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EXISTENCE OF CRITICAL ELLIPTIC SYSTEMS WITH BOUNDARY SINGULARITIES

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Abstract. In this paper, we are concerned with the existence of positive solutions of the following nonlinear elliptic system involving critical Hardy-Sobolev exponent

$$\begin{cases} -\Delta u = \frac{2\alpha}{\alpha+\beta}\frac{u^{\alpha-1}v^{\beta}}{|x|^s} - \lambda u^p & \text{in} \quad \Omega, \\ -\Delta v = \frac{2\beta}{\alpha+\beta}\frac{u^{\alpha}v^{\beta-1}}{|x|^s} - \lambda v^p & \text{in} \quad \Omega, \\ u > 0, v > 0 & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{$*$}$$

where $N \geq 4$ and Ω is a C^1 bounded domain in $\mathbb{R}^N$ with $0 \in \partial\Omega$. $0 < s < 2$, $\alpha + \beta = 2^*(s) = \frac{2(N-s)}{N-2}$, $\alpha, \beta > 1$, $\lambda > 0$ and $1 < p < \frac{N+2}{N-2}$. The case when 0 belongs to the boundary of Ω is closely related to the mean curvature at the origin on the boundary. We show in this paper that problem $(*)$ possesses at least a positive solution.

Keywords: existence, compactness, critical Hardy-Sobolev exponent, nonlinear system.

Mathematics Subject Classification: 35J57, 35B33, 35B40.

1. INTRODUCTION

In this paper, we are concerned with the existence of positive solutions of the following nonlinear elliptic system involving critical Hardy-Sobolev exponent

$$\begin{cases} -\Delta u = \frac{2\alpha}{\alpha+\beta}\frac{u^{\alpha-1}v^{\beta}}{|x|^s} - \lambda u^p & \text{in} \quad \Omega, \\ -\Delta v = \frac{2\beta}{\alpha+\beta}\frac{u^{\alpha}v^{\beta-1}}{|x|^s} - \lambda v^p & \text{in} \quad \Omega, \\ u > 0, v > 0 & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $N \geq 4$ and Ω is a C^1 bounded domain in $\mathbb{R}^N$ with $0 \in \partial\Omega$. We assume in this paper that $0 < s < 2$, $\alpha + \beta = 2^*(s) = \frac{2(N-s)}{N-2}$, $\alpha, \beta > 1$, $\lambda > 0$ and $1 < p < \frac{N+2}{N-2}$.

For the one equation case, the problem is related to the Caffarelli-Kohn-Nirenberg inequalities. It was discussed in [3] the existence of a minimizer of the best constant of the Caffarelli-Kohn-Nirenberg inequalities and related subject. In particular, it was shown that if $0 \in \Omega$, the best Hardy-Sobolev constant

$$\mu_{2^*(s),s}(\Omega) = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int\limits_\Omega |\nabla u|^2 dx}{\left(\int\limits_\Omega \frac{u^{2^*(s)}}{|x|^s} dx\right)^{\frac{2}{2^*(s)}}}, \tag{1.2}$$

is never attained unless $\Omega = \mathbb{R}^N$ and $\mu_{2^*(s),s}(\Omega) = \mu_{2^*(s),s}(\mathbb{R}^N)$. If $s = 0$, it is the best Sobolev constant

$$S = S(\Omega) = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int\limits_\Omega |\nabla u|^2 dx}{\left(\int\limits_\Omega |u|^{2^*} dx\right)^{\frac{2}{2^*}}},$$

where $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent and S is achieved if and only if $\Omega = \mathbb{R}^N$, see [13].

In contrast with the case $0 \in \Omega$, if $0 \in \partial\Omega$ the problem is closely related to the properties of the curvature of $\partial\Omega$ at 0. Ghoussoub and Kang showed in [5] that there exists a solution of the problem

$$-\Delta u = \frac{u^{2^*(s)-1}}{|x|^s} + \lambda u^p, \quad u > 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega,$$

where $\lambda > 0$, $1 < p < \frac{N+2}{N-2}$, $0 \in \partial\Omega$ and the mean curvature of $\partial\Omega$ at 0 is negative. Since the quantities $\|\nabla u\|_{L^2(\mathbb{R}^N)}$ and $\int_{\mathbb{R}^N} \frac{|u|^{2^*(s)}}{|x|^s} dx$ are invariant under scaling $u(x) \to r^{\frac{N-2}{2}} u(rx)$, the limiting problem of this equation is equivalent to the attainability of (1.2). The existence results of (1.3) were proved in [5] by the global compactness method. Moreover, Ghoussonb and Robert in [6] have proved that $\mu_{2^*(s),s}(\Omega)$ is achieved if $0 \in \partial\Omega$. In [9], Hsai *et al.* use the blow-up method to prove that the following elliptic equation involving two critical exponents

$$-\Delta u = \frac{u^{2^*(s)-1}}{|x|^s} + \lambda u^{\frac{N+2}{N-2}}, \quad u > 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega \tag{1.3}$$

possesses at least a positive solution.

In this paper, we deal with the exsistence of positive solutions of system (1.1). In [10], He and the first author have proved the existence of positive solutions of the problem (1.1) in non-contractible domains if $\lambda = 0$ and $s = 0$. In [14], the existence of sign-changing solutions was obtained for (1.1) with $s = 0$. Further results for the system we refer to the references in [10] and [14]. In (1.1), it involves the Hardy potential, that is $s \neq 0$, and the lower order terms are negative, which will push the energy up. We will prove that problem (1.1) possesses at least a positive solution by the blow up argument. The limiting problem after blowing up is as follows:

$$\begin{cases} -\Delta u = \frac{2\alpha}{\alpha+\beta}\frac{u^{\alpha-1}v^{\beta}}{|x|^s} & \text{in } \mathbb{R}^N_+, \\ -\Delta v = \frac{2\beta}{\alpha+\beta}\frac{u^{\alpha}v^{\beta-1}}{|x|^s} & \text{in } \mathbb{R}^N_+, \\ u>0, v>0 & \text{in } \mathbb{R}^N_+, \\ u=v=0 & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{1.4}$$

Denote

$$\mu_{\alpha,\beta,s}(\Omega) = \inf_{(u,v)\in(H_0^1(\Omega))^2\setminus\{0\}} \frac{\int\limits_\Omega (|\nabla u|^2+|\nabla v|^2)dx}{\big(\int\limits_\Omega \frac{u^\alpha v^\beta}{|x|^s}dx\big)^{\frac{2}{2^*(s)}}} \tag{1.5}$$

for a domain $\Omega \subset \mathbb{R}^N$. The solution of (1.4) will be obtained by showing that $\mu_{\alpha,\beta,s}(\mathbb{R}^N_+)$ is achieved. The minimizer of $\mu_{\alpha,\beta,s}(\mathbb{R}^N_+)$ is the least energy solution of (1.4) up to a constant. It was observed in [1] that $\mu_{\alpha,\beta,s}(\Omega)$ and $\mu_{\alpha+\beta,s}(\Omega)$ are closely related. Precisely, we have

$$\mu_{\alpha,\beta,s}(\Omega) = \Big[\Big(\frac{\alpha}{\beta}\Big)^{\frac{\beta}{\alpha+\beta}} + \Big(\frac{\alpha}{\beta}\Big)^{\frac{-\alpha}{\alpha+\beta}}\Big]\mu_{\alpha+\beta,s}(\Omega)$$

for $\alpha+\beta \le 2^*$. Moreover, if w_0 realizes $\mu_{\alpha+\beta,s}(\Omega)$, then $u_0 = Aw_0$ and $v_0 = Bw_0$ realizes $\mu_{\alpha,\beta,s}(\Omega)$ for any real constants A and B such that $\frac{A}{B} = \sqrt{\frac{\alpha}{\beta}}$.

