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ON INTERSECTIONS OF CANTOR SETS: HAUSDORFF MEASURE

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Abstract. We establish formulas for bounds on the Haudorff measure of the intersection of certain Cantor sets with their translates. As a consequence we obtain a formula for the Hausdorff dimensions of these intersections.

Keywords: Hausdorff measure, fractal, Cantor set, translation, intersection, digit expansion.

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1. INTRODUCTION

In this paper we study, and make precise, the "size" (or thickness) of the intersection of two Cantor sets; typically one will be a translate of the other. By "size" we mean an estimate on a suitable Hausdorff dimension. Indeed there are a number of surprises in the study of these intersection-sets. Our main theorem is Theorem 6.1, and it offers estimates both from below and above. The estimate from below involves a geometric exponent. In section 7, we give examples and applications. The "thickness" of a Cantor set on the real line is a measurement of its "size". To make thickness precise one introduces Hausdorff dimension and Hausdorff measure, thus leading to conditions that guarantee, in particular, nonempty intersection of two given Cantor sets. Such conditions are typically subtle; especially a priori estimates from below. The significance of the problem was noted by Furstenberg, see [9]. Departing from earlier literature, our present results deal with general classes of such intersection-fractals, thus going beyond various specialized studies in the prior literature. We offer an overview below of the earlier literature. While there are prior results in the literature, they are restricted to various classes; and to various special cases. While Cantor sets may, on the face of it, appear to be rather special, they occur in mathematical models involving fractals (e.g., iterated function systems (IFS) fractals and self-similar measures); they further play a role in number theory (e.g., in b-ary number representations, where

b is a base for a number system); in signal processing and in ergodic theory (e.g., in development of codes as beta-expansions); and in limit-theorems from probability (e.g., explicit properties of infinite Bernoulli convolutions.) Below we cite a sample of papers dealing with applications: [4,5,7,14,16–19,22,28,30–32]. The literature in the subject and in neighboring areas and applications is vast, and, in the list above, we limit ourselves to only a small sample of the literature.

2. STATEMENT OF THE PROBLEM AND RESULTS

Let $n \geq 3$ be an integer. Any real number $t \in [0,1]$ has at least one *n-ary representation*

$$t = 0._n t_1 t_2 \ldots = \sum_{k=1}^{\infty} \frac{t_k}{n^k}$$

where each t_k is one of the digits $0, 1, \ldots, n-1$. Deleting some element from the full digit set $\{0, 1, \ldots n-1\}$ we get a set of *digits* $D := \{d_k \mid k = 1, 2, \ldots, m\}$ with $m < n$ digits $d_k < d_{k+1}$ and a corresponding *deleted digits Cantor set*

$$C = C_{n,D} := \left\{ \sum_{k=1}^{\infty} \frac{x_k}{n^k} \mid x_k \in D \text{ for all } k \in \mathbb{N} \right\}. \tag{2.1}$$

In this paper we investigate the Hausdorff dimension and measure of the sets $C \cap (C + t)$, where $C + t := \{x + t \mid x \in C\}$. Since the problems we consider are invariant under translation we will assume $d_1 = 0$.

We say that D is *uniform,* if $d_{k+1} - d_k$, $k = 1, 2, \ldots, m-1$ is constant and greater than or equal to 2. We say D is *regular,* if D is a subset of a uniform digit set. Finally, we say that D is *sparse,* if $|\delta - \delta'| \geq 2$ for all $\delta \neq \delta'$ in

$$\Delta := D - D = \{d_j - d_k \mid d_j, d_k \in D\}.$$

Clearly, a uniform set is regular and a regular set is sparse. The set $D = \{0, 5, 7\}$ is sparse and not regular. We will abuse the terminology and say $C_{n,D}$ is uniform, regular, or sparse provided D has the corresponding property.

Previous studies of the sets $C \cap (C + t)$ include:

— When $C = C_{3,\{0,2\}}$ is the middle thirds Cantor set a formula for the Hausdorff dimension of $C \cap (C + t)$ can be found in [3] and in [24]. Such a formula can also be found in [7] if C is uniform and $d_m = n - 1$, and in [15] if C is regular. In Corollary 2.3 we establish a formula for the Hausdorff dimension for $C \cap (C + t)$ when C is sparse.

— Let F^+ be the set of all $t \geq 0$ such that $C \cap (C + t)$ is non-empty. For $0 \leq \beta \leq 1$, let $F_\beta := \{t \in F^+ \mid \dim (C \cap (C + t)) = \beta \log_n(m)\}$, where $\dim (C \cap (C + t))$ is the Hausdorff dimension of $C \cap (C + t)$. If C is the middle thirds Cantor set then $F^+ = [0,1]$ and it is shown in [3,11,24] that F_β is dense in F^+ for all $0 \leq \beta \leq 1$. This is extended to regular set and to sets $C_{n,D}$ such that D satisfies $d_{k+1} - d_k \geq 2$

and $d_m < n-1$ in [27]. It is also shown in [27] that F_β is not dense in F^+ for all $0 \le \beta \le 1$ for all deleted digits Cantor sets $C_{n,D}$. We address this problem for the Hausdorff measure in place of the Hausdorff dimension when D is sparse in Corollary 6.4.

— It is shown in [11,13] that, if C is the middle thirds Cantor set, then the Hausdorff dimension of $C \cap (C+t)$ is $\frac{1}{3}\log_3(2)$ for Lebesgue almost all t in the closed interval $[0,1]$. This is is extended to all deleted digits sets in [15].

— If C is the middle thirds Cantor set, then $C \cap (C+t)$ is self-similar if and only if the sequence $\{1-|y_k|\}$ is strong periodic where $t = \sum_{k=1}^{\infty} \frac{2y_k}{3^k}$ and $y_k \in \{-1,0,1\}$ for all k by [19]. Thus, $C \cap (C+t)$ is not, in general, a self-similar set.

— For the middle thirds Cantor set it is shown in [3, 24] that $C \cap (C+t)$ has $\log_3(2)$–dimensional Hausdorff measure 0 or $\frac{1}{2^k}$ for some integer k. This is extended to $\log_n(m)$–dimensional Hausdorff measure for uniform sets with $d_m = n-1$ in [7]. In Theorem 2.2 we estimate the s–dimensional Hausdorff measure of $C \cap (C+t)$, when D is sparse and s is the Hausdorff dimension of $C \cap (C+t)$.

Some of the cited papers only consider rational t and some consider Minkowski dimension in place of Hausdorff dimension. It is known, see e.g., [27] for an elementary proof, that the (lower) Minkowski dimensions of $C \cap (C+t)$ equals its Hausdorff dimension.

Palis [25] conjectured that for dynamically defined Cantor sets typically the corresponding set F^+ either has Lebesgue measure zero or contains an interval. The papers [6, 23] investigate this problem for random deleted digits sets and solve it in the affirmative in the deterministic case.

For n-ary representations $t = 0._n t_1 t_2 \dots$ with $t_k \in \{0,1,\dots,n-1\}$, let $\lfloor t \rfloor_k := \sum_{j=1}^{k} \frac{t_j}{n^j} = 0._n t_1 t_2 \dots t_k$ denote the *truncation* of t to the first k n-ary places. Note that the truncation of t is unique, unless t admits two different n-ary representations.

The case where t admits a finite n-ary representation is relatively simple. In fact, Theorem 4.1 shows that, if $t = 0._n t_1 t_2 \dots t_k$, then $C \cap (C+t)$ is a union of two, possibly empty, sets A and B, where A is a finite disjoint union of sets of the form $\frac{1}{n^k}(C+h)$ and B is a finite set. Consequently, we will focus on translations t that do not admit a finite n-ary representation.

Let

$$C_k := \{0._n x_1 x_2 \dots \mid x_j \in D \text{ for } 1 \le j \le k\}$$

for each k, then $C_0 = [0,1]$,

$$C_{k+1} \subset C_k, \text{ and } C = C_{n,D} = \bigcap_{k=0}^{\infty} C_k. \tag{2.2}$$

Let $0 \le t \le 1$ be fixed. Let $J = \frac{1}{n^k}(C_0 + h)$ be an interval contained in C_k for some integer h. We say J is in the *interval case*, if it is also an interval in $C_k + \lfloor t \rfloor_k$. And we say J is in the *potential interval* case, if $J + \frac{1}{n^k}$ is an interval in $C_k + \lfloor t \rfloor_k$.

Proposition 2.1. *Suppose D is sparse. Let $0 \leq t \leq 1$. If one of the intervals in C_k is in the interval case, then no interval in C_k is in the potential interval case. If one of the intervals in C_k is in the potential interval case, then no interval in C_k is in the interval case.*

Suppose D is sparse and $t = 0._n t_1 t_2 \ldots$. Let $\mu_t(0) = 1$ and inductively $\mu_t(k+1) = \mu_t(k) \cdot \#\left(D - t_{k+1}\right) \cap (D \cup (D+1))$ if one of the intervals in C_k is in the interval case, $\mu_t(k+1) = \mu_t(k) \cdot \#\left(D - n + t_{k+1}\right) \cap (D \cup (D-1))$ if one of the intervals in C_k is in the potential interval case, and $\mu_t(k+1) = 0$ if no interval in C_k is in the interval or potential interval case. Here $\#B$ denotes the number of elements in the finite set B. Let $\nu_t(k) := \log_m \mu_t(k)$, $\beta_t := \liminf_{k \to \infty} \frac{\nu_t(k)}{k}$, and $L_t := \liminf_{k \to \infty} m^{\nu_t(k) - k\beta_t}$. These numbers all depend on n and D, but we suppress this dependence in the notation. A special case of Theorem 6.1 is

Theorem 2.2. *Let $C = C_{n,D}$ be a deleted digits Cantor set. Suppose D is sparse, $0 < t < 1$ does not admit a finite n-ary representation, and $C \cap (C+t)$ is non-empty. If $s := \beta_t \log_n(m)$, then*

$$m^{-\beta_t} L_t \leq \mathscr{H}^s\left(C \cap (C+t)\right) \leq L_t,$$

where $\mathscr{H}^s\left(C \cap (C+t)\right)$ is the s-dimensional Hausdorff measure of $C \cap (C+t)$.

We also show, see Remark 6.2, that Lemma 5.4 leads to a smaller upper bound at the expense of a more complicated expression for this upper bound. We also present an example, Example 7.3, showing that this smaller upper bound need not be equal to the Hausdorff measure of $C \cap (C+t)$.

