RANDOM INTEGRAL EQUATIONS ON TIME SCALES

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Abstract. In this paper, we present the existence and uniqueness of random solution of a random integral equation of Volterra type on time scales. We also study the asymptotic properties of the unique random solution.

Keywords: random integral equations, time scale, existence, uniqueness, stability.

Mathematics Subject Classification: 34N05, 45D05, 45R99.

1. INTRODUCTION

The random integral equations of Volterra type, as a natural extension of deterministic ones, arise in many applications and have been investigated by many mathematicians. For details, the reader may see the monograph [22,27], the papers [7,12,21,26] and references therein. For the general theory of integral equations see, the monographs [8,11] and references therein. In recent years, it initiated the study of integral equations on time scales and obtained some significant results see [1,16,19,25]. The stochastic differential equations on time scales was first studied by Sanyal in his Ph.D. Thesis [24]. For other results about stochastic processes see [23].

The aim of this paper is to obtain the general conditions which ensure the existence and uniqueness of a random solution of a random integral equation of Volterra type on time scales and to investigate the asymptotic behavior of such a random solution. The paper is organized as follows: in Section 2 we set up the appropriate framework on random processes on time scales. We also introduce some functional spaces within which the study of random integral equations can be developed. In Section 3 we present the existence and uniqueness of random solutions. Finally, we establish an asymptotic stability result.
2. PRELIMINARIES

A **time scale** $\mathbb{T}$ is an arbitrary nonempty closed subset of the real number $\mathbb{R}$. Then the time scale $\mathbb{T}$ is a complete metric space with the usual metric on $\mathbb{R}$. Since a time scale $\mathbb{T}$ may or may not be connected, we need the concept of jump operators. The **forward (backward) jump operator** $\sigma(t) \in \mathbb{T}$ for $t < \sup \mathbb{T}$ (respectively $\rho(t)$ for $t > \inf \mathbb{T}$) is given by $\sigma(t) = \inf \{s \in \mathbb{T} : s > t\}$ (respectively $\rho(t) = \sup \{s \in \mathbb{T} : s < t\}$) for all $t \in \mathbb{T}$. If $\sigma(t) > t$, $t \in \mathbb{T}$, we say $t$ is *right scattered*. If $\rho(t) < t$, $t \in \mathbb{T}$, we say $t$ is *left scattered*. If $\sigma(t) = t$, $t \in \mathbb{T}$, we say $t$ is *right-dense*. If $\rho(t) = t$, $t \in \mathbb{T}$, we say $t$ is *left-dense*. Also, define the **graininess function** $\mu(t) : \mathbb{T} \rightarrow [0, \infty)$ as $\mu(t) := \sigma(t) - t$. We recall that a function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called *rd-continuous function* if $f$ is continuous at every right-dense point $t \in \mathbb{T}$, and $\lim_{s \rightarrow t^-} f(s)$ exists and is finite at every left-dense point $t \in \mathbb{T}$. We remark that every rd-continuous function is Lebesgue $\Delta$-integrable (see [14]). A *rd-continuous function* $f : \mathbb{T} \rightarrow \mathbb{R}$ is called *positively regressive* if $1 + \mu(t)f(t) > 0$ for all $t \in \mathbb{T}$. We will denote by $\mathbb{R}^+$ the set of all positively regressive functions. In the following, assume that $\mathbb{T}$ is unbounded. Without lost the generality, assume that $0 \in \mathbb{T}$ and let $\mathbb{T}_0 = [0, \infty) \cap \mathbb{T}$. Also, assume that there exists a strictly increasing sequence $(t_n)_n$ of elements of $\mathbb{T}_0$ such that $t_n \rightarrow \infty$ as $n \rightarrow \infty$. Denote by $\mathcal{L}$ the $\sigma$-algebra of $\Delta$-measurable subsets of $\mathbb{T}_0$ and by $\lambda$ the Lebesgue $\Delta$-measure of $\mathcal{L}$. Having the measure space $(\mathbb{T}_0, \mathcal{L}, \lambda)$ one can introduce the Lebesgue-Bochner integral for functions from $\mathbb{T}_0$ to a Banach space by simply employing the standard procedure from measure theory (see [3,18]). The Lebesgue-Bochner integral for functions from $\mathbb{T}_0$ to a Banach space was introduced by Neidhart in [18] and the Henstock-Kurzweil-Pettis integral was introduced by Cichoń in [10]. For details on the construction of the Lebesgue integral for real functions defined on a time scale, see [2,4,5,9,14,15]. Further, let $(\Omega, \mathcal{A}, P)$ be a complete probability space. A function $x : \Omega \rightarrow \mathbb{R}$ is called a *random variable* if $\{\omega \in \Omega : x(\omega) < a\} \in \mathcal{A}$ for all $a \in \mathbb{R}$. Let $1 \leq p < \infty$. A random variable $x : \Omega \rightarrow \mathbb{R}$ is said to be $p$-*integrable* if $\int_{\Omega} |x(\omega)|^p dP(\omega) < \infty$. Let $\mathcal{L}^p(\Omega)$ be the space of all $p$-integrable random variables. Then $\mathcal{L}^p(\Omega)$ is a vector space and the function $x \mapsto \|x\|_{\mathcal{L}^p(\Omega)}$ defined by