In the case $\Omega = \mathbb{R}^N_+$, it was proved in [6] that $\mu_{2^*(s),s}(\mathbb{R}^N_+)$ is achieved by a function $u \in H_0^1(\mathbb{R}^N_+)$. This implies that $\mu_{\alpha,\beta,s}(\mathbb{R}^N_+)$ is achieved if $\alpha+\beta = 2^*(s)$. Hence, there exists a least energy entire solution of system (1.4).

To deal with (1.1), we consider a related subcritical problem, and obtain a sequence of solutions of the subcritical problems. Then, we analyse the blow up behavior of the approximating sequence. Since the coefficient of lower order terms are negative, the energy of the corresponding functional becomes larger, it makes it difficult to find the upper compact bound. Our main result is as follows.

Theorem 1.1. *Suppose that the mean curvature of $\partial\Omega$ at 0 is negative, then system (1.1) has at least a positive solution.*

In Section 2, we find a suitable upper bound for the mountain pass level, then using this bound and the blow-up argument, we prove Theorem 1.1 in Section 3.

2. EXISTENCE OF POSITIVE SOLUTION IN Ω

We establish the upper bound for the mountain pass level. We recall that by [6], $\mu_{2^*(s),s}(\mathbb{R}^N_+)$ is achieved by a function $u \in H_0^1(\mathbb{R}^N_+)$. This implies that $\mu_{\alpha,\beta,s}(\mathbb{R}^N_+)$ is achieved if $\alpha+\beta = 2^*(s)$. Hence, there exists a least energy entire solution of system (1.4). Furthermore, it was shown in [12] that the following result holds.

Lemma 2.1. *Let $u \in H_0^1(\mathbb{R}_+^N)$ be an entire solution of the equation*

$$\begin{cases} -\Delta u = \frac{u^{2^*(s)-1}}{|y|^s} & in \quad \mathbb{R}_+^N, \\ u > 0 \quad in \quad \mathbb{R}_+^N, & u = 0 \quad on \quad \partial\mathbb{R}_+^N. \end{cases} \tag{2.1}$$

Then there is a constant C such that $|u(y)| \leq C(1+|y|)^{1-N}$ and $|\nabla u(y)| \leq$ $\leq C(1+|y|)^{-N}$.

Therefore, each component of the least energy solution of (1.4) enjoys the same properties in Lemma 2.1. It was proved in [5] that the following result holds.

Lemma 2.2. *If $N \geq 4$, then we have*

$$1 < \mu_{2^*(s),s}(\mathbb{R}^N) < \mu_{2^*(s),s}(\mathbb{R}_+^N).$$

We remark that Lemma 2.2 implies $\mu_{\alpha,\beta,s}(\mathbb{R}_+^N) > 1$. Indeed, since

$$\mu_{\alpha,\beta,s}(\mathbb{R}_+^N) = \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{\frac{-\alpha}{\alpha+\beta}}\right]\mu_{\alpha+\beta,s}(\mathbb{R}_+^N)$$

where $\alpha, \beta > 1$, by Lemma 2.2 we have $\mu_{\alpha+\beta,s}(\mathbb{R}_+^N) > 1$, and it is easily to verify that $(\frac{\alpha}{\beta})^{\frac{\beta}{\alpha+\beta}} + (\frac{\alpha}{\beta})^{\frac{-\alpha}{\alpha+\beta}} > 1$.

The energy functional for (1.1) is well defined on $H_0^1(\Omega)$ by

$$I_\lambda(u,v) = \int\limits_\Omega \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla v|^2 - \frac{2}{2^*(s)}\frac{u^\alpha v^\beta}{|x|^s} + \frac{\lambda}{p+1}u^{p+1} + \frac{\lambda}{p+1}v^{p+1}\right)dx.$$

It is well known that to find positive solutions of problem (1.1) is equivalent to finding nonzero critical points of functional I_λ in $H_0^1(\Omega) \times H_0^1(\Omega)$. Now, we bound the mountain pass level for the functional I_λ.

Lemma 2.3. *Suppose that Ω is a C^1 bounded domain in $\mathbb{R}^N$ with $0 \in \partial\Omega$, $\partial\Omega$ is C^2 at 0. If the mean curvature of $\partial\Omega$ at 0 is negative and $1 \leq p < \frac{N}{N-2}$. Then there exist nonnegative functions u_0 and v_0 in $H_0^1(\Omega) \setminus \{0\}$ such that $I_\lambda(u_0,v_0) < 0$ and*

$$\max_{0\leq t\leq 1} I_\lambda(tu_0,tv_0) < 2^{\frac{-2}{2^*(s)-2}}\left(\frac{1}{2} - \frac{1}{2^*(s)}\right)\mu_{\alpha,\beta,s}(\mathbb{R}_+^N)^{\frac{2^*(s)}{2^*(s)-2}}.$$

Proof. Let (u,v) be the minimizer of $\mu_{\alpha,\beta,s}(\mathbb{R}_+^N)$ such that

$$\int\limits_{\mathbb{R}_+^N} |\nabla u|^2 dx + \int\limits_{\mathbb{R}_+^N} |\nabla v|^2 dx = \mu_{\alpha,\beta,s}(\mathbb{R}_+^N), \qquad \int\limits_{\mathbb{R}_+^N} \frac{u^\alpha v^\beta}{|x|^s} dx = 1.$$

Then, there exist A, $B \in \mathbb{R}$ such that $u = Aw$, $v = Bw$ with $\frac{A}{B} = \sqrt{\frac{\alpha}{\beta}}$, where w is a minimizer of $\mu_{2^*(s),s}(\mathbb{R}_+^N)$. Since

$$|w(x)| \leq C(1+|x|)^{1-N}, \qquad |\nabla w(x)| \leq C(1+|x|)^{-N},$$

we obtain

$$|u(x)| \le C(1+|x|)^{1-N}, \quad |\nabla u(x)| \le C(1+|x|)^{-N} \tag{2.2}$$

and

$$|v(x)| \le C(1+|x|)^{1-N}, \quad |\nabla v(x)| \le C(1+|x|)^{-N}. \tag{2.3}$$

Moreover, (u, v) satisfies

$$-\Delta u = \tfrac{\alpha}{\alpha+\beta}\mu_{\alpha,\beta,s}(\mathbb{R}^N_+)\tfrac{u^{\alpha-1}v^{\beta}}{|x|^s}, \quad -\Delta v = \tfrac{\beta}{\alpha+\beta}\mu_{\alpha,\beta,s}(\mathbb{R}^N_+)\tfrac{u^{\alpha}v^{\beta-1}}{|x|^s} \qquad \text{in} \quad \mathbb{R}^N_+. \tag{2.4}$$