Corollary 2.3. *Let $C = C_{n,D}$ be a deleted digits Cantor set. If D is sparse, $0 < t < 1$ does not admit a finite n-ary representation, and $C \cap (C+t)$ is non-empty, then $C \cap (C+t)$ has Hausdorff dimension $\beta_t \log_n(m)$.*

As noted above the sets $C \cap (C+t)$ are usually not self-similar. In Example 7.4 we construct C and t such that $C \cap (C+t)$ has Hausdorff dimension $\beta \log_n(m)$ and $L = 0$ or $L = \infty$. In these cases $C \cap (C+t)$ is not self-similar and Theorem 2.2 provides a formula for the Hausdorff measure. We show, Theorem 6.6, that our proof of Theorem 2.2 can be modified to give the estimate $m^{-1} \leq \mathscr{H}^s(C) \leq 1$, where $s = \log_n(m)$. A formula for the Hausdorff measure of self-similar sets is not known except in very special circumstances. However, the papers [1,20,21] contain algorithms for calculating the Hausdorff measure of self-similar subsets of the real line satisfying an open set condition. Corollary 6.7 contain estimates on the Hausdorff measure of $C \cap (C+t)$ when t admits a finite n-ary representation.

In Section 7 we give examples showing that $\mathscr{H}^s\left(C \cap (C+t)\right)$ can but need not equal L_t. We also present an example showing that if D is not sparse, then $\mathscr{H}^s\left(C \cap (C+t)\right)$ need not be in the interval $\left[m^{-\beta_t} L_t, L_t\right]$.

We refer to [8] for background information on Hausdorff dimension, Hausdorff measure and self-similar sets. Parts of this paper are based on the second named authors' thesis [26].

After this work was completed, we became aware of earlier works [10,29], on these problems. These papers consider a class of Cantor sets similar to but larger than uniform deleted digits sets with $d_m = n - 1$. We refer to this class as *homogeneous Cantor sets* and refer to the cited papers for the exact definition. The first of these papers, [10], establishes an estimate for homogeneous Cantor sets, similar to our Theorem 2.2. The second of these papers, [29], shows that, for a smaller class of homogeneous Cantor sets, the upper bound in [10] is in fact equal to the Hausdorff measure. Generically, these results do not apply to the cases where D is not uniform.

3. A CONSTRUCTION OF $C \cap (C + t)$

In this section we assume $n \geq 3$ is given and that $D = \{d_k \mid k = 1, 2, \ldots\}$ is some digits set. We indicate how a natural method of construction of C can be used to analyze $C \cap (C + t)$. This construction forms the basis for our analysis of $C \cap (C + t)$.

The middle thirds Cantor set is often constructed by starting with the closed interval $C_0 = [0, 1]$ and for each $k \geq 0$ letting C_{k+1} be obtained from C_k be removing the open middle of each interval in C_k. We show that $C = C_{n,D}$ can be constructed in a similar manner.

The *refinement* of the interval $[a, b]$ is the set

$$\bigcup_{j=1}^{m} \left[a + \frac{d_j}{n} (b - a), a + \frac{d_j + 1}{n} (b - a) \right].$$

The set C_{k+1} is obtained from C_k by refining each n-ary interval in C_k. For the middle thirds Cantor set refinement of C_k is the same as removing the open middle third from each interval in C_k.

Since we are interested in studying $C \cap (C + t)$ only t such that $C \cap (C + t)$ is not empty are of interest. Consequently we introduce the set

$$F := \{t \mid C \cap (C + t) \neq \varnothing\}.$$

It is easy to see that F is compact and $F = C - C$. As a result, $F^+ = F \cap [0, \infty)$ and $F = (-F) \cup F$. Since $C \cap (C - t)$ is translate of $C \cap (C + t)$ it is sufficient to consider $t \geq 0$.

Remark 3.1. It is shown in [27] that F is the compact set $\{0._n t_1 t_2 \ldots \mid t_k \in \Delta\}$. In particular, F is a self-similar set. Note the representations $0._n t_1 t_2 \ldots$ with $t_k \in \Delta$ allows the digits t_k to be positive for some k and negative for other k. We will not need this construction of F in this paper.

Fix $t = 0._n t_1 t_2 \ldots$ in $[0, 1]$. We split our analysis of $C \cap (C + t)$ into three steps. First, we consider the method of construction for the sets $C_k \cap (C_k + \lfloor t \rfloor_k)$. Second, we establish a relationship between $C_k \cap (C_k + \lfloor t \rfloor_k)$ and $C_k \cap (C_k + t)$. Thirdly, this allows us to use that (2.2) implies

$$C \cap (C+t) = \bigcap_{k=0}^{\infty} (C_k \cap (C_k + t)) \tag{3.1}$$

to investigate $C \cap (C+t)$.

4. ANALYSIS OF $C_k \cap (C_k + \lfloor t \rfloor_k)$

Given any $h \in \mathbb{Z}$ we say that the interval $J = \frac{1}{n^k}(C_0 + h)$ is an *n-ary interval* of length $\frac{1}{n^k}$. We will simply say n-ary interval when k is understood from the context. In particular, if U is a compact set, the phrase *an n-ary interval of U* refers to an n-ary interval of length $\frac{1}{n^k}$ contained in U where k is the smallest such k. In particular, C_k consists of m^k disjoint n-ary intervals.

Fix $t = 0._n t_1 t_2 \ldots$ in $[0,1]$. To construct $C_k \cap (C_k + \lfloor t \rfloor_k)$ we begin by generating C_{k+1} by refining each n-ary interval of C_k. Note that $\lfloor t \rfloor_k = \frac{h}{n^k}$ for some positive integer h so that $C_k + \lfloor t \rfloor_k$ also consists of n-ary intervals. Thus, $C_{k+1} + \lfloor t \rfloor_{k+1}$ is generated by first refining each n-ary interval of $C_k + \lfloor t \rfloor_k$ and then translating these refined intervals by the positive factor $\frac{t_{k+1}}{n^{k+1}}$. We say that $C_k \cap (C_k + \lfloor t \rfloor_k)$ *transitions to* $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$ by first generating the sets C_{k+1} and $C_{k+1} + \lfloor t \rfloor_{k+1}$ and then taking their intersection.

Let $J \subset C_k$ be an arbitrary n-ary interval. Then J can be classified using combinations of the following four cases: (1) J also in an n-ary interval in $C_k + \lfloor t \rfloor_k$, (2) the left hand end point of J is the right hand end point of some n-ary interval in $C_k + \lfloor t \rfloor_k$, (3) the right hand end point of J is the left hand end point of some n-ary interval in $C_k + \lfloor t \rfloor_k$, or (4) J does not have any points in common with $C_k + \lfloor t \rfloor_k$. More specifically, let J be an n-ary interval in C_k.

1. We say J is in the *interval case,* if there exists an n-ary interval $K \subset C_k + \lfloor t \rfloor_k$ such that $J = K$.
2. We say J is in the *potential interval case,* if there exists an n-ary interval $K \subset C_k + \lfloor t \rfloor_k$ such that $J = K + \frac{1}{n^k}$.
3. We say J is in the *potentially empty case,* if there exists an n-ary interval $K \subset C_k + \lfloor t \rfloor_k$ such that $J = K - \frac{1}{n^k}$.
4. We say J is in the *empty case,* if $J \cap (C_k + \lfloor t \rfloor_k) = \varnothing$.

Any n-ary interval in C_k is in one or more of the four cases described above. An n-ary interval J in C_k may both in the interval case and in the potential interval case, i.e., there exists n-ary intervals $K_I, K_P \subset (C_k + \lfloor t \rfloor_k)$ such that $K_P + \frac{1}{n^k} = J = K_I$. It is also possible for J to be in both the interval case and potentially empty case, or to be both in the potential interval case and in the potentially empty case. However, the intersections corresponding to the potentially empty cases do not contribute points to $C \cap (C+t)$, when $0 < t - \lfloor t \rfloor_k$. Hence, we will not identify these cases with special terminology. Finally, any J in the empty case cannot also be in any of the other cases.

The idea of our method is to take n-ary interval in C_k and use the above classification to investigate the intersection $J \cap C \cap (C+t)$. The basic question is whether or not this intersection is non-empty? whether or not repeated refinement of J "leads to" points in $C \cap (C+t)$?

4.1. FINITE n-ARY REPRESENTATIONS

We show that, if $t \in F^+$ admits a finite n-ary representation, then $C \cap (C+t)$ is a union of finite sets and sets similar to C.

Theorem 4.1. *Suppose $t = 0._n t_1 t_2 \dots t_k$ is in F^+. Then*

$$C \cap (C+t) = A \cup B,$$

where A is empty or $A = \bigcup_j \frac{1}{n^k}(C + h_j)$ for a finite set of integers h_j and B is a finite, perhaps empty, set. More precisely, each n-ary interval in C_k that is in the interval case gives rise to a term in the union in A. If $d_m < n-1$, then B is empty. If $d_m = n-1$, then:

(i) *each n-ary interval in C_k that is in the potential interval case and not in the potentially empty case gives rise to one point in B,*
(ii) *each n-ary interval in C_k that is in the potentially empty case and not in the potential interval case gives rise to one point in B,*
(iii) *each n-ary interval in C_k that both is in the potential interval case and in the potentially empty case gives rise to two point in B.*

Proof. Let J_0 be an n-ary interval in C_k and let h be the integer for which $J_0 = \frac{1}{n^k}(C_0 + h)$.

Suppose J_0 is in the interval case. For $j \geq 0$ let J_{j+1} be obtained from J_j by refining each interval in J_j. Since $C_{\ell+1}$ is obtained from C_ℓ by refining each interval in C_ℓ, it follows that $J_j = \frac{1}{n^k}(C_j + h)$ for all $j \geq 0$. So (2.2) implies

$$\bigcap_{j=0}^{\infty} J_j = \frac{1}{n^k}(C + h). \tag{4.1}$$

Consider the transition from $C_k \cap (C_k + \lfloor t \rfloor_k)$ to $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$. By assumption $J_0 \subseteq C_k$ and $J_0 \subseteq C_k + \lfloor t \rfloor_k$. Applying the refinement process to all intervals gives $J_1 \subseteq C_{k+1}$ and $J_1 \subseteq C_{k+1} + \lfloor t \rfloor_k$. Since $\lfloor t \rfloor_k = \lfloor t \rfloor_{k+1}$ we conclude $J_1 \subseteq C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right) = C_{k+1} \cap (C_{k+1} + t)$. Repeating this argument shows that $J_j \subseteq C_{k+j} \cap (C_{k+j} + t)$ for all $j \geq 0$. Hence combining (4.1) and (3.1) we conclude

$$\frac{1}{n^k}(C + h) \subseteq C \cap (C+t).$$

Thus any interval in C_k that is in the interval case gives rise to a "small copy" of C in $C \cap (C+t)$.