$$
\|x\|_{\mathcal{L}^p(\Omega)} = \left( \int_{\Omega} |x(\omega)|^p dP(\omega) \right)^{1/p}
$$

is a seminorm on $\mathcal{L}^p(\Omega)$. If $x \in \mathcal{L}^1(\Omega)$, then

$$
E[x] := \int_{\Omega} x(\omega) dP(\omega)
$$

is called the *expected value* of random variable $x$. A random variable $x$ is called a $P$-*essentially bounded* if there exists a $M > 0$ and $A \in \mathcal{A}$ with $P(A) = 0$ such that $|x(\omega)| \leq M$ for all $\omega \in \Omega \setminus A$. Let $\mathcal{L}^\infty(\Omega)$ be the space of all $P$-essentially bounded random variables. Then

$$
\|x\|_{\mathcal{L}^\infty(\Omega)} = P{-}\text{ess sup}_{\omega \in \Omega} |x(\omega)|
$$
is a seminorm on $\mathcal{L}^\infty(\Omega)$, where

$$P\text{-ess sup}_{\omega \in \Omega} |x(\omega)| := \inf\{M > 0 : |x(\omega)| \leq M \text{ P-a.e. } \omega \in \Omega\}. $$

When a random variable $x$ is $p$-integrable or $P$-essentially bounded it is convenient to use notation $\hat{x}$ to denote the equivalent class of random variables which coincide with $x$ for $P$-a.e. $\omega \in \Omega$. Let us denote by $L^p(\Omega)$ the space of all equivalence classes of random variables that are $p$-integrable and by $L^\infty(\Omega)$ the space of all equivalence classes of random variables that are $P$-essentially bounded. If $x \in L^p(\Omega)$, $1 \leq p \leq \infty$, we denote by $\hat{x}$ its equivalence class, that is, $y \in \hat{x}$ if and only if $y(\omega) = x(\omega)$ for $P$-a.e. $\omega \in \Omega$. Moreover, we have that $\|y\|_{L^p(\Omega)} = \|x\|_{L^p(\Omega)}$. Thus we can define a norm $\|\cdot\|_{L^p(\Omega)}$ on $L^p(\Omega)$ by means of the formula $\|\hat{x}\|_{L^p(\Omega)} = \|x\|_{L^p(\Omega)}$, $1 \leq p \leq \infty$.

Then $L^p(\Omega)$, $1 \leq p \leq \infty$, is a Banach space with respect to the norm $\|\cdot\|_{L^p(\Omega)}$.

Since, for $1 \leq p \leq \infty$, $L^p(\Omega)$ is a Banach space, then all elementary properties of the calculus (such as continuity, differentiability, and integrability) for abstract functions defined on a subset of $\mathbb{T}$ with values into a Banach space remain also true for the functions defined a subset of $\mathbb{T}$ with values into $L^p(\Omega)$, $1 \leq p \leq \infty$.