Without loss of generality, we may assume that in a neighborhood of 0, the boundary $\partial\Omega$ can be represented by $x_N = \varphi(x')$, where $x' = (x_1, \dots x_{N-1})$, $\varphi(0) = 0$, $\nabla'\varphi(0) = 0$, $\nabla' = (\partial_1, \dots, \partial_{N-1})$ and the outward normal of $\partial\Omega$ at 0 is $-e_N = (0, 0, \dots, -1)$. Define

$$\psi(x) = (x', x_N - \varphi(x')).$$

We choose a small positive number r_0 so that there exist neighborhoods U and $\tilde{U}$ of 0, such that $\psi(U) = B_{r_0}(0)$, $\psi(U \cap \Omega) = B^+_{r_0}(0)$, $\psi(\tilde{U}) = B_{\frac{r_0}{2}}(0)$, $\psi(\tilde{U} \cap \Omega) = B^+_{\frac{r_0}{2}}(0)$. For $\varepsilon > 0$, we define

$$\tilde{u}_\varepsilon(x) = \varepsilon^{-\frac{N-2}{2}}\eta(x)u\left(\frac{\psi(x)}{\varepsilon}\right) = \eta(x)u_\varepsilon, \qquad \tilde{v}_\varepsilon(x) = \varepsilon^{-\frac{N-2}{2}}\eta(x)v\left(\frac{\psi(x)}{\varepsilon}\right) = \eta(x)u_\varepsilon,$$

where $\eta \in C_0^\infty(U)$ is a positive cut-off function with $\eta \equiv 1$ in $\tilde{U}$. In what follows, we estimate each term in $I_\lambda(t\tilde{u}_\varepsilon, t\tilde{v}_\varepsilon)$. Apparently,

$$\int_\Omega |\nabla \tilde{u}_\varepsilon|^2\, dx = \int_\Omega (|\nabla\eta|^2 u_\varepsilon^2 + \eta^2|\nabla u_\varepsilon|^2 + 2\nabla\eta\nabla u_\varepsilon \eta u_\varepsilon)\, dx.$$

Since

$$\int_\Omega \eta u_\varepsilon \nabla\eta\nabla u_\varepsilon\, dx = -\int_\Omega |\nabla\eta|^2 u_\varepsilon^2\, dx - \int_\Omega \nabla\eta\eta\nabla u_\varepsilon u_\varepsilon\, dx - \int_\Omega \eta(\Delta\eta)|u_\varepsilon|^2\, dx,$$

we obtain

$$\int_\Omega |\nabla\tilde{u}_\varepsilon|^2\, dx = \int_{\Omega\cap U} \eta^2|\nabla u_\varepsilon|^2\, dx - \int_{\Omega\cap U} \eta(\Delta\eta)|u_\varepsilon|^2\, dx.$$

By the change of the variable $y = \frac{\psi(x)}{\varepsilon} \in B^+_{\frac{r_0}{\varepsilon}}(0)$ and (3.1), (3.2), we obtain

$$\left|\int_{\Omega\cap U} \eta(\Delta\eta)u_\varepsilon^2\, dx\right| \le C\varepsilon^2 \int_{B^+_{\frac{r_0}{\varepsilon}}(0)\setminus B^+_{\frac{r_0}{2\varepsilon}}(0)} \eta(\psi^{-1}(\varepsilon y))\big|\Delta\eta(\psi^{-1}(\varepsilon y))\big|u^2(y)\, dy =$$

$$= O(\varepsilon^2)$$

and

$$\int\limits_{\Omega\cap U} \eta^2|\nabla u_\varepsilon(x)|^2\,dx = \varepsilon^{2-N}\int\limits_{B^+_{\frac{r_0}{\varepsilon}}} \eta^2(\psi^{-1}(\varepsilon y))|\nabla_x u(y)|^2\varepsilon^N dy.$$

Since

$$|\nabla_x u(y)|^2 = \frac{1}{\varepsilon^2}|\nabla_y u(y)|^2 - \frac{2}{\varepsilon^3}\partial_N u(y)\nabla' u(y)\nabla'[\varphi(\varepsilon y')] + \frac{1}{\varepsilon^4}[\partial_N u(y)]^2(\nabla'[\varphi(\varepsilon y')])^2,$$

we deduce that

$$\begin{aligned}\int\limits_{B^+_{\frac{r_0}{\varepsilon}}} &\eta^2(\psi^{-1}(\varepsilon y))|\nabla_x u(y)|^2\varepsilon^2\,dy \le \\ &\le \int\limits_{\mathbb{R}^N_+}|\nabla_y u(y)|^2 dy - 2\int\limits_{B^+_{\frac{r_0}{\varepsilon}}}\eta^2(\psi^{-1}(\varepsilon y))\partial_N u(y)\nabla' u(y)(\nabla'\varphi)(\varepsilon y')\,dy + \\ &\quad + \int\limits_{B^+_{\frac{r_0}{\varepsilon}}}\eta^2(\psi^{-1}(\varepsilon y))|\partial_N u(y)|^2|(\nabla'\varphi)(\varepsilon y')|^2\,dy = I_1 + I_2 + I_3.\end{aligned} \tag{2.5}$$

Using the facts

$$|\nabla'\varphi(y')| = O(|y'|), \quad \varphi(y') = \sum_{i=1}^{N-1}\alpha_i y_i^2 + o(1)(|y'|^2),$$

(2.2) and (2.3), we see that

$$I_3 \le C\int\limits_{\mathbb{R}^N}(1+|y|)^{-2N}|\varepsilon y|^2\,dy = O(\varepsilon^2).$$

Integrating by parts, we infer that

$$\begin{aligned}I_2 &= \frac{4}{\varepsilon}\int\limits_{B^+_{\frac{r_0}{\varepsilon}}}\eta(\psi^{-1}(\varepsilon y))\nabla'\eta(\psi^{-1}(\varepsilon y))\partial_N u(y)\nabla' u(y)\varphi(\varepsilon y')dy + \\ &\quad + \frac{2}{\varepsilon}\int\limits_{B^+_{\frac{r_0}{\varepsilon}}}\eta^2(\psi^{-1}(\varepsilon y))\nabla'\partial_N u(y)\nabla' u(y)\varphi(\varepsilon y')dy + \\ &\quad + \frac{2}{\varepsilon}\int\limits_{B^+_{\frac{r_0}{\varepsilon}}}\eta^2(\psi^{-1}(\varepsilon y))\partial_N u(y)\sum_{i=1}^{N-1}\partial_{ii}u(y)\varphi(\varepsilon y')dy = I_{21} + I_{22} + I_{23}.\end{aligned}$$

By (2.2) and (2.3),

$$|I_{21}| \leq C\varepsilon^2 \int\limits_{B^+_{\frac{r_0}{\varepsilon}}(0)\setminus B_{\frac{r_0}{2\varepsilon}}(0)} (1+|y|)^{-2N}|y|^2\,dy \leq C\varepsilon^N.$$

In the same way, $I_{22} = O(\varepsilon^N)$. By (2.4),

$$\sum_{i=1}^{N-1} \partial_{ii}u(y) = \Delta u - \partial_{NN}u(y) = -\frac{\alpha}{\alpha+\beta}\mu_{\alpha,\beta,s}(\mathbb{R}^N_+)\frac{u^{\alpha-1}v^{\beta}}{|y|^s} - \partial_{NN}u(y).$$

Therefore,

$$I_{23} = -\frac{2}{\varepsilon}\int\limits_{B^+_{\frac{r_0}{\varepsilon}}} \eta^2(\psi^{-1}(\varepsilon y))\partial_N u(y)\frac{\alpha}{\alpha+\beta}\mu_{\alpha,\beta,s}(\mathbb{R}^N_+)\frac{u^{\alpha-1}v^{\beta}}{|y|^s}\varphi(\varepsilon y')\,dy-$$
$$-\frac{2}{\varepsilon}\int\limits_{B^+_{\frac{r_0}{\varepsilon}}} \eta^2(\psi^{-1}(\varepsilon y))\partial_N u(y)\partial_{NN}u(y)\varphi(\varepsilon y')dy := F_1 + F_2.$$