Suppose J_0 is in the potential interval case. Then $K_0 := J_0 - \frac{1}{n^k}$ is an n-ary interval in $C_k + \lfloor t \rfloor_k$. The refinements of J_0 and K_0 are

$$J_1 = \bigcup_{p=1}^{m} \frac{1}{n^{k+1}} (C_0 + nh + d_p) \text{ and } K_1 = \bigcup_{p=1}^{m} \frac{1}{n^{k+1}} (C_0 + nh + d_p - n).$$

Since C_0 is a closed interval of length one, $0 \le d_q \le d_{q+1} \le n-1$, $J_1 \cap K_1$ is non-empty iff $d_0 = 1 + d_m - n$ iff $d_m = n - 1$. In the affirmative case $J_0 \cap K_0 = J_1 \cap K_1$. Since $t = \lfloor t \rfloor_k = \lfloor t \rfloor_{k+1}$ we have

$$C \cap (C + t) \supseteq (J_0 \cap C) \cap (K_0 \cap (C + t)) \supseteq (J_1 \cap C) \cap (K_1 \cap (C + t)).$$

Hence, $J_0 \cap K_0$ is a point in $C \cap (C + t)$ iff $d_m = n - 1$.

The case where J_0 is in the potentially empty case is similar to the case where J_0 is in the potential intervals case.

Finally, suppose J_0 is in the empty case. Since $t = \lfloor t \rfloor_k$ it follows from (2.2) that $J_0 \cap (C + t) \subseteq J_0 \cap (C_k + \lfloor t \rfloor_k)$. But the right hand side is the empty set by assumption. □

Remark 4.2. The sets A and B in Theorem 4.1 need not be disjoint.

4.2. INFINITE n-ARY REPRESENTATIONS

Theorem 4.1 provides us with complete information about $C \cap (C + t)$, when t admits a finite n-ary representation. Consequently, it remains to investigate $C \cap (C + t)$ when t does not admit such a finite representation, i.e., when

$$0 < t - \lfloor t \rfloor_k < \frac{1}{n^k} \quad \text{for all} \quad k \ge 1. \tag{4.2}$$

Our next result shows that, if t does not admit a finite n-ary representation, then only interval and potential interval cases can contribute points to $C \cap (C + t)$.

Lemma 4.3. *Suppose $0 < t - \lfloor t \rfloor_k < \frac{1}{n^k}$ for some k. If J is an n-ary interval in C_k and J is neither in the interval case nor in the potential interval case, then $J \cap (C_k + t)$ is empty, in particular, the intersection $J \cap C \cap (C + t)$ is empty.*

Proof. Suppose $t = 0._n t_1 t_2 \ldots$ satisfies $0 < t - \lfloor t \rfloor_k < \frac{1}{n^k}$ for some k. Let J be an n-ary interval in C_k and let K be an n-ary interval in $C_k + \lfloor t \rfloor_k$. Pick integers h_J and h_K such that $J = \frac{1}{n^k} (C_0 + h_J)$ and $K = \frac{1}{n^k} (C_0 + h_K)$

Suppose J is in the potentially empty case and K is such that $J = K - \frac{1}{n^k}$. Then $h_J = h_K - 1$. Hence, $0 < t - \lfloor t \rfloor_k$ implies

$$J \cap (K + (t - \lfloor t \rfloor_k)) = \frac{1}{n^k} \left((C_0 + h_J) \cap \left(C_0 + h_J + 1 + (t - \lfloor t \rfloor_k) n^k \right) \right) = \varnothing,$$

since C_0 is an interval of length one. By (3.1) this intersection does not contribute any points to $C \cap (C + t)$.

Suppose J is in the empty case. Let $K \subset C_k + \lfloor t \rfloor_k$ be an arbitrary n-ary interval. Since J is a minimum distance of $\frac{1}{n^k}$ from K, then $K + (t - \lfloor t \rfloor_k)$ is at least a distance $\frac{1}{n^k} - (t - \lfloor t \rfloor_k) > 0$ from J. Hence, $J \cap (C_k + t) = \varnothing$ and J does not contain any points of $C \cap (C + t)$. □

Remark 4.4. The arguments from the proof of Lemma 4.3 also give some information about the interval and potential interval cases when t does not admit a finite n-ary representation. More precisely, suppose J is in the interval case and $K \subset C_k + \lfloor t \rfloor_k$ is an n-ary interval such that $K = J$. Since $t - \lfloor t \rfloor_k < \frac{1}{n^k}$ then $J \cap (K + (t - \lfloor t \rfloor_k))$ is an interval of length $\frac{1}{n^k} - (t - \lfloor t \rfloor_k) > 0$ contained in $C_k \cap (C_k + t)$ which therefore may contain points of $C \cap (C + t)$.

Suppose J is in the potential interval case and K is an n-ary interval in $C_k + \lfloor t \rfloor_k$ such that $K + \frac{1}{n^k} = J$. Since $0 < (t - \lfloor t \rfloor_k)$, then $J \cap (K + t - \lfloor t \rfloor_k)$ is an interval of length $t - \lfloor t \rfloor_k$ and this intersection may therefore contain points of $C \cap (C + t)$.

5. ANALYSIS OF THE TRANSITION FROM $C_k \cap (C_k + \lfloor t \rfloor_k)$ TO $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$

It follows from Lemma 4.3 that, when we investigate $C \cap (C + t)$, it is sufficient to consider intervals in C_k that are in the interval case or in the potential interval case. Fix $t = 0._n t_1 t_2 \ldots$ in $[0, 1]$. Consequently, we begin by considering what happens to an n-ary interval J in C_k that is in the interval case or the potential intervals case when we transition from $C_k \cap (C_k + \lfloor t \rfloor_k)$ to $C_{k+1} \cap \left(C_{k+1} + \lfloor t \rfloor_{k+1}\right)$.

Lemma 5.1. *Let $J \subset C_k$ and $K \subset C_k + \lfloor t \rfloor_k$ be n-ary intervals and let $t = 0._n t_1 t_2 \ldots$ be some point in $[0, 1]$. Consider the refinements J' and K' of J and K, respectively.*

1. *Suppose $J = K$. (Interval case)*
   a) *If t_{k+1} is in Δ, then exactly $\#D \cap (D + t_{k+1})$ of the intervals in J' are in the interval case relative to K'. By this we mean that the intersection*

   $$J' \cap \left(K' + \tfrac{t_{k+1}}{n^{k+1}}\right)$$

   contains $\#D \cap (D + t_{k+1})$ intervals of length $1/n^{k+1}$.
   b) *If t_{k+1} is in $\Delta - 1$, then exactly $\#D \cap (D + t_{k+1} + 1)$ of the intervals in J' are in the potential interval case relative to K'. By this we mean that the intersection*

   $$J' \cap \left(K' + \tfrac{t_{k+1}}{n^{k+1}} + \tfrac{1}{n^{k+1}}\right)$$

   contains $\#D \cap (D + t_{k+1} + 1)$ intervals of length $1/n^{k+1}$.
   c) *If t_{k+1} is neither in Δ nor in $\Delta - 1$, then all intervals in J' are either in the empty case or in the potentially empty case. More precisely, no interval in J' is in the interval case or in the potential interval case relative to K'.*
2. *Suppose $J = K + \frac{1}{n^k}$. (Potential interval case)*
   a) *If t_{k+1} is in $n - \Delta$, then exactly $\#D \cap (D + n - t_{k+1})$ of the intervals in J' are in the interval case relative to K'.*

b) *If* t_{k+1} *is in* $n-\Delta-1$*, then exactly* $\#D\cap(D+n-t_{k+1}-1)$ *of the intervals in* J' *are in the potential interval case relative to* K'*.*
c) *If* t_{k+1} *is neither in* $n-\Delta$ *nor in* $n-\Delta-1$*, then all intervals in the refinement of* J *are either in the empty case or in the potentially empty case.*

Proof. Let h_J and h_K be integers such that $J=\frac{1}{n^k}(C_0+h_J)$ and $K=\frac{1}{n^k}(C_0+h_K)$ and let

$$J(p):=\frac{1}{n^{k+1}}(C_0+h_Jn+d_p)$$

and

$$K(q):=\frac{1}{n^{k+1}}(C_0+h_Kn+d_q)$$

for $p,q=1,2,\ldots,m$. Then the refinements of J and K are $\bigcup_{p=1}^m J(p)$ and $\bigcup_{q=1}^m K(q)$.

Suppose $J=K$, then $h_J=h_K$. Hence $J(p)=K(q)+\frac{t_{k+1}}{n^{k+1}}$ iff $d_p=d_q+t_{k+1}$ and $J(p)=K(q)+\frac{t_{k+1}}{n^{k+1}}+\frac{1}{n^{k+1}}$ iff $d_p=d_q+t_{k+1}+1$. This establishes the interval case.

Suppose $J=K+\frac{1}{n^k}$, then $h_J=h_K+1$. So $J(p)=K(q)+\frac{t_{k+1}}{n^{k+1}}$ iff $n+d_p=d_q+t_{k+1}$ and $J(p)=K(q)+\frac{t_{k+1}}{n^{k+1}}+\frac{1}{n^{k+1}}$ iff $n+d_p=d_q+t_{k+1}+1$. This establishes the potential interval case. □

To describe our analysis of the sets $C_k\cap(C_k+\lfloor t\rfloor_k)$ we introduce appropriate terminology.