Thereby, if $X : \mathbb{T}_0 \to L^p(\Omega)$ is strongly measurable then the function $t \mapsto \|X(t)\|_{L^p(\Omega)}$ is Lebesgue measurable on $\mathbb{T}_0$. Also, a strongly measurable function $X : \mathbb{T}_0 \to L^p(\Omega)$ is Bochner $\Delta$-integrable on $\mathbb{T}_0$ if and only if the function $t \mapsto \|X(t)\|_{L^p(\Omega)}$ is Lebesgue $\Delta$-integrable on $\mathbb{T}_0$ (see [3]).

Let $1 \leq p \leq \infty$. A function $X : \mathbb{T}_0 \to L^p(\Omega)$ is called $rd$-continuous function if $X$ is continuous at every right-dense point $t \in \mathbb{T}_0$, and $\lim_{s \to t^-} X(s)$ exists in $L^p(\Omega)$ at every left-dense point $t \in \mathbb{T}_0$.

Of particular importance is the fact that every $rd$-continuous function $X : \mathbb{T}_0 \to L^p(\Omega)$ is Bochner $\Delta$-integrable on $\mathbb{T}_0$ (see [3, Theorem 6.3]).

If $X : \mathbb{T}_0 \to L^p(\Omega)$ is a strongly measurable function then for each fixed $t \in \mathbb{T}_0$, $X(t) \in L^p(\Omega)$ is an equivalence class. If for each $t \in \mathbb{T}_0$ we select a particular function $x(t, \cdot) \in X(t)$ then we obtain a function $x(\cdot, \cdot) : \mathbb{T}_0 \times \Omega \to \mathbb{R}$ such that $\omega \mapsto x(t, \omega)$ is a random variable for each $t \in \mathbb{T}_0$. This resulting function is called a representation of $X$. In fact, such a representation is so called a random process. However, is not immediate that this representation function is even a $\mathcal{L} \times \mathcal{A}$-measurable function. In this sense, we have the following result.

**Lemma 2.1.** (a) ([13, Theorem III.1.17]). Let $(\mathbb{T}_0 \times \Omega, \mathcal{L} \times \mathcal{A}, \lambda \times P)$ be the product space of the measure space $(\mathbb{T}_0, \mathcal{L}, \lambda)$ and $(\Omega, \mathcal{A}, P)$. Let $1 \leq p \leq \infty$ and let $X : \mathbb{T}_0 \to L^p(\Omega)$ be a Bochner $\Delta$-integrable function. Then there exists a $\mathcal{L} \times \mathcal{A}$-measurable function $x(\cdot, \cdot) : \mathbb{T}_0 \times \Omega \to \mathbb{R}$ which is uniquely determined except a set of $\lambda \times P$-measure zero, such that $\hat{x}(t, \cdot) = X(t)$ for $\lambda$-a.e. $t \in \mathbb{T}_0$. Moreover, $x(\cdot, \omega)$ is Lebesgue $\Delta$-integrable on $\mathbb{T}_0$ for $P$-a.e. $\omega \in \Omega$, and integral $\int_{\mathbb{T}_0} x(t, \omega) \Delta t$, as a function of $\omega$, is equal to the element $\int_{\mathbb{T}_0} X(t) \Delta t$ of $L^p(\Omega)$, that is,

$$\int_{\mathbb{T}_0} x(t, \cdot) \Delta t = \left(\int_{\mathbb{T}_0} X(t) \Delta t\right)(\cdot).$$
(b) ([13, Lemma III.11.10]). Let \(1 \leq p < \infty\) and let \(x(\cdot, \cdot) : \mathbb{T}_0 \times \Omega \rightarrow \mathbb{R}\) be a \(\mathcal{L} \times \mathcal{A}\)-measurable function such that \(x(t, \cdot) \in \mathcal{L}^p(\Omega)\) for \(\lambda\)-a.e. \(t \in \mathbb{T}_0\). Then the function \(X : \mathbb{T}_0 \rightarrow \mathcal{L}^p(\Omega)\), defined by \(X(t) = \hat{x}(t, \cdot)\), is strongly measurable on \(\mathbb{T}_0\).

A \(\mathcal{L} \times \mathcal{A}\)-measurable function \(x(\cdot, \cdot) : \mathbb{T}_0 \times \Omega \rightarrow \mathbb{R}\) will be called a measurable random process.