Since $u = Aw$,

$$F_1 = -\frac{C_0}{\varepsilon}\int\limits_{B^+_{\frac{r_0}{\varepsilon}}} \eta^2(\psi^{-1}(\varepsilon y))\frac{\partial_N w(y)^{2^*(s)}}{|y|^s}\varphi(\varepsilon y')\,dy,$$

where $C_0 = \frac{2\alpha}{(2^*(s))^2}\mu_{\alpha,\beta,s}(\mathbb{R}^N_+)A^{\alpha}B^{\beta}$. Integrating by parts, we obtain

$$F_1 = \frac{C_0}{\varepsilon}\int\limits_{B^+_{\frac{r_0}{\varepsilon}}} \frac{2\eta(\psi^{-1}(\varepsilon y))\partial_N\eta(\psi^{-1}(\varepsilon y))\varphi(\varepsilon y')}{|y|^s}w^{2^*(s)}\,dy+$$
$$+\frac{C_0}{\varepsilon}\int\limits_{B^+_{\frac{r_0}{\varepsilon}}} \frac{\eta^2(\psi^{-1}(\varepsilon y))\partial_N\varphi(\varepsilon y')}{|y|^s}w^{2^*(s)}\,dy-$$
$$-\frac{C_0 s}{\varepsilon}\int\limits_{B^+_{\frac{r_0}{\varepsilon}}} \frac{\eta^2(\psi^{-1}(\varepsilon y))\varphi(\varepsilon y')y_N}{|y|^{s+2}}w^{2^*(s)}\,dy = F_{11}+F_{12}+F_{13}.$$

We may verify as above that

$$F_{11} = O(\varepsilon^{\frac{N^2-N-Ns+2}{N-2}}), \quad F_{12} = O(\varepsilon^{\frac{N^2-N-Ns+2}{N-2}}).$$

Now, we estimate F_2. Integrating by parts, we deduce

$$\begin{aligned} F_2 &= \frac{1}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}} \partial_N[\eta^2(\psi^{-1}(\varepsilon y))\varphi(\varepsilon y')](\partial_N u)^2\, dy + \\ &\quad + \frac{1}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}\cap\partial\mathbb{R}^N_+} \eta^2(\psi^{-1}(\varepsilon y))\varphi(\varepsilon y')(\partial_N u)^2\nu^N\, dS_y = \\ &= \frac{1}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}} 2\eta(\psi^{-1}(\varepsilon y))\partial_N[\eta(\psi^{-1}(\varepsilon y))]\varphi(\varepsilon y')(\partial_N u)^2\, dy + \\ &\quad + \frac{1}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}} \eta^2(\psi^{-1}(\varepsilon y))\partial_N[\varphi(\varepsilon y')](\partial_N u)^2 dy + \\ &\quad + \frac{1}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}\cap\partial\mathbb{R}^N_+} \eta^2(\psi^{-1}(\varepsilon y))\varphi(\varepsilon y')(\partial_N u)^2\, dS_y = \\ &= F_{21} + F_{22} + F_{23}. \end{aligned}$$

It can be shown that $F_{21} = O(\varepsilon^{N-1})$, $F_{22} = O(\varepsilon^{N-1})$. Hence,

$$I_2 = F_{13} + F_{23} + O(\varepsilon^{N-1}).$$

Since $\eta(\psi^{-1}(\varepsilon y)) \equiv 1$ in $B^+_{\frac{r_0}{2\varepsilon}}$, we have

$$\begin{aligned} F_{13} = &-\frac{C_0 s}{\varepsilon}\int_{B^+_{\frac{r_0}{\varepsilon}}\setminus B^+_{\frac{r_0}{2\varepsilon}}} \frac{\eta^2(\psi^{-1}(\varepsilon y))\varphi(\varepsilon y')y_N}{|y|^{s+2}} w^{2^*(s)}\, dy - \\ &- \frac{C_0 s}{\varepsilon}\int_{B^+_{\frac{r_0}{2\varepsilon}}} \frac{\varphi(\varepsilon y')y_N}{|y|^{s+2}} w^{2^*(s)}\, dy = J_1 + J_2. \end{aligned}$$

We have

$$J_1 \le C\varepsilon \int_{B^+_{\frac{r_0}{\varepsilon}}\setminus B^+_{\frac{r_0}{2\varepsilon}}} \frac{|y|^3(1+|y|)^{(1-N)2^*(s)}}{|y|^{s+2}}\, dy \le C\varepsilon^{\frac{N(N-s)}{N-2}}.$$

In the same way,

$$J_2 = -\frac{C_0 s}{\varepsilon}\int\limits_{\mathbb{R}^N_+}\frac{\varphi(\varepsilon y')y_N}{|y|^{s+2}}w^{2^*(s)}\,dy - \frac{C_0 s}{\varepsilon}\int\limits_{\mathbb{R}^N_+\setminus B^+_{\frac{r_0}{\varepsilon}}}\frac{\varphi(\varepsilon y')y_N}{|y|^{s+2}}w(y)^{2^*(s)}dy =$$

$$= -\frac{C_0 s}{\varepsilon}\int\limits_{\mathbb{R}^N_+}\frac{\varphi(\varepsilon y')y_N}{|y|^{s+2}}w^{2^*(s)}dy + O(\varepsilon^{\frac{N(N-s)}{N-2}}) =$$

$$= -\varepsilon C_0 s\sum_{i=1}^{N-1}\alpha_i\int\limits_{\mathbb{R}^N_+}\frac{y_i^2 y_N w(y)^{2^*(s)}}{|y|^{s+2}}dy(1+o(1)) + O(\varepsilon^{\frac{N(N-s)}{N-2}}) =$$

$$= -\frac{s\varepsilon c_1}{N-1}\int\limits_{\mathbb{R}^N_+}\frac{|y'|^2 y_N w(y)^{2^*(s)}}{|y|^{s+2}}dy\sum_{i=1}^{N-1}\alpha_i(1+o(1)) + O(\varepsilon^{\frac{N(N-s)}{N-2}}) =$$

$$= -C_0 K_1 H(0)(1+o(1))\varepsilon + O(\varepsilon^{\frac{N(N-s)}{N-2}}),$$

where

$$H(0) = \frac{1}{N-1}\sum_{i=1}^{N-1}\alpha_i, \quad K_1 = s\int\limits_{\mathbb{R}^N_+}\frac{|y'|^2 y_N w^{2^*(s)}}{|y|^{s+2}}\,dy.$$

Similarly,

$$F_{23} = \frac{1}{\varepsilon}\int\limits_{(B^+_{\frac{r_0}{\varepsilon}}\setminus B^+_{\frac{r_0}{2\varepsilon}})\cap\partial\mathbb{R}^N_+}\eta^2(\psi^{-1}(\varepsilon y))\varphi(\varepsilon y')(\partial_N u(y))^2\,dS_y +$$

$$+\frac{1}{\varepsilon}\int\limits_{B^+_{\frac{r_0}{2\varepsilon}}\cap\partial\mathbb{R}^N_+}\varphi(\varepsilon y')(\partial_N u(y))^2\,dS_y = L_1 + L_2.$$

There holds

$$L_1 \le \frac{C}{\varepsilon}\int\limits_{\{\frac{r_0}{2}<|\varepsilon y'|\le r_0\}}|(\partial_N u)(y',0)|^2|\varphi(\varepsilon y')|\,dy' \le$$

$$\le C\varepsilon\int\limits_{\{\frac{r_0}{2}<|\varepsilon y'|\le r_0\}}|y'|^{-2N+2}\,dy' = O(\varepsilon^N).$$