— $C_k\cap(C_k+\lfloor t\rfloor_k)$ is in the *interval case,* if there exists an n-ary interval $J\subset C_k$ in the interval case and no n-ary interval $K\subset C_k$ is in the potential interval case or simultaneous case.
— $C_k\cap(C_k+\lfloor t\rfloor_k)$ is in the *potential interval case,* if there exists $J\subset C_k$ in the potential interval case and no n-ary interval $K\subset C_k$ is in the interval case or simultaneous case.
— $C_k\cap(C_k+\lfloor t\rfloor_k)$ is in the *simultaneous case,* if there exist $J_I,J_P\subset C_k$ such that J_I is in the interval case and J_P is in the potential interval case.
— $C_k\cap(C_k+\lfloor t\rfloor_k)$ is in the *irrecoverable case,* if J is in the empty or potentially empty case for all n-ary intervals $J\subset C_k$.

Our next goal is to introduce a function whose values determine whether $C_k\cap(C_k+\lfloor t\rfloor_k)$ is in the interval, potential interval, simultaneous, or irrecoverable case. Since $C_0\cap(C_0+\lfloor t\rfloor_0)=[0,1]$, then we begin in the interval case and can examine transitions inductively. The following constructions are motivated by Lemma 5.1. Let $i:=\sqrt{-1}$ and let

$$\xi:\{0,\pm1,i\}\times\{0,1,\ldots,n-1\}\to\{0,\pm1,\pm i\}$$

be determined by

$$\xi(0,h) := 0,$$

$$\xi(1,h) := \begin{cases} 1 & \text{if } h \text{ is in } \Delta \text{ but not in } \Delta - 1, \\ -1 & \text{if } h \text{ is in } \Delta - 1 \text{ but not in } \Delta, \\ i & \text{if } h \text{ is both in } \Delta \text{ and } \Delta - 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\xi(-1,h) := \begin{cases} -1 & \text{if } h \text{ is in } n - \Delta \text{ but not in } n - \Delta - 1, \\ 1 & \text{it } h \text{ is in } n - \Delta - 1 \text{ but not in } n - \Delta, \\ -i & \text{if } h \text{ is both in } n - \Delta \text{ and in } n - \Delta - 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\xi(i,h) := \begin{cases} -i & \text{if } h \text{ is in } \Delta \cup (n - \Delta) \text{ but not in } (\Delta - 1) \cup (n - \Delta - 1), \\ i & \text{if } h \text{ is in } (\Delta - 1) \cup (n - \Delta - 1) \text{ but not in } \Delta \cup (n - \Delta), \\ 1 & \text{if } h \text{ is both in } \Delta \cup (n - \Delta) \text{ and in } (\Delta - 1) \cup (n - \Delta - 1), \\ 0 & \text{otherwise.} \end{cases}$$

The function $\xi(z,h)$ is completely determined by D and n. Let $\sigma_t : \mathbb{N}_0 \to \{0, \pm 1, i\}$ be determined by

$$\sigma_t(0) := 1 \text{ and inductively}$$
$$\sigma_t(k+1) := \xi(\sigma_t(k), t_{k+1}) \cdot \sigma_t(k) \text{ for } k \geq 0.$$

By construction of ξ we have $\sigma_t(k) \in \{0, \pm 1, i\}$ for all $k \geq 0$.

Lemma 5.2. *Let $t = 0._n t_1 t_2 \ldots$ be some point in $[0,1]$. Then $C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the interval case iff $\sigma_t(k) = 1$, the potential interval case iff $\sigma_t(k) = -1$, the simultaneous case iff $\sigma_t(k) = i$, and the irrecoverable case iff $\sigma_t(k) = 0$.*

Proof. This is a simple consequence of Lemma 5.1 and our construction of σ. □

We now show that D is sparse iff every $t \geq 0$ in F has an n-ary representation such that for all $k \geq 0$ the set $C_k \cap (C_k + \lfloor t \rfloor_k)$ is either in the interval case or in the potential interval case.

Theorem 5.3. *Let $C = C_{n,D}$ be a deleted digits Cantor set. Then*

$$F^+ = \{t \in [0,1] \mid \sigma_t(k) = \pm 1 \textit{ for all } k \in \mathbb{N}\}$$

iff D is sparse.

Proof. Suppose D is sparse, then $\Delta \cap (\Delta - 1) = \varnothing$ and $(n - \Delta) \cap (n - \Delta - 1) = \varnothing$. Hence, our construction of ξ and σ shows that $\sigma_t(k) \in \{0, \pm 1\}$ for all k and all $t \in F^+$. We must show that $\sigma_t(k) \neq 0$ for all k and all $t \in F^+$.

Suppose $t \in F^+$ such that $\sigma_t(k) = 0$ for some k. By Lemma 5.2 all n-ary intervals in C_k are in the potentially empty or the empty case. Since $t \in F^+$ at least one n-ary interval, J say, in C_k is in the potentially empty case and $t_j = 0$ for all $j > k$. Since $0 \in \Delta$ it follows from the construction of σ that $t \neq 0$. Hence, there is a $k \geq 1$ such that $t_k > 0$ and $t_j = 0$ for all $j > k$. Let $s_j = t_j$ when $j < k$, $s_k = t_k - 1$, and $s_j = d_m$ for all $j > k$. Then $t = 0.s_1 s_2 \ldots$. We must show that $\sigma_s(j) \neq 0$ for all j. Now $\sigma_s(j) = \sigma_t(t) \in \{\pm 1\}$ for all $j < k$. Hence it remains to consider $j \geq k$.

The potentially empty cases in $C_k \cap (C_k + \lfloor t \rfloor_k)$ are interval cases in $C_k \cap \left(C_k + \lfloor t \rfloor_k - \frac{1}{n^k}\right)$. Some of the empty cases in $C_k \cap (C_k + \lfloor t \rfloor_k)$ may give potentially empty cases in $C_k \cap \left(C_k + \lfloor t \rfloor_k - \frac{1}{n^k}\right)$, but they cannot give interval cases in $C_k \cap \left(C_k + \lfloor t \rfloor_k - \frac{1}{n^k}\right)$. Consequently, $\sigma_s(k) = 1$.

Since $C_j \cap \left(C_j + \lfloor t \rfloor_j\right) = C_j \cap (C_j + t)$ for all $j \geq k$ and $t \in F$ it follows from (3.1) that $C_j \cap \left(C_j + \lfloor t \rfloor_j\right)$ is non-empty for all $j \geq k$.

Since $t = 0._n t_1 \ldots t_k$ is in F^+ and no intervals in C_k are in the interval case Theorem 4.1 implies $d_m = n - 1$. Since $\sigma_s(k) = 1$ and $s_j = d_m = n - 1 \in \Delta$, it follows from Lemma 5.1 that $\sigma_s(j) = 1$ for all $j > k$.

Conversely, suppose D is not sparse, then $\Delta \cap (\Delta - 1) \neq \varnothing$. Let $\delta \in \Delta \cap (\Delta - 1)$. Consider $t := \frac{\delta}{n}$. Then $\sigma_t(1) = i$. Hence $C_1 \cap (C_1 + \lfloor t \rfloor_1) = C_1 \cap (C_1 + t)$ contains at least one n-ary interval J which is in the interval case. The n-ary intervals in $C_1 \cap (C_1 + t)$ refine to $\frac{1}{n}(C + h)$ for some integer h. By (3.1) $\frac{1}{n}(C + h) \subseteq C \cap (C + t)$. In particular, $C \cap (C + t) \neq \varnothing$ so that $t \in F^+$. □

Theorem 5.3 shows that the simultaneous case does not occur when D is sparse. In particular, we have established Proposition 2.1.

In the following two lemmas we establish two key results required to establish Theorem 2.2. In Lemma 5.4 we show that $\mu_t(k)$ counts the number of n-ary intervals of C_k in either the interval or the potential interval case. In Lemma 5.5 we show that the intervals counted by $\mu_t(k)$ have points in common with $C \cap (C + t)$, hence that we do not "over" count.

Lemma 5.4. *Let $C = C_{n,D}$ be given. Suppose $t \in F^+$ does not admit a finite n-ary representation and $\sigma_t(k) = \pm 1$ for all $k \geq 0$. Then $C_k \cap (C_k + t)$ is a union of $\mu_t(k)$ intervals, each of length*

$$\ell_k := \begin{cases} \frac{1}{n^k} - (t - \lfloor t \rfloor_k) & \text{when } \sigma_t(k) = 1, \\ t - \lfloor t \rfloor_k & \text{when } \sigma_t(k) = -1. \end{cases}$$

Proof. Let $t \in F^+$ be given. Suppose t does not admit a finite n-ary representation and $\sigma_t(k) = \pm 1$ for all k. Every n-ary interval in C_k is either in the interval, the potential, interval, or the potentially empty case. By Lemma 4.3, if J is an n-ary interval in C_k that is in the potentially empty or the empty case, then $J \cap (C_k + t)$ is empty. Hence, it is sufficient to consider n-ary intervals in C_k that either are in the interval or the potential intervals case. By definition of σ_t, no n-ary interval in C_k is both in the interval and the potential interval case.

Since the length of the intervals is determined by Lemma 5.2 and Remark 4.4, we only need to show that $C_k \cap (C_k + t)$ contains $\mu_t(k)$ intervals for $k \geq 0$. Since $C_0 \cap (C_0 + t) = [t, 1]$ is one interval and $\mu_t(0) = 1$, the claim holds for $k = 0$.

Assume the claim holds for some integer $k \geq 0$. Then $C_k \cap (C_k + t)$ consists of $\mu_t(k)$ intervals. Suppose $\sigma_t(k) = 1$. Then C_k contains $\mu_t(k)$ n-ary intervals J_j in the interval case and no intervals in the potential interval case. Since $t \in F^+$ it follows from part (1) of Lemma 5.1 and Lemma 4.3 that $t_{k+1} \in \Delta$ or $t_{k+1} \in \Delta - 1$. If $t_{k+1} \in \Delta$, then each J_j gives $\#D \cap (D + t_{k+1})$ intervals in $C_k \cap (C_k + t)$ by part (1)(a) of Lemma 5.1 and Remark 4.4. Hence $C_{k+1} \cap (C_{k+1} + t)$ contains $\mu_t(k) \cdot \#D \cap (D + t_{k+1})$ intervals. On the other hand, if $D \cap (D + t_{k+1} + 1)$ is nonempty, then t_{k+1} is an element of $\Delta \cap (\Delta - 1)$ which contradicts the assumption that $\sigma_t(k+1) \neq i$. Hence $(D - t_{k+1}) \cap (D \cup (D+1)) = D \cap (D + t_{k+1})$. Consequently, $\mu_t(k+1) = \mu_t(k) \cdot \#D \cap (D + t_{k+1})$ by the definition of μ_t. The case $t_{k+1} \in \Delta - 1$ is similar to $t_{k+1} \in \Delta$.