**Remark 2.2.** Let \(x(\cdot, \cdot) : \mathbb{T}_0 \times \Omega \rightarrow \mathbb{R}\) be a measurable random process such that, for each fixed \(t \in \mathbb{T}_0\), \(x(t, \cdot) \in \mathcal{L}^p(\Omega)\). If we denote \(\hat{x}(t, \cdot)\) by \(X(t)\), then \(X(t) : \Omega \rightarrow \mathbb{R}\) is a random variable such that \(X(t) \in \mathcal{L}^p(\Omega)\) and \(x(t, \omega) = X(t)(\omega)\) for \(P\)-a.e. \(\omega \in \Omega\).

In the following, using a common abuse of notation in measure theory, we will denote \(x(t, \cdot)\) by \(X(t)\) for each fixed \(t \in \mathbb{T}_0\). In this way, a measurable random process \(x(\cdot, \cdot) : \mathbb{T}_0 \times \Omega \rightarrow \mathbb{R}\) such that \(x(t, \cdot) \in \mathcal{L}^p(\Omega)\) for all \(t \in \mathbb{T}_0\) can be identified with a strongly measurable function \(X : \mathbb{T}_0 \rightarrow \mathcal{L}^p(\Omega)\).

Let us denote by \(C_c = C(\mathbb{T}_0, \mathcal{L}^p(\Omega))\) the space of continuous functions \(X : \mathbb{T}_0 \rightarrow \mathcal{L}^p(\Omega)\) with the compact open topology. We recall that if \(K\) is a compact subset of \(\mathbb{T}_0\) and \(U\) is an open subset of \(\mathcal{L}^p(\Omega)\) and we put

\[ S(K, U) = \{X : K \rightarrow \mathcal{L}^p(\Omega) \mid X(K) \subset U\}, \]

then the sets

\[ S(K_1, \ldots, K_n; U_1, \ldots, U_n) = \bigcap_{i=1}^{n} S(K_i, U_i), \]

where \(n \in \mathbb{N}\), form a basis for the compact open topology. In fact, this topology coincides with the topology of uniform convergence on any compact subset of \(\mathbb{T}_0\). The space \(C_c\) is a locally convex space [28] whose topology is defined by means of the following family of seminorms:

\[ \|X\|_n = \sup_{t \in K_n} \|X(t)\|_{\mathcal{L}^p(\Omega)}, \]

where \(K_n = [0, t_n] \subset \mathbb{T}_0\), \(n \in \mathbb{N}\) and \((t_n)_n\) is a strictly increasing sequence of elements of \(\mathbb{T}_0\) such that \(t_n \rightarrow \infty\) as \(n \rightarrow \infty\).

A distance function can be defined on \(C_c\) by

\[ d_c(X, Y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|X - Y\|_{\mathcal{L}^p(\Omega)}}{1 + \|X - Y\|_{\mathcal{L}^p(\Omega)}}. \]

The topology induced by this distance function is the same topology of uniform convergence on any compact subset of \(\mathbb{T}_0\).

Further, consider a continuous function \(g : \mathbb{T}_0 \rightarrow (0, \infty)\). By \(C_g = C_g(\mathbb{T}_0, \mathcal{L}^p(\Omega))\) we denote the space of all continuous functions from \(\mathbb{T}_0\) into \(\mathcal{L}^p(\Omega)\) such that

\[ \sup_{t \in \mathbb{T}_0} \left\{ \frac{\|X(t)\|_{\mathcal{L}^p(\Omega)}}{g(t)} : t \in \mathbb{T}_0 \right\} < \infty. \]
Then

$$\|X\|_{C_g} := \sup_{t \in \mathbb{T}_0} \frac{\|X(t)\|_{L^p(\Omega)}}{g(t)} \quad (2.1)$$

is a norm of $C_g$.

