t_0}$ is bounded in $\left(H^1(\Omega)\right)^2$ hence precompact in $\left(L^2(\Omega)\right)^2$, and we conclude that the ω-limit set

$$\begin{aligned} \omega(S_0, I_0) \;\; = \;\; & \Big\{(U,V) \in \left(H^1(\Omega) \cap L^\infty(\Omega)\right)^2 : \exists t_k \to \infty \\ & (\phi_S(S), \phi_I(I))(\,\cdot\,,\ t_k) \to (U,V)(\cdot) \text{ on } \left(L^2(\Omega)\right)^2\Big\} \end{aligned}$$

is well-defined. Now we give a characterization of $\omega(S_0, I_0)$.

Proposition 5.1. *Let $(U,V) \in \omega(S_0, I_0)$. Then (U,V) is a solution of the homogeneous Neumann problem*

$$\begin{cases} -\Delta U = -\psi_I(V)\left(\gamma\psi_S(U) - \delta\right), \\ -\Delta V = \psi_I(V)\left(\gamma\psi_S(U) - \delta\right), \end{cases} \quad \text{in } \Omega, \tag{5.1}$$

$$\frac{\partial U}{\partial n} = \frac{\partial V}{\partial n} = 0 \quad \text{in } \partial\Omega. \tag{5.2}$$

Proof. Let $(U,V)\in\omega(S_0,I_0)$. Then there exists $t_k\to\infty$ such that

$$(U,V)(\cdot)=\lim_{k\to\infty}(\phi_S(S),\phi_I(I))(\cdot,\ t_k)\ \text{ in }\ \left(L^2(\Omega)\right)^2.$$

Let us consider two sequences U_k, V_k in $L^2(\Omega\times(-1,1))$ defined as

$$\begin{array}{r}U_k(x,s)=S(x,t_k+s),\\ V_k(x,s)=I(x,t_k+s),\end{array}\quad x\in\Omega,\ -1<s<1,\ k>0.$$

For each $s\in(-1,1)$,

$$\begin{aligned}\int_\Omega|\phi_S(U_k)(x,s)-\phi_S(S)(x,t_k)|^2\,dx&=\int_\Omega|\phi_S(S)(x,t_k+s)-\phi_S(S)(x,t_k)|^2\,dx=\\ &=\int_\Omega\left|\int_{t_k}^{t_k+s}(\phi_S(S))_t\,dt\right|^2dx\le\\ &\le\int_\Omega\int_{t_k}^{t_k+s}|(\phi_S(S))_t|^2\,dx_t^2dt\le\\ &\le\int_\Omega\int_{t_k}^{\infty}|(\phi_S(S))_t|^2\,dxdt.\end{aligned}$$

Hence

$$\|\phi_S(U_k)-\phi_S(S)(\cdot,t_k)\|_{L^2(\Omega\times(-1,1))}\le\left[\int_\Omega\int_{t_k}^{\infty}|(\phi_S(S))_t|^2\,dxdt\right]^{\frac12}.$$

By Lemma 3.4, we get

$$\lim_{k\to\infty}\int_\Omega\int_{t_k}^{\infty}|(\phi_S(S))_t|^2\,dxdt=0.$$

Then for a subsequence still denoted U_k, V_k: $\phi_S(U_k)\to U$, $\phi_I(V_k)\to V$ in $L^2(\Omega\times(-1,1))$ and almost everywhere in $\Omega\times(-1,1)$ as $k\to\infty$, and then by the Lebesgue dominated convergence theorem we have

$$U_k\to\psi_S(U),\quad V_k\to\psi_I(V),\quad V_k(\gamma U_k-\delta)\to\psi_I(V)(\gamma\psi_S(U)-\delta)$$

in $L^2(\Omega\times(-1,1))$.

Next, let $\xi \in C^2\left(\bar{\Omega}\right)$ be such that $\frac{\partial \xi}{\partial \eta} = 0$ on $\partial\Omega$ and $\rho \in C_0^1\left((-1,1)\right)$, $\rho \geq 0$, $\int_{-1}^{1} \rho\left(s\right) ds = 1$. We set $\varphi\left(x,t\right) = \rho\left(t - t_k\right) \xi\left(x\right)$ and use φ as a test function in the definition of S with $T = t_k + 1$ and $t_k \geq 1$. We get

$$\int_0^T \int_\Omega \left[S\rho_t\left(t - t_k\right) \xi\left(x\right) + \phi_S(S)\rho\left(t - t_k\right) \Delta\xi - I\left(\gamma S - \delta\right) \rho\left(t - t_k\right) \xi\right] dxdt = 0,$$

i.e.