The case $\sigma_t(k) = -1$ is handled using arguments similar to those used for $\sigma_t(k) = 1$ above, replacing Δ by $n - \Delta$ and $\Delta - 1$ by $n - \Delta - 1$. □

Lemma 5.5. *Let $C = C_{n,D}$ be given. Suppose $t \in F^+$ does not admit a finite n-ary representation and $\sigma_t(k) = \pm 1$ for all $k \geq 0$. For each k, every n-ary interval of C_k in the interval or potential interval case contains points of $C \cap (C + t)$.*

Proof. Let $J_0 = \frac{1}{n^k}(C_0 + h)$ be an n-ary interval of C_k. Suppose J_0 is in the interval case. Let x_k be the right hand endpoint of J_0. Since $0 < t - \lfloor t \rfloor_k < \frac{1}{n^k}$ and J_0 has length $\frac{1}{n^k}$ then $x_k \in J_0 \cap (J_0 + t - \lfloor t \rfloor_k)$. Now $J_0 \cap (J_0 + t - \lfloor t \rfloor_k) \subseteq C_k \cap (C_k + t)$ follows from $J_0 \subseteq C_k + \lfloor t \rfloor_k$. Consequently, x_k is in $C_k \cap (C_k + t)$.

Supposing J_0 is in the potential interval case and x_k be the left hand endpoint of J_0, an argument similar to the one above shows that x_k is in $C_k \cap (C_k + t)$.

Suppose J_0 is in the interval case. Then $\sigma_t(k) = 1$ by assumption and all n-ary intervals in C_k are either in the interval case or one of the empty cases. Since $t \in F^+$ it follows from Lemma 5.1 and Lemma 4.3 that at least one subinterval J_1 in the refinement of J_0 is either in the interval or the potential interval case. Similarly, if J_0 is in the potential interval case it follows that one of the subintervals J_1 in the refinement of J_0 is in the interval or potential interval case.

By induction we get a sequence x_j of points and a sequence of intervals J_j such that $J_{j+1} \subset J_j$ and $x_j \in C_j \cap (C_j + t) \subseteq J_j$. By the nested interval theorem $x_j \to x \in \bigcap J_j \subset J_0$. By (3.1) $x \in C \cap (C + t)$. □

Theorem 5.3 shows that the assumptions of the previous Lemmas are met whenever t does not admit finite n-ary representation and D is sparse. Example 7.5 demonstrates we may "over" count when t does not meet the $\sigma_t(k) = \pm 1$ requirement.

6. ESTIMATING THE HAUSDORFF MEASURE OF $C \cap (C + t)$

Let $\mathscr{H}^s(K)$ denote the s-dimensional Hausdorff measure of a compact set K and let $|K| := \sup\{|x - y| \mid x, y \in K\}$ denote the diameter. Given $\varepsilon > 0$, a collection of closed intervals $\{U_\alpha\}$ is an *ε-cover* of K if $K \subset \bigcup U_\alpha$ and $\varepsilon > |U_\alpha| > 0$. Define

$$\mathscr{H}^s_\varepsilon(K) := \inf\left\{\sum |U_\alpha|^s\right\}$$

to be the approximation to the Hausdorff measure of K by ε-covers so that

$$\mathscr{H}^s(K) = \lim_{\varepsilon \to 0} \mathscr{H}^s_\varepsilon(K). \tag{6.1}$$

The approximating measure $\mathscr{H}^s_\varepsilon(K)$ can be equivalently defined using a collection of arbitrary open or closed sets, each having appropriate diameter. The closed intervals definition is natural for this paper based on the construction of $C \cap (C+t)$.

The Hausdorff dimension of C is $\log_n(m)$ and $0 < \mathscr{H}^{\log_n(m)}(C) < \infty$ since C is self-similar by [12]. Since $C \cap (C+t) \subseteq C$, then $0 \leq \dim(C \cap (C+t)) \leq \log_n(m)$ for any real t and if $0 < \dim(C \cap (C+t)) < \log_n(m)$ then t does not admit finite n-ary representation by Theorem 4.1. Our goal is to estimate the Hausdorff measure of $C \cap (C+t)$.

6.1. INFINITE n-ARY REPRESENTATIONS

We use the counting method of Lemma 5.4 to estimate the Hausdorff measure of $C \cap (C+t)$ whenever t does not admit finite n-ary representation.

Theorem 6.1. *Let $C = C_{n,D}$ be given. Suppose t is an element of F^+ which does not admit finite n-ary representation and $\sigma_t(k) = \pm 1$ for all k. If $L_t := \liminf_{k\to\infty}\{m^{\nu_t(k)-k\cdot\beta_t}\}$ and $s := \beta_t \log_n(m)$, then*

$$m^{-\beta_t} \cdot L_t \leq \mathscr{H}^s(C \cap (C+t)) \leq L_t.$$

Proof. We begin by showing $\mathscr{H}^s(C \cap (C+t)) \leq L_t$. Let $N \in \mathbb{N}_0$ be given and $k \geq N$ be arbitrary so that $n^{-N} \geq n^{-k}$.

Lemma 5.4 shows that $C_k \cap (C_k + \lfloor t \rfloor_k)$ consists of $m^{\nu_t(k)}$ closed n-ary intervals which cover $C \cap (C+t)$. Let V_i denote the i^{th} such interval of length $\frac{1}{n^k}$ so that $\{V_i\}_{i=1}^{m^{\nu_t(k)}}$ is the collection of intervals or potential intervals chosen from $C_k \cap (C_k + \lfloor t \rfloor_k)$. Then

$$\mathscr{H}^s_{n^{-N}}(C \cap (C+t)) \leq \sum_{i=1}^{m^{\nu_t(k)}} |V_i|^s = m^{\nu_t(k)} \cdot \left(\frac{1}{n^k}\right)^{\beta_t \log_n(m)} = m^{\nu_t(k)-k\cdot\beta_t}. \tag{6.2}$$

Since $k \geq N$ is arbitrary, then $\mathscr{H}^s_{n^{-N}}(C \cap (C+t)) \leq \liminf_{k\to\infty}\{m^{\nu_t(k)-k\cdot\beta_t}\}$ and $\mathscr{H}^s(C \cap (C+t)) \leq L_t$ by equation (6.1). Thus, if $L_t = 0$ then $\mathscr{H}^s(C \cap (C+t)) = 0$ and we are finished.

Suppose $0 < L_t < \infty$. Then for arbitrarily small $\delta > 0$, there exists $N(\delta) \in \mathbb{N}$ such that $L_t - \delta \leq m^{\nu_t(k)-k\cdot\beta_t}$ for all $k \geq N(\delta)$. Let $\varepsilon = n^{-N(\delta)}$.

Let $\{U_\alpha\}$ be an arbitrary closed ε-cover of $C \cap (C+t)$. By compactness of $C \cap (C+t)$, there exists a finite subcover $\{U_i\}_{i=1}^r$ for some integer r. For each $1 \leq i \leq r$, let h_i denote the integer satisfying

$$\left(\frac{1}{n}\right)^{h_i+1} \leq |U_i| < \left(\frac{1}{n}\right)^{h_i}.$$

Let $k \geq \max\{h_i + 1 \mid 1 \leq i \leq r\}$ be arbitrary. For each $1 \leq i \leq r$, define $\mathcal{U}_i$ to be the collection of n-ary intervals $J \subset C_k \cap (C_k + \lfloor t \rfloor_k)$ such that J is in either the potential interval or the interval case and $J \cap U_i \neq \varnothing$. Since $\sigma_t(k) = \pm 1$ by assumption, then each $J \in \mathcal{U}_i$ contains points of $C \cap (C+t)$ by Lemma 5.5. Thus, $\bigcup_{i=1}^{r} \mathcal{U}_i = C_k \cap (C_k + \lfloor t \rfloor_k)$.

For any j, since the set $C_j \cap \left(C_j + \lfloor t \rfloor_j\right)$ contains $m^{\nu_t(j)}$ intervals which all transition the same way, then each interval $K \subset C_{h_i} \cap \left(C_{h_i} + \lfloor t \rfloor_{h_i}\right)$ transitions to $m^{\nu_t(k)-\nu_t(h_i)}$ intervals or potential intervals of $C_k \cap (C_k + \lfloor t \rfloor_k)$.

If there exists an n-ary interval J such that both J and $J - \frac{1}{n^{h_i}}$ are intervals in $C_{h_i} \cap \left(C_{h_i} + \lfloor t \rfloor_{h_i}\right)$ then J is in both the interval and potential interval case. However, $\sigma_t(h_i) = \sqrt{-1}$ by Lemma 5.2, which contradicts our assumption. Thus, any pair of n-ary intervals of $C_{h_i} \cap \left(C_{h_i} + \lfloor t \rfloor_{h_i}\right)$ are separated by at least $\frac{1}{n^{h_i}}$. Due to the diameter $\frac{1}{n^{h_i}} > |U_i|$, each U_i intersects at most one interval of $C_{h_i} \cap \left(C_{h_i} + \lfloor t \rfloor_{h_i}\right)$. Thus,

$$m^{\nu_t(k)} = \#\left(\bigcup_{i=1}^{r} \mathcal{U}_i\right) \leq \sum_{i=1}^{r} \#\mathcal{U}_i \leq \sum_{i=1}^{r} m^{\nu_t(k)-\nu_t(h_i)}.$$

Hence, $1 \leq \sum_{i=1}^{r} m^{-\nu_t(h_i)}$. Furthermore, $(L_t - \delta)\, m^{-\nu_t(h_i)} \leq m^{-h_i \cdot \beta_t}$ since $h_i \geq N(\delta)$ by choice of ε.

$$\sum_{i=1}^{r} |U_i|^s \geq \sum_{i=1}^{r} \left(\frac{1}{n}\right)^{(h_i+1)\beta_t \log_n(m)} \geq m^{-\beta_t} \cdot \sum_{i=1}^{r} m^{-\beta_t \cdot h_i} \geq$$
$$\geq m^{-\beta_t}(L_t - \delta) \sum_{i=1}^{r} m^{-\nu_t(h_i)} \geq m^{-\beta_t}(L_t - \delta). \tag{6.3}$$