**Lemma 2.3.** $(C_g, \|\cdot\|_{C_g})$ is a Banach space.

**Proof.** Let $(X_n)$ be a Cauchy sequence in $C_g$. Then for each $\varepsilon > 0$ there exists a $N = N(\varepsilon) > 0$ such that $\|X_n - X\|_{C_g} < \varepsilon$ for all $n, m \geq N$. Hence, by (2.1), it follows that

$$\|X_n(t) - X_m(t)\|_{L^p(\Omega)} < \varepsilon g(t), \quad (2.2)$$

for all $t \in \mathbb{T}_0$ and $n, m \geq N$. Since $L^p(\Omega)$ is a complete metric space, it follows that, for any fixed $t \in \mathbb{T}_0$, $(X_n(t))$ is a convergent sequence in $L^p(\Omega)$. Therefore, for any fixed $t \in \mathbb{T}_0$, there exists $X(t) \in L^p(\Omega)$ such that $X(t) = \lim_{n \to \infty} X_n(t)$ in $L^p(\Omega)$. Moreover, it follows from (2.2) that $X(t) = \lim_{n \to \infty} X_n(t)$ in $L^p(\Omega)$, uniformly on any compact subset of $\mathbb{T}_0$. Hence, $X$ is a continuous function from $\mathbb{T}_0$ into $L^p(\Omega)$. Further, we show that $X \in C_g$. Let us keep $n$ fixed and take $m \to \infty$ in (2.2). Then we obtain that $X_n - X \in C_g$ for all $n \geq N$. Since $X = (X - X_n) + X_n$ and $X - X_n, X_n \in C_g$, it follows that $X \in C_g$. 

**Remark 2.4.** The topology of $C_g$ is stronger than the topology of $C_c$. Indeed, if $X_n \to X$ in $C_g$ as $n \to \infty$, then for each $\varepsilon > 0$ there exists $N = N(\varepsilon) > 0$ such that $\|X_n(t) - X(t)\|_{L^p(\Omega)} < \varepsilon g(t)$, for all $t \in \mathbb{T}_0$ and $n \geq N(\varepsilon)$. Since $g$ is bounded on any compact subset of $\mathbb{T}_0$, it allows that $X_n(t) \to X(t)$ as $n \to \infty$, uniformly on any compact subset of $\mathbb{T}_0$. In other words, convergence in $C_g$ implies convergence in $C_c$. If $g(t) = 1$ on $\mathbb{T}_0$, then $C_g$ becomes the space $C = C(\mathbb{T}_0, L^p(\Omega))$ of all continuous and bounded functions from $\mathbb{T}_0$ into $L^p(\Omega)$. The norm on $C$ is given by

$$\|X\|_c = \sup_{t \in \mathbb{T}_0} \|X(t)\|_{L^p(\Omega)}.$$ 

Note that the following inclusions hold $C \subset C_g \subset C_c$.

Let $(B, D)$ be a pair of Banach spaces such that $B, D \subset C_c$ and let $\mathcal{T}$ be a linear operator from $C_c$ to itself. The pair of Banach spaces $(B, D)$ is called *admissible* with respect to the operator $\mathcal{T} : C_c \to C_c$ if $\mathcal{T}(B) \subset D$ ([13]).

**Remark 2.5.** If the pair $(B, D)$ is admissible with respect to the linear operator $\mathcal{T} : C_c \to C_c$ then, by Lemma 2.1.1 from [21], it follows that $\mathcal{T}$ is a continuous operator from $B$ to $D$. Therefore, there exists a $M > 0$ such that

$$\|\mathcal{T}X\|_D \leq M \|X\|_B, \quad X \in B.$$
3. RANDOM INTEGRAL EQUATION OF VOLTERRA TYPE

In this section we study the existence and uniqueness of a random solution of a random integral equation of Volterra type.

\[
x(t, \omega) = h(t, \omega) + \int_{t_0}^{t} k(t, s, \omega) f(s, x(s, \omega), \omega) \Delta s, \quad t \in \mathbb{T}_0,
\]

where \(P\)-a.e. \(\omega \in \Omega\), \(x(\cdot, \cdot) : \mathbb{T}_0 \times \Omega \to \mathbb{R}\) is the unknown random process, \(h : \mathbb{T}_0 \times \Omega \to \mathbb{R}\) is a measurable random process, \(f : \mathbb{T}_0 \times \mathbb{R} \times \Omega \to \mathbb{R}\) is a random function, \(k : \Gamma \times \Omega \to \mathbb{R}\) is the random kernel, \(\lambda \in \mathbb{R}^*\), and \(\Gamma := \{(t, s) \in \mathbb{T}_0 \times \mathbb{T}_0 : t_0 \leq s \leq t < \infty\}\).