Using the fact

$$\int\limits_{\mathbb{R}^{N-1}\setminus(B^+_{\frac{r_0}{2\varepsilon}}\cap\partial\mathbb{R}^N_+)}\varphi(\varepsilon y')(\partial_N u(y))^2\,dS_y = O(\varepsilon^N),$$

one finds

$$
\begin{aligned}
L_2 &= \frac{1}{\varepsilon}\int\limits_{\mathbb{R}^{N-1}} \varphi(\varepsilon y')(\partial_n u(y))^2\,dS_y - \frac{1}{\varepsilon}\int\limits_{\mathbb{R}^{N-1}\setminus(B^+_{\frac{r_0}{2\varepsilon}}\cap\partial\mathbb{R}^N_+)} \varphi(\varepsilon y')(\partial_N u(y))^2\,dS_y = \\
&= \frac{1}{\varepsilon}\int\limits_{\mathbb{R}^{N-1}} \varphi(\varepsilon y')(\partial_N u(y))^2 dSy + O(\varepsilon^{N-1}) = \\
&= \varepsilon\sum_{i=1}^{N-1}\alpha_i \int\limits_{\mathbb{R}^{N-1}} [(\partial_N u)(y',0)]^2 y_i^2 dy'(1+o(1)) + O(\varepsilon^{N-1}) = \\
&= K_2 H(0)(1+o(1))\varepsilon + O(\varepsilon^{N-1}),
\end{aligned}
$$

where $K_2 = \int\limits_{\mathbb{R}^{N-1}} |(\partial_N u)(y',0)|^2|y'|^2\,dy'$. Consequently,

$$
\int\limits_{\Omega} |\nabla\tilde{u}_\varepsilon|^2\,dx = \int\limits_{\mathbb{R}^N_+} |\nabla u|^2\,dy - (C_0K_1 - K_2)H(0)(1+o(1))\varepsilon + O(\varepsilon^2),
$$

and similarly,

$$
\int\limits_{\Omega} |\nabla\tilde{v}_\varepsilon|^2\,dx = \int\limits_{\mathbb{R}^N_+} |\nabla v|^2\,dy - (C_1K_1 - K_2)H(0)(1+o(1))\varepsilon + O(\varepsilon^2).
$$

where $C_1 = \frac{2\beta}{(2^*(s))^2}\mu_{\alpha,\beta,s}(\mathbb{R}^N_+)A^\alpha B^\beta$.

Next, let $y = \frac{\psi(x)}{\varepsilon}$. We estimate

$$
\int\limits_{\Omega} \frac{\tilde{u}^\alpha_\varepsilon\tilde{v}^\beta_\varepsilon}{|x|^s}dx \geq \int\limits_{\Omega\cap\tilde{U}} \frac{\tilde{u}^\alpha_\varepsilon\tilde{v}^\beta_\varepsilon}{|x|^s}\,dx = \int\limits_{\Omega\cap\tilde{U}} \frac{u^\alpha_\varepsilon v^\beta_\varepsilon}{|x|^s}\,dx = \int\limits_{B^+_{\frac{r_0/2}{\varepsilon}}} \frac{u^\alpha(y)v^\beta(y)}{|\frac{\psi^{-1}(\varepsilon y)}{\varepsilon}|^s}\,dy,
$$

since $\eta \equiv 1$ in $\Omega\cap\tilde{U}$. The facts

$$
\frac{1}{|\frac{\phi^{-1}(\varepsilon y)}{\varepsilon}|^s} = \frac{1}{|y|^s}\left(1 - \frac{s y_N\varphi(\varepsilon y')}{\varepsilon|y|^2} - \frac{s\varphi^2(\varepsilon y')}{2\varepsilon^2|y|^2}\right) + \frac{1}{|y|^s}O\left(\Big(\frac{2y_N\varphi(\varepsilon y')}{\varepsilon|y|^2} + \frac{\varphi^2(\varepsilon y')}{\varepsilon^2|y|^2}\Big)^2\right)
$$

and

$$
\int\limits_{\mathbb{R}^N_+\setminus B^+_{\frac{r_0}{2\varepsilon}}} \frac{u^\alpha v^\beta}{|y|^s}\,dy = O(\varepsilon^{\frac{N(N-s)}{N-2}})
$$

enable us to show that

$$\int\limits_{\Omega\cap\tilde{U}} \frac{\tilde{u}_\varepsilon^\alpha \tilde{v}_\varepsilon^\beta}{|x|^s}\,dx = \int\limits_{B^+_{\frac{r_0}{2\varepsilon}}} \frac{u^\alpha v^\beta}{|y|^s}\,dy - \frac{s}{\varepsilon}\int\limits_{B^+_{\frac{r_0}{2\varepsilon}}} \frac{y_N\varphi(\varepsilon y')u^\alpha(y)v^\beta(y)}{|y|^{s+2}}\,dy + O(\varepsilon^2) =$$

$$= \int\limits_{\mathbb{R}^N_+} \frac{u^\alpha v^\beta}{|y|^s}\,dy - \frac{s}{\varepsilon}\int\limits_{B^+_{\frac{r_0}{2\varepsilon}}} \frac{y_N\varphi(\varepsilon y')u^\alpha v^\beta}{|y|^{s+2}}\,dy + O(\varepsilon^2).$$

Moreover,

$$-\frac{s}{\varepsilon}\int\limits_{B^+_{\frac{r_0}{2\varepsilon}}} \frac{y_N\varphi(\varepsilon y')u^\alpha v^\beta}{|y|^{s+2}}\,dy = -\frac{s}{\varepsilon}A^\alpha B^\beta \int\limits_{B^+_{\frac{r_0}{2\varepsilon}}} \frac{y_N\varphi(\varepsilon y')w^{2^*(s)}}{|y|^{s+2}}\,dy =$$

$$= -s\varepsilon\sum_{i=1}^{N-1}\alpha_i A^\alpha B^\beta \int\limits_{\mathbb{R}^N_+} \frac{y_N y_i^2 w^{2^*(s)}}{|y|^{s+2}}\,dy(1+o(1)) + O(\varepsilon^{\frac{N(N-s)}{N-2}}) =$$

$$= -\frac{s\varepsilon}{N-1}A^\alpha B^\beta \int\limits_{\mathbb{R}^N_+} \frac{y_N|y'|^2 w^{2^*(s)}}{|y|^{s+2}}\,dy \sum_{i=1}^{N-1}\alpha_i(1+o(1)) + O(\varepsilon^{\frac{N(N-s)}{N-2}}).$$

Hence,

$$\int\limits_{\Omega\cap\tilde{U}} \frac{\tilde{u}_\varepsilon^\alpha \tilde{v}_\varepsilon^\beta}{|x|^s}\,dx = \int\limits_{\mathbb{R}^N_+} \frac{u^\alpha v^\beta}{|y|^s}\,dy - K_3 H(0)(1+o(1))\varepsilon + O(\varepsilon^2),$$

where $K_3 = sA^\alpha B^\beta \int\limits_{\mathbb{R}^N_+} \frac{y_N|y'|^2 w^{2^*(s)}}{|y|^{s+2}}\,dy = A^\alpha B^\beta K_1$.