Since $\{U_\alpha\}$ is an arbitrary ε-cover of $C \cap (C+t)$ then $\mathscr{H}^s_\varepsilon(C \cap (C+t)) \geq m^{-\beta_t}(L_t - \delta)$. Furthermore, $\varepsilon = n^{-N(\delta)} \to 0$ as $\delta \to 0$ so that

$$m^{-\beta_t} L_t = \lim_{\delta \to 0}\left(m^{-\beta_t}(L_t - \delta)\right) \leq \lim_{\varepsilon \to 0} \mathscr{H}^s_\varepsilon(C \cap (C+t)) = \mathscr{H}^s(C \cap (C+t)).$$

Suppose $L_t = \infty$. Then for each $j \in \mathbb{N}$ there exists $N(j) \in \mathbb{N}$ such that $j \leq m^{\nu_t(k) - k \cdot \beta_t}$ for all $k \geq N(j)$. Choose ε such that $n^{-N\left(\lceil m^{\beta_t} \cdot j \rceil\right)} > \varepsilon > 0$. Thus we can replace $(L_t - \delta)$ by $\lceil m^{\beta_t} \cdot j \rceil$ in equation (6.3) so that

$$\sum_{i=1}^{r} |U_i|^s \geq m^{-\beta_t} \cdot \lceil m^{\beta_t} \cdot j \rceil \sum_{i=1}^{r} m^{-\nu_t(h_i)} \geq j.$$

Hence, $\mathscr{H}^s(C \cap (C+t)) \geq \lim_{j \to \infty}(j) = \infty$. □

Theorem 6.1 shows that $C \cap (C+t)$ is an s-set [8] whenever $0 < L_t < \infty$ and $C \cap (C+t)$ is not self-similar for any t such that L_t is either zero or infinite. Furthermore, if $C = C_{n,D}$ is sparse and $t \in F^+$ does not admit finite n-ary representation, then $m^{-\beta_t} \cdot L_t \leq \mathscr{H}^s(C \cap (C+t)) \leq L_t$ by Theorem 5.3.

Remark 6.2. The proof of Theorem 6.1 calculates the upper bound L_t using the collection of n-ary intervals chosen from $C_k \cap (C_k + \lfloor t \rfloor_k)$. When D is sparse, then $C_k \cap (C_k + t)$ consists of $m^{\nu_t(k)}$ intervals of length $\ell_k \leq \frac{1}{n^k}$ which also cover $C \cap (C+t)$ by Lemma 5.4. Choosing this cover, we can replace $\frac{1}{n^k}$ by ℓ_k in equation (6.2) and define $\widetilde{L}_t := \liminf_{k\to\infty} \left\{ m^{\nu_t(k)} \left(\ell_k\right)^{\beta_t \log_n(m)} \right\}$ so that

$$\mathscr{H}^s \left(C \cap (C+t)\right) \leq \widetilde{L}_t \leq L_t.$$

This may calculate a more accurate upper bound for the Hausdorff measure of $C \cap (C+t)$, however it is more difficult to calculate $\widetilde{L}_t$ since ℓ_k depends directly on t. Example 7.3 shows that the Hausdorff measure may be strictly smaller than $\widetilde{L}_t$.

Corollary 6.3. *Let $C = C_{n,D}$ be given. If $t \in F^+$ does not admit finite n-ary representation and $\sigma_t(k) = \pm 1$ for all k, then the Hausdorff dimension of $C \cap (C+t)$ is $\beta_t \log_n(m)$.*

Proof. The dimension is determined by Theorem 6.1 whenever $0 < L_t < \infty$. We need to show the result when L_t is zero or infinite. Let $\varepsilon > 0$ be given and $\{U_i\}_{i=1}^r$ an arbitrary ε-cover of $C \cap (C+t)$ as in the proof of Theorem 6.1. Let $N(\varepsilon) \in \mathbb{N}$ be such that $\varepsilon > n^{-N(\varepsilon)}$.

Suppose $L_t = \infty$. Choose an arbitrary value γ such that $\beta_t < \gamma$ and choose δ such that $\gamma - \beta_t > \delta > 0$. By definition of β_t there exists a subsequence $\{h_j\}$ and integer $M(\delta)$ such that $\frac{\nu_t(h_j)}{h_j} < \beta_t + \delta < \gamma$ for all $j \geq M(\delta)$. Then for any $j \geq \max\{N(\varepsilon), M(\delta)\}$ we can replace β_t by γ in the proof of Theorem 6.1 so that

$$\mathscr{H}_\varepsilon^{\gamma \log_n(m)} \left(C \cap (C+t)\right) \leq \liminf_{j\to\infty} \left\{ m^{\nu_t(h_j) - h_j \cdot \gamma} \right\} = \liminf_{k\to\infty} \left\{ m^{\left(\frac{\nu_t(h_j)}{h_j} - \gamma\right) h_j} \right\} \leq$$
$$\leq \liminf_{k\to\infty} \left\{ m^{(\beta_t + \delta - \gamma) h_j} \right\} = 0.$$

Since $\varepsilon > 0$ is arbitrary, then $\mathscr{H}^{\gamma \log_n(m)} \left(C \cap (C+t)\right) = 0$ for any $\gamma > \beta_t$.

Suppose $L_t = 0$. Choose an arbitrary value γ such that $0 \leq \gamma < \beta_t$. Let $\Gamma_t := \liminf_{k\to\infty} \left\{ m^{\nu_t(k) - k\cdot\gamma} \right\}$. Choose δ such that $\beta_t - \gamma > \delta > 0$ and choose $M(\delta)$ such that $\Gamma_t - \delta \leq m^{\nu_t(k) - k \cdot \gamma}$ for all $k \geq M(\delta)$. Thus, we can replace β_t by γ and L_t by Γ_t in the proof of Theorem 6.1 so that $\mathscr{H}^{\gamma \log_n(m)} \left(C \cap (C+t)\right)$ is infinite whenever $\Gamma_t = \infty$.

Since $m^{(\beta_t - \delta - \gamma)} > 1$, then for any $k \geq \max\{N(\varepsilon), M(\delta)\}$,

$$m^{\nu_t(k) - k\cdot\gamma} = m^{\left(\frac{\nu_t(k)}{k} - \gamma\right) k} \geq m^{(\beta_t - \delta - \gamma) k} \geq m^{(\beta_t - \delta - \gamma) N(\varepsilon)}.$$

Hence, $\Gamma_t \geq \liminf_{N(\varepsilon)\to\infty} \left\{ m^{(\beta_t - \delta - \gamma) N(\varepsilon)} \right\} = \infty$ so that $\mathscr{H}^{\gamma \log_n(m)} \left(C \cap (C+t)\right) = \infty$ for any $0 \leq \gamma < \beta_t$. □

Corollary 6.4. *Let $C = C_{n,D}$ be sparse and $\beta, y \in \mathbb{R}$ such that $0 < \beta < 1$ and $0 < y < \infty$. Define*

$$F_{\beta,y} := \left\{ x \mid m^{-2\beta} \cdot y \leq \mathscr{H}^{\beta \cdot \log_n(m)} \left(C \cap (C+x)\right) \leq y \right\}.$$

Then $F_{\beta,y}$ is dense in F.

Proof. Choose $0 < \beta < 1$ and $0 < y < \infty$. It is sufficient to show that $F^+_{\beta,y}$ is dense in F^+. Let $t \in F^+$ and $\varepsilon > 0$ be given. We will construct the necessary $x = 0._n x_1 x_2 \ldots$.

Let $k \in \mathbb{N}$ such that $\varepsilon > \left(\frac{1}{n}\right)^{k-1} > 0$. Choose $x_j = t_j$ for all $1 \le j \le k-1$ so that $|x - t| < \varepsilon$ regardless of any choice of remaining digits x_j for $j \ge k$. If $\sigma_x(k-1) = 1$ then choose $x_k = 0$ so that $\sigma_x(k) = 1$. Otherwise, if $\sigma_x(k-1) = -1$ then choose $x_k = n - d_m$ so that $\sigma_x(k) = 1$. Thus $\sigma_x(k) = 1$ and we begin in the interval case.

Since k is finite, then $0 < m^{\nu_t(k)-k\beta} < \infty$. If $x_j = 0$ then $\mu_x(x_j) = m$ so that $\nu_x(j+1) = \nu_x(j) + 1$ and $m^{\nu_x(j)-j\cdot\beta} < m^{\nu_x(j+1)-(j+1)\beta}$. Similarly, if $x_j = d_m$ then $\mu(x_j) = 1$ so that $\nu_x(j+1) = \nu_x(j)$ and $m^{\nu_x(j)-j\cdot\beta} > m^{\nu_x(j+1)-(j+1)\beta}$. For all $j \ge k$, choose the remaining digits of x such that

$$x_{j+1} = \begin{cases} 0 & \text{if } m^{\nu_x(j)-j\cdot\beta} \le y, \\ d_m & \text{if } m^{\nu_x(j)-j\cdot\beta} > y. \end{cases}$$

Thus, if $x_{j+1} = d_m$ then $m^{\nu_x(j+1)-(j+1)\cdot\beta} = m^{-\beta} m^{\nu_x(j)-j\cdot\beta} > y \cdot m^{-\beta}$ so that

$$y \cdot m^{-\beta} \le \liminf_{j\to\infty} \left\{ m^{\nu_x(j)-j\cdot\beta} \right\} \le y.$$

Therefore, $y \cdot m^{-2\beta} \le \mathscr{H}^{\beta \log_n(m)}(C \cap (C + x)) \le y$ by Theorem 6.1. □

It would be ideal to construct x such that $L_x = y$ in the proof of Corollary 6.4, however this is not always possible. Example 6.5 shows a class of sparse Cantor sets $C_{n,D}$ such that L_t is either infinite or some element of a countable, nowhere dense subset of $\mathbb{R}$ for all $t \in F^+$.