$$\int_{t_k-1}^{t_k+1} \int_\Omega \left[S\rho_t\left(t - t_k\right) \xi + \phi_S(S)\rho\left(t - t_k\right) \Delta\xi - I\left(\gamma S - \delta\right) \rho\left(t - t_k\right) \xi\right] dxdt = 0.$$

Setting $s = t - t_k$, we get

$$\int_{-1}^{1} \int_\Omega U_k\rho_t\left(s\right) \xi\left(x\right) + \phi_S(U_k)\rho\left(s\right) \Delta\xi - V_k\left(\gamma U_k - \delta\right) \rho\left(s\right) \xi dxds = 0.$$

Passing to the limit as $k \to \infty$ and since $\int_{-1}^{1} \rho'\left(s\right) ds = 0$ we obtain

$$\left(\int_{-1}^{1} \rho\left(s\right) ds\right)\left(\int_\Omega U\left(x\right) \Delta\xi\left(x\right) - \psi_I(V)\left(\gamma\psi_S(U) - \delta\right) \xi dx\right) = 0.$$

We conclude that for any $\xi \in C^2\left(\bar{\Omega}\right)$, $\dfrac{\partial \xi}{\partial \eta} = 0$ on $\partial\Omega$. We have

$$\int_\Omega U\left(x\right) \Delta\xi - \psi_I(V)\left(\gamma\psi_S(U) - \delta\right) \xi dx = 0.$$

Thus U is a weak solution of

$$\begin{cases} -\Delta U = -\psi_I(V)\left(\gamma\psi_S(U) - \delta\right) & \text{in } \Omega, \\ \dfrac{\partial U}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

In the same way, we prove that V is a weak solution of

$$\begin{cases} -\Delta V = \psi_I(V)\left(\gamma\psi_S(U) - \delta\right) & \text{in } \Omega, \\ \dfrac{\partial V}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

□

5.2. PROOF OF THEOREM 2.4

The semi-orbits $\{(S(\cdot,t),I(\cdot,t)):t\geq 0\}$ are relatively compact in $(C(\overline{\Omega}))^2$, because they are bounded in $(L^\infty(Q_\infty))^2$ by Lemma 3.1 and one may use results of Di Benedetto [7].

Let $(U,V)\in\omega\,(S_0,I_0)$. Then (U,V) is a solution of (5.1)–(5.2). The function $U+V$ is a solution of

$$\begin{cases} -\Delta\,(U+V)=0 & \text{in } \Omega, \\ \dfrac{\partial}{\partial n}\,(U+V)=0 & \text{on } \partial\Omega. \end{cases}$$

Then $U+V$ is constant.

Now multiply the equation for V by V and integrate over Ω to obtain

$$\int\limits_\Omega |\nabla V|^2\,dx+\int\limits_\Omega V\psi_I(V)\,(\delta-\gamma\psi_S(U))\,dx=0. \tag{5.3}$$

Since $\delta-\gamma\psi_S(U)\geq 0$, we conclude that V is constant, and then also U is constant. Putting these conclusions in (5.3) one has

$$V\psi_I(V)\,(\delta-\gamma\psi_S(U))=0\rightarrow V=0 \quad \text{or} \quad U=\phi_S\left(\frac{\delta}{\gamma}\right).$$

Now using estimate (3.7), we get

$$\psi_S(U)+\psi_I(V)=\frac{1}{|\Omega|}\int\limits_\Omega S_0\,(x)+I_0\,(x)\,dx.$$

Keeping in mind $S\leq\frac{\delta}{\gamma}$, i.e. $U\leq\phi_S(\frac{\delta}{\gamma})$, then either

$$V=0\,\rightarrow\,U=\phi_S\left(\frac{1}{|\Omega|}\int\limits_\Omega S_0\,(x)+I_0\,(x)\,dx\right)\leq\phi_S\left(\frac{\delta}{\gamma}\right)$$

and this is possible only if

$$\frac{1}{|\Omega|}\int\limits_\Omega S_0\,(x)+I_0\,(x)\,dx\leq\frac{\delta}{\gamma},$$

or

$$U=\phi_S\left(\frac{\delta}{\gamma}\right)\rightarrow\psi_I(V)=\frac{1}{|\Omega|}\int\limits_\Omega S_0\,(x)+I_0\,(x)\,dx-\frac{\delta}{\gamma}\geq 0$$

and this is possible only if

$$\frac{1}{|\Omega|}\int\limits_\Omega S_0\,(x)+I_0\,(x)\,dx\geq\frac{\delta}{\gamma}.$$

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