Finally, let $y = \frac{\psi(x)}{\varepsilon} \in B^+_{\frac{r_0}{\varepsilon}}(0)$. We deduce that

$$\int\limits_{\Omega} \tilde{u}_\varepsilon^{p+1}\,dx = \varepsilon^{\frac{(2-N)(p+1)}{2}} \int\limits_{\Omega\cap U} \eta^2(x)\left[u\left(\frac{\psi(x)}{\varepsilon}\right)\right]^{p+1} dx =$$

$$= \varepsilon^{\frac{(2-N)(p+1)}{2}+N} \int\limits_{B^+_{\frac{r_0}{\varepsilon}}} u^{p+1}\,dy =$$

$$= \varepsilon^{\frac{N+2}{2}-\frac{(N-2)p}{2}} \int\limits_{\mathbb{R}^N_+} u^{p+1}\,dy + O(\varepsilon^{\frac{N(p+1)}{2}}).$$

Similarly,

$$\int\limits_{\Omega} \tilde{v}_\varepsilon^{p+1}\,dx = \varepsilon^{\frac{N+2}{2}-\frac{(N-2)p}{2}} \int\limits_{\mathbb{R}^N_+} v^{p+1}\,dy + O(\varepsilon^{\frac{N(p+1)}{2}}).$$

Since $q < \frac{N}{N-2}$, $\frac{N+2}{2} - \frac{(N-2)p}{2} > 1$. For $t \geq 0$, we have

$$I_\lambda(t\tilde{u}_\varepsilon, t\tilde{v}_\varepsilon) = \frac{t^2}{2}\left(\int\limits_{\mathbb{R}^N_+} |\nabla u|^2\, dy + \int\limits_{\mathbb{R}^N_+} |\nabla v|^2\, dy\right) - \frac{2t^{2^*(s)}}{2^*(s)} \int\limits_{\mathbb{R}^N_+} \frac{u^\alpha v^\beta}{|y|^s}\, dy +$$
$$+ \frac{H(0)}{2}\left[(2K_2 - C_0K_1 - C_1K_1)t^2 + \frac{4}{2^*(s)}(K_3 + o(1))t^{2^*(s)}\right]\varepsilon + O(\varepsilon^2) =$$
$$= f_1(t) + \frac{H(0)}{2}\varepsilon f_2(t) + O(\varepsilon^2),$$

where

$$f_1(t) = \frac{t^2}{2}\mu_{\alpha,\beta,s}(\mathbb{R}^N_+) - \frac{2t^{2^*(s)}}{2^*(s)}.$$

We may verify that

$$\max_{0\leq t\leq 1} f_1(t) = f_1(t_0) = 2^{\frac{-2}{2^*(s)-2}}\left(\frac{1}{2} - \frac{1}{2^*(s)}\right)\mu_{\alpha,\beta,s}(\mathbb{R}^N_+)^{\frac{2^*(s)}{2^*(s)-2}},$$

with $t_0 = (\frac{1}{2}\mu_{\alpha,\beta,s}(\mathbb{R}^N_+))^{\frac{1}{2^*(s)-2}}$. Since $K_1 > 0$,

$$f_2(t_0) = (2K_2 - C_0K_1 - C_1K_1)t_0^2 + \frac{4}{2^*(s)}K_3 t_0^{2^*(s)} =$$
$$= \left(2K_2 - \frac{2}{2^*(s)}A^\alpha B^\beta K_1\right)t_0^2 + \frac{4}{2^*(s)}A^\alpha B^\beta K_1 t_0^{2^*(s)} =$$
$$= 2K_2 t_0^2 + \frac{4}{2^*(s)}A^\alpha B^\beta K_1\left(1 - \frac{1}{\mu_{\alpha,\beta,s}(\mathbb{R}^N_+)}\right)t_0^{2^*(s)}.$$

$f_2(t_0) > 0$ if and only if $\mu_{\alpha,\beta,s}(\mathbb{R}^N_+) > 1$.

Since $H(0) < 0$, by choosing T large enough, we have $I_\lambda(T\tilde{u}_\varepsilon, T\tilde{v}_\varepsilon) < 0$ for $t \geq T$ and $\varepsilon \geq 0$ small. Let $u_0 = T\tilde{u}_\varepsilon$, $v_0 = T\tilde{v}_\varepsilon$. We obtain

$$\max_{0\leq t\leq 1} I_\lambda(tu_0, tv_0) < 2^{\frac{-2}{2^*(s)-2}}\left(\frac{1}{2} - \frac{1}{2^*(s)}\right)\mu_{\alpha,\beta,s}(\mathbb{R}^N_+)^{\frac{2^*(s)}{2^*(s)-2}}$$

and

$$I_\lambda(u_0, v_0) < 0.$$

This completes the proof of Lemma 2.1. □

3. EXISTENCE OF POSITIVE SOLUTION IN Ω

Now we will use the blow up argument to prove Theorem 1.1.

For any $\varepsilon > 0$, by applying Lemma 2.1 and the mountain pass theorem, we have a positive solution pair $(u_\varepsilon, v_\varepsilon)$ of the following subcritical system

$$\begin{cases} -\Delta u_\varepsilon = \frac{2\alpha}{\alpha+\beta-\varepsilon}\frac{u_\varepsilon^{\alpha-1}v_\varepsilon^{\beta-\varepsilon}}{|x|^s} - \lambda u_\varepsilon^{p-\varepsilon} & \text{in} \quad \Omega, \\ -\Delta v_\varepsilon = \frac{2\beta}{\alpha+\beta-\varepsilon}\frac{u_\varepsilon^{\alpha}v_\varepsilon^{\beta-1-\varepsilon}}{|x|^s} - \lambda v_\varepsilon^{p-\varepsilon} & \text{in} \quad \Omega, \\ u_\varepsilon > 0, v_\varepsilon > 0 & \text{in} \quad \Omega, \\ u_\varepsilon = v_\varepsilon = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.1}$$

The mountain pass level c_ε of (3.1) satisfies

$$c_\varepsilon = I_\lambda^\varepsilon(u_\varepsilon, v_\varepsilon) < 2^{\frac{-2}{2^*(s)-2}}\left(\frac{1}{2} - \frac{1}{2^*(s)}\right)\mu_{\alpha,\beta,s}(\mathbb{R}_+^N)^{\frac{2^*(s)}{2^*(s)-2}}, \tag{3.2}$$

where

$$I_\varepsilon(u_\varepsilon, v_\varepsilon) = \int\limits_\Omega \left(\frac{1}{2}|\nabla u_\varepsilon|^2 + \frac{1}{2}|\nabla v_\varepsilon|^2 - \frac{2}{2^*(s)-\varepsilon}\frac{u_\varepsilon^\alpha v_\varepsilon^{\beta-\varepsilon}}{|x|^s}\right)dx+$$
$$+\int\limits_\Omega \left(\frac{\lambda}{p+1-\varepsilon}u_\varepsilon^{p+1-\varepsilon} + \frac{\lambda}{p+1-\varepsilon}v_\varepsilon^{p+1-\varepsilon}\right)dx.$$

It can be easily shown that both $\|u_\varepsilon\|_{H_0^1(\Omega)}$ and $\|v_\varepsilon\|_{H_0^1(\Omega)}$ are uniformly bounded for $\varepsilon > 0$ small. Thus, there is a subsequence $\{(u_j, v_j)\}$ of $\{(u_\varepsilon, v_\varepsilon)\}$ such that

$$\begin{aligned} &u_j \rightharpoonup u, \quad v_j \rightharpoonup v \quad \text{in} \quad H_0^1(\Omega), \\ &u_j \to u, \quad v_j \to v \quad \text{in} \quad L^{p+1}(\Omega), \\ &u_j \rightharpoonup u, \quad v_j \rightharpoonup v \quad \text{in} \quad L^{2^*(s)}(\Omega, |x|^{-s}dx), \end{aligned} \tag{3.3}$$

with $u, v \geq 0$ and (u, v) is a solution of system (1.1). If (u, v) is a nontrivial solution, by the strong maximum principle, $u, v > 0$, then we are done.