Example 6.5. Let $n \ge 3$ and $D = \{0, d\}$ be given for some $2 \le d < n$ so that $C = C_{n,D}$ is sparse. Choose $\beta = \frac{a}{b}$ for some integers $0 \le a \le b$ and $b \ne 0$. Then $\frac{\mu_t(j)}{\mu_t(j-1)} = 1, 2$ for any $t \in F^+$ and $j \in \mathbb{N}_0$. Define $p_k := \#\{j \le k \mid \mu(j) = 2\mu_t(j-1)\}$ and $q_k := \#\{j \le k \mid \mu_t(j) = \mu_t(j-1)\}$ for each k so that $p_k, q_k \in \mathbb{N}_0$ and $k = p_k + q_k$. Thus,

$$\nu_t(k) - k\beta = p_k - (p_k + q_k)\beta = \frac{1}{b}(p_k b - a(p_k + q_k)) \in \frac{1}{b}\mathbb{Z}.$$

If $\liminf_{k\to\infty}\{\nu_t(k) - k\beta\} = -\infty$ then $L_t = 0$ and if $\liminf_{k\to\infty}\{\nu_t(k) - k\beta\} = \infty$ then $L_t = \infty$. Otherwise, any subsequence $\nu_t(k_j) - k_j\beta \to r$ is a bounded sequence of $\frac{1}{b}\mathbb{Z}$. Hence, if L_t is finite then $L_t \in \left\{2^{\frac{r}{b}} \mid r \in \mathbb{Z}\right\}$ and there is no real x such that $1 < L_x < \sqrt[b]{2}$ for this choice of $C_{n,D}$.

6.2. FINITE n-ARY REPRESENTATIONS

According to Theorem 4.1, if $t \in F^+$ admits finite n-ary representation then $C \cap (C + t)$ is either finite, or a finite collection of sets $\frac{1}{n^k}(C + h_j)$. Therefore, the Hausdorff $\log_n(m)$-dimensional measure is either zero or can be expressed in terms of $\mathscr{H}^s(C)$ for $s := \log_n(m)$.

The exact Hausdorff measure of many Cantor sets in $[0,1]$ can be calculated by methods of [1,20,21]; this includes deleted digits Cantor sets $C = C_{n,D}$. The proof of Theorem 8.6 in [8] estimates the Hausdorff measure of an arbitrary self-similar set and gives the bounds $\frac{1}{3n} \leq \mathscr{H}^s(C_{n,D}) \leq 1$. The basic idea of the proof of Theorem 6.1 leads to bounds on $\mathscr{H}^s(C_{n,D})$, we include these bounds for completeness. This is much simpler than the proof of Theorem 6.1 since the needed versions of Lemma 5.4 and Lemma 5.5 are trivial.

Theorem 6.6. *Let $C = C_{n,D}$ be given and $s := \log_n(m)$. Then $\frac{1}{m} \leq \mathscr{H}^s(C) \leq 1$.*

Proof. Let V_i denote the i^{th} n-ary interval of C_k so that $C_k = \bigcup_{i=1}^{m^k} V_i$ is a cover of C. Then $\sum_{i=1}^{m^k} |V_i|^s = m^k \cdot \left(n^{-k}\right)^{\log_n(m)} = 1$ for all $k \in \mathbb{N}_0$ so that $\mathscr{H}^s(C) \leq \mathscr{H}^s_{n^{-k}}(C) \leq 1$.

The proof of the lower bound is similar to the proof of Theorem 6.1 with minor variations. Let $\varepsilon > 0$ be given and $\{U_i\}_{i=1}^r$ be an arbitrary closed ε-cover of C for some integer r. For each $1 \leq i \leq r$, let h_i denote the integer satisfying $n^{-h_i-1} \leq |U_i| < n^{-h_i}$.

Let $k \geq \max\{h_i + 1 \mid 1 \leq i \leq r\}$ be arbitrary and, for each $1 \leq i \leq r$, define $\mathcal{U}_i$ to be the collection of n-ary intervals J selected from C_k such that $J \cap U_i \neq \varnothing$. Each $J \in \mathcal{U}_i$ contains points of C by the Nested Intervals Theorem so that $\bigcup_{i=1}^r \mathcal{U}_i = C_k$ and each interval $K \subset C_{h_i}$ contains m^{k-h_i} n-ary intervals of C_k.

Since $\frac{1}{n^{h_i}} > |U_i|$, then each U_i intersects at most two intervals of C_{h_i}. Suppose U_i intersects both K and $K - \frac{1}{n^{h_i}}$ for some n-ary interval $K \subset C_{h_i}$ and let $K(p) \subset C_k \cap K$ denote the n-ary subintervals of K for $1 \leq p \leq m^{k-h_i}$. Note that if $U_i \cap K(p) \neq \varnothing$ for some p then $U_i \cap \left(K(p) - \frac{1}{n^{h_i}}\right)$ is empty unless $K(p)$ contains an endpoint of U_i and $|U_i| > \frac{n-1}{n^{h_i+1}}$. Thus, U_i intersects at most $m^{k-h_i} + 1$ intervals of C_k so that $\#\mathcal{U}_i \leq m^{k-h_i} + 1$ for all $1 \leq i \leq r$ and

$$m^k = \#\left(\bigcup_{i=1}^r \mathcal{U}_i\right) \leq \sum_{i=1}^r \#\mathcal{U}_i \leq \sum_{i=1}^r \left(m^{k-h_i} + 1\right).$$

Therefore, $1 - r \cdot m^{-k} \leq \sum_{i=1}^r m^{-h_i}$ so that

$$\sum_{i=1}^r |U_i|^s \geq \sum_{i=1}^r \left(\frac{1}{n}\right)^{(h_i+1)s} \geq \frac{1}{m} \cdot \sum_{i=1}^r m^{-h_i} \geq \frac{1}{m}\left(1 - r \cdot m^{-k}\right). \tag{6.4}$$

Since $\{U_i\}_{i=1}^r$ is an arbitrary ε-cover of C and equation (6.4) holds for any sufficiently large k, then $\mathscr{H}^s_\varepsilon(C) \geq \lim_{k\to\infty}\left\{\frac{1}{m}\left(1 - rm^{-k}\right)\right\} = \frac{1}{m}$ for any $\varepsilon > 0$. Hence, $\frac{1}{m} \leq \mathscr{H}^s(C)$. □

Let $n = 9$, $D = \{0, d, 8\}$ for some integer $0 < d < 8$, and $s := \log_9(3) = \frac{1}{2}$. If $d = 4$ then D is uniform and $\mathscr{H}^s\left(C_{9,\{0,4,8\}}\right) = 1$. However, if $d = 2$ then D is regular and it is shown in example 7.3 that $\mathscr{H}^s\left(C_{9,\{0,2,8\}}\right) < 1$.

Corollary 6.7. *Let $C = C_{n,D}$ be arbitrary, $s := \log_n(m)$, and $t \in F^+$ such that $t = 0._n t_1 t_2 \ldots t_k$. Then $C \cap (C + t) = A \cup B$ and the following hold:*

1. *If A is nonempty, then $A = \bigcup_{j=1}^{a} \frac{1}{n^k}(C + h_j)$ for some integer a and* $\frac{a}{m^{k+1}} \leq \mathscr{H}^s(C \cap C + t) \leq \frac{a}{m^k}$. In particular, if D is sparse then $a = \mu_t(k)$.
2. *If A is empty, then $\mathscr{H}^0(C \cap (C + t)) = \#B$. If D is sparse then $\#B = \mu_{t+n^{-k}}(k) + \mu_{t-n^{-k}}(k)$.*

Proof. The general statements follow immediately from Theorem 6.6 and Theorem 4.1. We only need show the result when D is sparse. Without loss of generality, assume that k is the minimal element of $\{j \mid t = 0._n t_1 \dots t_j\}$.

Suppose A is nonempty and $s = \log_n(m)$. Since $\frac{1}{n^j} > t - \lfloor t \rfloor_j > 0$ for any $1 \leq j < k$, we can apply Lemma 4.3 so that $C_k \cap (C_k + \lfloor t \rfloor_k) = C_k \cap (C_k + t)$ consists of $\mu_t(k)$ disjoint intervals. Since each such interval refines to $\frac{1}{n^k}(C + h_j)$ and $\mathscr{H}^s(B \setminus A) = 0$, it follows that $a = \mu_t(k)$.

Suppose A is empty so that B contains a finite number of isolated points by definition of F. Any n-ary interval $J \subset C_k$ in the potential interval case is also an n-ary interval of $C_k + \lfloor t \rfloor_k + \frac{1}{n^k}$. Thus, J is in the interval case of $C_k \cap \left(C_k + \lfloor t \rfloor_k + \frac{1}{n^k}\right)$ and B contains $\mu_{t+n^{-k}}(k)$ points corresponding to potential intervals. Similarly, if $J \subset C_k \cap (C_k + \lfloor t \rfloor_k)$ is in the potentially empty case then J is an interval case of $C_k \cap \left(C_k + \lfloor t \rfloor_k - \frac{1}{n^k}\right)$ and B contains $\mu_{t-n^{-k}}(k)$ points corresponding to potentially empty cases.

Since $d - d' \geq 2$ for all $d, d' \in D \subset \Delta$, then no point of B can be in both the potential interval and potentially empty cases. Hence, $\#B = \mu_{t+n^{-k}}(k) + \mu_{t-n^{-k}}(k)$. $\square$

7. EXAMPLES

We use the results of the previous sections to estimate the Hausdorff measure of $C \cap (C + t)$. The following examples demonstrate when the Hausdorff measure is equal to both $\widetilde{L}_t$ and L_t (Example 7.1), equal to $\widetilde{L}_t$ but less than L_t (Example 7.2), or less than both $\widetilde{L}_t$ and L_t (Example 7.3).

Example 7.1. Let $C = C_{n,D}$ be sparse such that $\mathscr{H}^s(C) = 1$ for $s = \log_n(m)$. This is true for the class of uniform sets such that $d_m = n-1$ by [8]. Choose $t = 0._n t_1 t_2 \dots t_k$ for some k such that $\sigma_k(t) = 1$. Then $\nu_t(k + j) = \nu_t(k) + j$ for all $j \geq 0$ and $\beta_t = 1$ so that

$$L_t = \liminf_{j \to \infty} \left\{ m^{\nu_t(k+j) - (k+j)\beta_t} \right\} = m^{\nu_t(k) - k}.$$

Since $C \cap (C + t) = \bigcup_j \frac{1}{n^k}(C + h_j)$ consists of $m^{\nu_t(k)}$ disjoint copies of $\frac{1}{n^k}C$, then

$$\mathscr{H}^s(C \cap (C + t)) = m^{\nu_t(k) - k} \cdot \mathscr{H}^s(C) = L_t.$$

Example 7.2. Let $C = C_{3,\{0,2\}}$ denote the Middle Thirds Cantor set and let $t := 0._3\overline{20} = \frac{3}{4}$. Then $\nu_t(k) = \left\lfloor \frac{k+1}{2} \right\rfloor$ for all k so that $\nu_t(2k) = k$ and $\nu_t(2k+1) = k+1$. Thus, $\beta_t = \frac{1}{2}$ so that $\nu_t(2k) - 2k\beta_t = 0$ and $\nu_t(2k+1) - (2k+1)\beta_t = \frac{1}{2}$. Hence, $L_t = \liminf_{k \to \infty} \left\{1, \sqrt{2}, 1, \dots\right\} = 1$.