Now, we prove (u, v) is nontrivial. It will be shown by the blowing up argument. Suppose on the contrary that $u = v = 0$ in Ω. Let

$$M_j = u_j(x_j) = \max_{\bar\Omega} u_j(x), \quad N_j = v_j(y_j) = \max_{\bar\Omega} v_j(x).$$

Then, we have either $m_j \to \infty$ or $n_j \to \infty$ as $j \to \infty$. Indeed, on the contrary we would have $m_j \leq C$ and $n_j \leq C$ for a positive constant C. By the Sobolev embedding,

$$\int\limits_\Omega \frac{u_j^\alpha v_j^{\beta-\varepsilon_j}}{|x|^s}\,dx \leq C\int\limits_\Omega \frac{u_j^\alpha}{|x|^s}dx \to 0$$

as $j \to \infty$. This implies

$$\int\limits_\Omega (|\nabla u_j|^2 + |\nabla v_j|^2)\,dx = 2\int\limits_\Omega \frac{u_j^\alpha v_j^{\beta-\varepsilon_j}}{|x|^s}dx - \lambda\int\limits_\Omega u_j^{p+1-\varepsilon_j}dx - \lambda\int\limits_\Omega v_j^{p+1-\varepsilon_j}dx \to 0,$$

that is, $u_j \to 0$, $v_j \to 0$ strongly in $H_0^1(\Omega)$. It yields

$$0 = \lim_{j\to\infty} \frac{1}{2} \int_\Omega (|\nabla u_j|^2 + |\nabla v_j|^2)dx = c > 0,$$

a contradiction.

We will show that $M_j = O(1)N_j$, and $x_j \to 0$, $y_j \to 0$ at the same time, which implies that the origin is the only blow up point. Suppose $N_j \le M_j \to \infty$ and denote

$$\tilde u_j(y) = M_j^{-1} u_j(k_j y + x_j), \quad \tilde v_j(y) = M_j^{-1} v_j(k_j y + x_j), \quad \text{for} \quad y \in \Omega_j,$$

where $k_j = M_j^{-\frac{2^*(s)-2-\varepsilon_j}{2-s}}$, $\Omega_j = \{x \in \mathbb{R}^N : x_j + k_j x \in \Omega\}$. Then $(\tilde u_j, \tilde v_j)$ satisfies

$$\begin{cases} -\Delta \tilde u_j = \frac{2\alpha}{\alpha+\beta-\varepsilon_j} (\frac{k_j}{|x_j|})^s \frac{\tilde u_j^{\alpha-1} \tilde v_j^{\beta-\varepsilon_j}}{\left|\frac{x_j}{|x_j|} + \frac{k_j}{|x_j|}x\right|^s} - \lambda k_j^2 M_j^{p-1-\varepsilon_j} \tilde u_j^{p-\varepsilon_j} & \text{in} \quad \Omega_j, \\ -\Delta \tilde v_j = \frac{2(\beta-\varepsilon_j)}{\alpha+\beta-\varepsilon_j} (\frac{k_j}{|x_j|})^s \frac{\tilde u_j^{\alpha} \tilde v_j^{\beta-1-\varepsilon_j}}{\left|\frac{x_j}{|x_j|} + \frac{k_j}{|x_j|}x\right|^s} - \lambda k_j^2 M_j^{p-1-\varepsilon_j} \tilde v_j^{p-\varepsilon_j} & \text{in} \quad \Omega_j, \\ 0 \le \tilde u_j, \tilde v_j \le 1, & \text{in} \quad \Omega_j, \\ \tilde u_j = \tilde v_j = 0 & \text{on} \quad \partial\Omega_j. \end{cases} \tag{3.4}$$

We claim that $|x_j| = O(k_j)$ and $x_j \to 0$ as $j \to \infty$. Suppose on the contrary that $\limsup_{j\to\infty} \frac{|x_j|}{k_j} = \infty$. Since $M_j \to \infty$, $k_j \to 0$ as $j \to \infty$. Furthermore, we have

$$k_j^2 M_j^{p-1-\varepsilon_j} = k_j^{2-\frac{(2-s)(p-\varepsilon_j-1)}{2^*(s)-2-\varepsilon_j}} \to 0 \quad \text{as} \quad j \to \infty$$

due to the facts $k_j \to 0$ and $2 - \frac{(2-s)(p-\varepsilon_j-1)}{2^*(s)-2-\varepsilon_j} > 0$, i.e, $p < \frac{N+2}{N-2}$. Because $(\tilde u_j, \tilde v_j)$ is uniformly bounded in $C^{2,\alpha}_{loc}$, we may assume that $\tilde u_j \to u, \tilde v_j \to v$ in C^2_{loc}.

Suppose $x_j \to x_0 \in \bar\Omega$. There are two cases: (i) $x_0 \in \Omega$ or $x_0 \in \partial\Omega$ and $\frac{dist(x_j,\partial\Omega)}{k_j} \to \infty$; and (ii) $x_0 \in \partial\Omega$ and $\frac{dist(x_j,\partial\Omega)}{k_j} \to \sigma \ge 0$.

In the case (i), we have $\Omega_j \to \mathbb{R}^N$ as $j \to \infty$ and (u, v) satisfies

$$\begin{cases} -\Delta u = 0 & \text{in} \quad \mathbb{R}^N, \\ -\Delta v = 0 & \text{in} \quad \mathbb{R}^N, \\ 0 \le u, v \le 1, & u(0) = 1. \end{cases}$$

Furthermore, we have

$$\int_{\Omega_j} \tilde u_j^{\frac{2N}{N-2}} \, dy = k_j^{\frac{N\varepsilon_j}{2^*(s)-2-\varepsilon_j}} \int_\Omega u_j^{\frac{2N}{N-2}} \, dx \le C, \quad \text{and} \quad \int_{\Omega_j} \tilde v_j^{\frac{2N}{N-2}} \, dy \le C,$$

which yields

$$\int_{\mathbb{R}^N} u^{\frac{2N}{N-2}} \, dy < \infty, \quad \int_{\mathbb{R}^N} v^{\frac{2N}{N-2}} \, dy < \infty.$$

However, by the Liouville theorem, $u \equiv v \equiv 1$ for $x \in \mathbb{R}^N$. This is a contradiction.

In the case (ii), after an orthogonal transformation, we have $\Omega_j \to \mathbb{R}^N_+ = \{x = (x_1, \dots, x_N) \mid x_N > 0\}$ as $j \to \infty$ and $\tilde{u}_j$, $\tilde{v}_j$ converge to some u, v uniformly in every compact subset of $\mathbb{R}^N_+$. Apparently, $u(0) = 1$ and $0 \le v(0) \le 1$. Hence, (u, v) satisfies

$$\begin{cases} -\Delta u = 0 \quad \text{in} \quad \mathbb{R}^N_+, \\ -\Delta v = 0 \quad \text{in} \quad \mathbb{R}^N_+, \\ 0 \le u, v \le 1 \quad \text{in} \quad \mathbb{R}^N_+, \\ u = v = 0 \quad \text{on} \quad \partial\mathbb{R}^N_+. \end{cases}$$

By the boundary condition and the maximum principle, $u \equiv v \equiv 0$ for $x \in \mathbb{R}^N_+$ which violates $u(0) = 1$. Consequently, $\limsup_{j\to\infty} \frac{|x_j|}{k_j} < \infty$. Since $k_j \to 0$, we have $x_j \to 0$ as $j \to \infty$.