Since $\ell_{2k} = \frac{1}{9^k} - \frac{1}{9^k}\left(\frac{3}{4}\right) = \frac{1}{4\cdot 9^k}$ and $\ell_{2k+1} = \frac{1}{3\cdot 9^k} - \frac{1}{3\cdot 9^k}\left(\frac{1}{4}\right) = \frac{1}{4\cdot 9^k}$, then for $s := \log_9(2)$,

$$\widetilde{L}_t = \liminf_{k\to\infty}\left\{2^{\nu_t(k)-k}\left(\frac{1}{4}\right)^s\right\} = \liminf_{k\to\infty}\left\{\left(\frac{9}{4}\right)^s, \left(\frac{1}{4}\right)^s, \left(\frac{9}{4}\right)^s, \dots\right\} = \left(\frac{1}{4}\right)^s.$$

Therefore, $\mathscr{H}^s(C\cap(C+t)) \leq \widetilde{L}_t < L_t$. An upcoming paper, by the co-authors, shows that the Hausdorff measure is exactly 4^{-s} for this choice of $C_{n,D}$ and t.

Example 7.3. Let $n = 9$ and $D = \{0, 2, 8\}$ so that $C = C_{n,D}$ is regular. Choose $t := 0$ so that for all k, $\nu_t(k) = k$, $\ell_k = \frac{1}{n^k}$, $\beta_t = 1$, and $\widetilde{L}_t = L_t = \liminf\left\{m^{\nu_t(k)-k\beta_t}\right\} = 1$. Since $C\cap(C+t) = C$, we will show that $\mathscr{H}^s(C) < 1$ for $s := \log_9(3) = \frac{1}{2}$.

Let $\varepsilon > 0$ be given and choose k such that $\varepsilon > \frac{1}{n^{k-1}}$. Let $J = \frac{1}{n^{k-1}}(C_0 + h_j)$ be an arbitrary n-ary interval of C_{k-1}. Then the refinement of J consists of three subintervals $J(1) = \frac{1}{n^k}(C_0 + h_j n)$, $J(2) = \frac{1}{n^k}(C_0 + h_j n + 2)$, and $J(3) = \frac{1}{n^k}(C_0 + h_j n + 8)$. Choose $U_{2j-1} = \frac{1}{n^k}(3C_0 + h_j n)$ so that $J(1)\cup J(2) \subset U_{2j-1}$ and choose $U_{2j} = J(3)$. Since there are 3^{k-1} such intervals J and $\varepsilon > |J| > |U_{2j-1}| > |U_{2j}|$, then the collection $\{U_j\}_{j=1}^{2\cdot 3^{k-1}}$ is an ε-cover of C. Therefore,

$$\begin{aligned}\mathscr{H}^s_\varepsilon(C) &\leq \sum_{j=1}^{2\cdot 3^{k-1}} |U_j|^s = \sum_{j=1}^{3^{k-1}} |U_{2j-1}|^s + \sum_{j=1}^{3^{k-1}} |U_{2j}|^s = \\ &= 3^{k-1}\cdot\left(\frac{3}{9^k}\right)^s + 3^{k-1}\cdot\left(\frac{1}{9^k}\right)^s = \frac{\sqrt{3}+1}{3} < \widetilde{L}_t.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, then $\frac{1}{3} \leq \mathscr{H}^s(C) \leq \frac{\sqrt{3}+1}{3}$ according to Theorem 6.6.

Theorem 6.1 shows that the Hausdorff measure of $C\cap(C+t)$ is equal to L_t whenever L_t is zero or infinite. In the following example we construct $x, y \in F$ such that $L_x = \infty$ and $L_y = 0$ so that the sets $C\cap(C+x)$ and $C\cap(C+y)$ are not self-similar.

Example 7.4. Let $n = 11$, $D = \{0, 7, 10\}$, and $t := 0._{11}\overline{70}$ so that $\nu_t(k) = \left\lfloor\frac{k+1}{2}\right\rfloor$ for all k. Thus, $\nu_t(2k) = k$ and $\nu_t(2k+1) = k+1$ so that $\beta_t = \frac{1}{2}$ and $s := \frac{1}{2}\log_{11}(3)$. Define $x := 0._{11}x_1x_2\dots$ and $y := 0._{11}y_1y_2\dots$ such that

$$x_k = \begin{cases} 0 & \text{if } k = 1 + 2j^2 \text{ for some integer } j,\\ t_k & \text{otherwise,}\end{cases}$$

$$y_k = \begin{cases} 7 & \text{if } k = 2j^2 \text{ for some integer } j,\\ t_k & \text{otherwise.}\end{cases}$$

Since $\mu\left(t_{2j^2}\right)=1$ and $\mu\left(y_{2j^2}\right)=0$ for each integer j, and $\mu\left(t_k\right)=\mu\left(y_k\right)$ otherwise, then $\nu_y\left(2j^2\right)=\nu_t\left(2j^2\right)-j$ for each $j>0$. Thus, if $2j^2\leq k<2\left(j+1\right)^2$ for some j then $\nu_y\left(k\right)=\nu_t\left(k\right)-j$ so that $\beta_y=\beta_t=\frac{1}{2}$. Furthermore,

$$\begin{aligned}L_y &\leq \liminf_{j\to\infty}\left\{3^{\nu_y\left(2j^2\right)-\beta_y 2j^2}\right\}=\liminf_{j\to\infty}\left\{3^{\nu_t\left(2j^2\right)-j-j^2}\right\}=\\ &=\liminf_{j\to\infty}\left\{3^{-j}\right\}=0.\end{aligned}$$

Therefore, $\mathscr{H}^s\left(C\cap\left(C+y\right)\right)=L_y=0$ by Theorem 6.1.

Similarly, $\mu\left(t_{1+2j^2}\right)=0$ and $\mu\left(x_{1+2j^2}\right)=1$ for each integer j, and $\mu\left(t_k\right)=\mu\left(x_k\right)$ otherwise. Thus, $\nu_x\left(1+2j^2\right)=\nu_t\left(1+2j^2\right)+j$ for each $j>0$ and $\nu_x\left(k\right)=\nu_t\left(k\right)+j$ whenever $2j^2\leq k<2\left(j+1\right)^2$. Therefore, $\beta_x=\beta_t=\frac{1}{2}$ and for each k,

$$3^{\nu_x(k)-\beta_x(k)}=3^{\nu_t(k)+j-\beta_x(k)}\geq 3^{\frac{1}{2}k+j-\frac{1}{2}k}=3^j.$$

Hence, $\mathscr{H}^s\left(C\cap\left(C+x\right)\right)=L_x\geq\liminf_{j\to\infty}\left\{3^j\right\}=\infty$.

Theorem 6.1 requires that t does not admit finite n-ary representation and that $\sigma_t\left(k\right)=\pm1$ for all k. The infinite representation requirement allows us to ignore the potentially empty and empty cases by Lemma 4.3. The requirement that $\sigma_t\left(k\right)=\pm1$ for all k allows us to not only count the total number of intervals and potential intervals of C_k using the function $\mu_t\left(k\right)$, but also guarantees that all intervals and potential intervals contain points in $C\cap\left(C+t\right)$.

Note that L_t is calculated by counting all interval and potential interval cases at each step k. The following example demonstrates when potential interval cases do not lead to points in $C\cap(C+t)$, thus showing the necessity of Lemma 5.4 and Lemma 5.5 to the calculations in Theorem 6.1.

Example 7.5. Let $D=\{0,2,4,7,10,\ldots,4+3r\}$ for some integer $r>2$ and $n>4+3\left(r+1\right)$ so that $C=C_{n,D}$ is not sparse. Let $t:=0._n\overline{2}$ so that $\sigma_t\left(k\right)=i$ for all k. For each k, $C_k\cap\left(C_k+\lfloor t\rfloor_k\right)$ contains 2^k interval cases and $r\cdot 2^{k-1}$ potential interval cases, however the potential interval cases never contain points in $C\cap\left(C+t\right)$ since 2 is neither in $n-\Delta$ nor $n-\Delta-1$. By calculation,

$$\begin{aligned}\beta_t &=\liminf_{k\to\infty}\left\{\frac{\log_m\left(2+r\right)+\log_m\left(2^{k-1}\right)}{k}\right\}=\log_m\left(2\right),\\ L_t &=\liminf_{k\to\infty}\left\{\left(2+r\right)\cdot 2^{k-1}\cdot m^{-\beta_t k}\right\}=\frac{2+r}{2}.\end{aligned}$$

Thus, $m^{-\beta_t}=\frac{1}{2}$ and $\left[m^{-\beta_t}L_t,L_t\right]=\left[\frac{2+r}{4},\frac{2+r}{2}\right]$ by the same method as Theorem 6.1. We will show that the Hausdorff measure at most $1<\frac{2+r}{4}$:

Since potential interval cases never contain points in $C\cap\left(C+t\right)$, we can instead perform the same calculations using only the interval cases as a cover of $C\cap\left(C+t\right)$. Thus, $\beta_t=\log_m\left(2\right)$ and $s:=\log_n\left(2\right)$ so that $\mathscr{H}^s\left(C\cap\left(C+t\right)\right)\leq\liminf_{k\to\infty}\left\{2^k\cdot m^{-\beta_t k}\right\}=1$. Thus, the calculation of L_t gives an incorrect result even though β_t is calculated properly.

8. OPEN QUESTIONS

It is known that integral self-affine sets must have rational Lebesgue measure [2] so, perhaps, the range of $t \mapsto \mathscr{H}^s(C \cap (C+t))$ is not all of the interval $[0, \infty)$. See also Example 6.5.

It is likely that our methods provided an estimate of the Hausdorff measure of $C_{n,D_1} \cap (C_{n,D_2}+t)$, simply by replacing the sparcity condition by the assumption that $|\delta - \delta'| \geq 2$ for all $\delta \neq \delta'$ in $D_1 - D_2$.

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