Next, we show that $\liminf_{j\to\infty} \frac{|x_j|}{k_j} > 0$. Were it not the case, we would have, up to a subsequence, that $\lim_{j\to\infty} \frac{|x_j|}{k_j} = 0$. Then $(\tilde{u}_j, \tilde{v}_j)$ satisfies

$$\begin{cases} -\Delta \tilde{u}_j = \frac{2\alpha}{\alpha+\beta-\varepsilon_j} \frac{\tilde{u}_j^{\alpha-1}\tilde{v}_j^{\beta-\varepsilon_j}}{|\frac{x_j}{k_j}+x|^s} - \lambda k_j^2 M_j^{p-1-\varepsilon_j} \tilde{u}_j^{p-\varepsilon_j} & \text{in} \quad \Omega_j, \\ -\Delta \tilde{v}_j = \frac{2(\beta-\varepsilon_j)}{\alpha+\beta-\varepsilon_j} \frac{\tilde{u}_j^{\alpha}\tilde{v}_j^{\beta-1-\varepsilon_j}}{|\frac{x_j}{k_j}+x|^s} - \lambda k_j^2 M_j^{p-1-\varepsilon_j} \tilde{v}_j^{p-\varepsilon_j} & \text{in} \quad \Omega_j, \\ 0 \le \tilde{u}_j, \tilde{v}_j \le 1 & \text{in} \quad \Omega_j, \\ \tilde{u}_j = \tilde{v}_j = 0 & \text{on} \quad \partial\Omega_j. \end{cases} \tag{3.5}$$

Up to a rotation, we have $\Omega_j \to \mathbb{R}^N_+$ and $\tilde{u}_j$, $\tilde{v}_j$ converge to some u, v uniformly in compact subsets of $\mathbb{R}^N_+$ respectively, where (u, v) satisfies

$$\begin{cases} -\Delta u = \frac{2\alpha}{\alpha+\beta} \frac{u^{\alpha-1}v^{\beta}}{|y|^s} \quad \text{in} \quad \mathbb{R}^N_+, \\ -\Delta v = \frac{2\beta}{\alpha+\beta} \frac{u^{\alpha}v^{\beta-1}}{|y|^s} \quad \text{in} \quad \mathbb{R}^N_+, \\ 0 \le u, v \le 1 \quad \text{in} \quad \mathbb{R}^N_+, \quad u = v = 0 \quad \text{on} \quad \partial\mathbb{R}^N_+. \end{cases}$$

The boundary condition violates $u(0) = 1$. Hence, $\liminf_{j\to\infty} \frac{|x_j|}{k_j} > 0$.

Now, we complete the proof of Theorem 1.1 by showing that problem (1.1) has a nontrivial solution. We may assume $\frac{dist(x_j, \partial\Omega)}{k_j} \to \sigma \ge 0$. By an affine transformation, we find $(\tilde{u}_j, \tilde{v}_j)$ converges to (u, v) uniformly in any compact subset of $\mathbb{R}^N_+$ and (u, v) satisfies

$$\begin{cases} -\Delta u = \frac{2\alpha}{\alpha+\beta} \frac{u^{\alpha-1}v^{\beta}}{|y|^s} \quad \text{in} \quad \mathbb{R}^N_+, \\ -\Delta v = \frac{2\beta}{\alpha+\beta} \frac{u^{\alpha}v^{\beta-1}}{|y|^s} \quad \text{in} \quad \mathbb{R}^N_+, \\ u, v > 0 \quad \text{in} \quad \mathbb{R}^N_+, \quad u = v = 0 \quad \text{on} \quad \partial\mathbb{R}^N_+ \end{cases} \tag{3.6}$$

with $u(0, \dots, \sigma) = 1$. By the definition of $\mu_{\alpha,\beta,s}(\Omega)$, we have

$$\mu_{\alpha,\beta,s}(\Omega_j) \le \frac{\int\limits_{\Omega} (|\nabla \tilde{u}_j|^2 + |\nabla \tilde{v}_j|^2)\, dx}{\Big(\int\limits_{\Omega} \frac{\tilde{u}_j^{\alpha}\tilde{v}_j^{\beta-\varepsilon}}{|x|^s}\, dx\Big)^{\frac{2}{2^*(s)}}},$$

and then

$$\mu_{\alpha,\beta,s}(\mathbb{R}^N_+) \le \frac{\int\limits_{\mathbb{R}^N_+} (|\nabla u|^2 + |\nabla v|^2)\, dy}{\Big(\int\limits_{\mathbb{R}^N_+} \frac{u^\alpha v^\beta}{|y|^s}\, dy\Big)^{\frac{2}{2^*(s)}}} = 2\Big(\int\limits_{\mathbb{R}^N_+} \frac{u^\alpha v^\beta}{|y|^s}\, dy\Big)^{\frac{2^*(s)-2}{2^*(s)}},$$

that is,

$$\int\limits_{\mathbb{R}^N_+} (|\nabla u|^2 + |\nabla v|^2)\, dy = 2\int\limits_{\mathbb{R}^N_+} \frac{u^\alpha v^\beta}{|y|^s}\, dy \ge 2^{\frac{-2}{2^*(s)-2}} \mu_{\alpha,\beta,s}(\mathbb{R}^N_+)^{\frac{2^*(s)}{2^*(s)-2}}. \tag{3.7}$$

Furthermore, noting that

$$\begin{aligned} \lim_{j\to\infty} \int\limits_{\Omega} (|\nabla u_j|^2 + |\nabla v_j|^2)\, dx &= \lim_{j\to\infty} k_j^{-\frac{(N-2)\varepsilon_j}{2^*(s)-2-\varepsilon_j}} \int\limits_{\Omega_j} (|\nabla \tilde{u_j}|^2 + |\nabla \tilde{v_j}|^2)\, dy \ge \\ &\ge \lim_{j\to\infty} \int\limits_{\Omega_j} (|\nabla \tilde{u_j}|^2 + |\nabla \tilde{v_j}|^2)\, dy \ge \int\limits_{\mathbb{R}^N_+} (|\nabla u|^2 + |\nabla v|^2)\, dy, \end{aligned} \tag{3.8}$$

we derive from (3.2), (3.7), (3.8) that

$$c = \Big(\frac{1}{2} - \frac{1}{2^*(s)}\Big) \lim_{j\to\infty} \int\limits_{\Omega} (|\nabla u_j|^2 + |\nabla v_j|^2)\, dx \ge \Big(\frac{1}{2} - \frac{1}{2^*(s)}\Big) 2^{\frac{-2}{2^*(s)-2}} \mu_{\alpha,\beta,s}(\mathbb{R}^N_+)^{\frac{2^*(s)}{2^*(s)-2}},$$

which yields a contradiction to (3.2). Thus, (u, v) is a nontrivial solution of (1.1).

Now we show $M_j = O(N_j)$. Indeed, since u is nontrivial, so is v. Otherwise, we would have

$$\begin{cases} \Delta u = 0 \quad \text{in} \quad \mathbb{R}^N_+, \\ 0 \le u \le 1, u(0, \dots, \sigma) = 1 \quad \text{in} \quad \mathbb{R}^N_+, \\ u = 0 \quad \text{on} \quad \partial\mathbb{R}^N_+. \end{cases}$$

By the strong maximum principle, u would be a constant because it attains its maximum value inside $\mathbb{R}^N_+$. This yields a contradiction between $u(0, \dots, \sigma) = 1$ and the boundary condition. Therefore, there exists $y_0 \in \mathbb{R}^N_+$ such that $v(y_0) \ne 0$. Hence,

$$\tilde{v}_j(y_0) = m_j^{-1} v_j(x_j + k_j y_0) \to v(y_0) > 0$$

implies

$$1 \ge \frac{n_j}{m_j} \ge \frac{v_j(x_j + k_j y_0)}{m_j} \ge v(y_0) - \varepsilon > 0$$

for $\varepsilon > 0$ small and j large, namely, $N_j = O(1)M_j$ as $j \to \infty$. Replacing M_j by N_j in the above blow up process, we may deduce that $|y_j| = O(\tilde{k}_j)$, where $\tilde{k}_j = N_j^{-\frac{2^*(s)-2-\varepsilon_j}{2-s}}$. So we also have $y_j \to 0$. Consequently, the origin is the only blow up point and problem (1.1) has a positive nontrivial solution. The proof of Theorem 1.1 is complete. □

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