In what follows, we will use the notations \(X(t) = x(t, \cdot)\), \(H(t) = h(t, \cdot)\), \(K(t, s) = k(t, s, \cdot)\), \(F(t, X(t)) = f(t, x(t, \cdot), \cdot)\).

Let us consider the following assumptions:

(h1) \(K(t, s) \in L^\infty(\Omega)\) for all \((t, s) \in \Gamma\), \(K(\cdot, \cdot) : \Gamma \to L^\infty(\Omega)\) continuous in its first variable and \(rd\)-continuous in its second variable, there exists \(k_0 > 0\) and \(\alpha > 0\) with \(-\alpha \in \mathbb{R}^+\) such that

\[
\|K(t, s)\|_{L^\infty(\Omega)} \leq k_0 e^{-\alpha(t, \sigma(s))}
\]

for \((t, s) \in \Gamma\).

(h2) \(f(\cdot, x, \cdot) : \mathbb{T}_0 \times \Omega \to \mathbb{R}\) is a \(L \times A\)-measurable function for each \(x \in \mathbb{R}\), and there exist an \(a > 0\) and a positive random variable \(L : \Omega \to \mathbb{R}\) such that \(P(\{\omega \in \Omega : L(\omega) > a\}) = 0\) and

\[
|f(t, x, \omega) - f(t, y, \omega)| \leq L(\omega) |x - y|
\]

for all \(t \in \mathbb{T}_0\) and \(x, y \in \mathbb{R}\).

(h3) \(F(t, 0) \in L^p(\Omega)\) for all \(t \in \mathbb{T}_0\) and there exists \(\beta \in (0, \alpha)\) with \(-\beta \in \mathbb{R}^+\) such that

\[
r := \sup_{t \in \mathbb{T}_0} \frac{\|F(t, 0)\|_{L^p(\Omega)}}{e^{-\beta(t, 0)}} < \infty.
\]

In what follows, consider \(g(t) := e^{-\beta(t, 0)}, t \in \mathbb{T}_0\), where \(0 < \beta < \alpha\). Also, we will use the notation \(C_\beta\) instead of \(C_g\).

Lemma 3.1. If (h2) and (h3) hold, then

\[
\sup_{t \in \mathbb{T}_0} \frac{\|F(t, X(t))\|_{L^p(\Omega)}}{e^{-\beta(t, 0)}} \leq a \|X\|_{C_\beta} + r < \infty
\]

for every \(X \in C_\beta\), and

\[
\|F(t, X(t)) - F(t, Y(t))\|_{L^p(\Omega)} \leq a \|X(t) - Y(t)\|_{L^p(\Omega)}
\]

for all \(t \in \mathbb{T}_0\) and \(X, Y \in C_\beta\).
Proof. If we denote \( \omega \in \Omega : L(\omega) \leq a \) by \( \Omega_a \), then from (h2) we have that \( P(\Omega_a) = 1 \). If \( X, Y \in C_{\beta} \), using the Minkowski’s inequality, (h2) and (h3), we have

\[
\| F(t, X(t)) \|_{L^p(\Omega)} = \| f(t, x(t, \cdot), \cdot) \|_{L^p(\Omega)} \leq \\
\left( \int_{\Omega} |f(t, x(t, \omega), \omega) - f(t, 0, \omega)|^p dP(\omega) \right)^{1/p} + \left( \int_{\Omega} |f(t, 0, \omega)|^p dP(\omega) \right)^{1/p} \leq \\
\left( \int_{\Omega_a} |L(\omega)|^p |x(t, \omega)|^p dP(\omega) \right)^{1/p} + \| F(t, 0) \|_{L^p(\Omega)} \leq \\
a \| X(t) \|_{L^p(\Omega)} + \| F(t, 0) \|_{L^p(\Omega)}.
\]

Dividing both sides of the last inequality by \( e^{-\beta(t, 0)} > 0 \) and taking the supremum with respect to \( t \in T_0 \), we obtain (3.2). Also,

\[
\| F(t, X(t)) - F(t, X(t)) \|_{L^p(\Omega)} = \| f(t, x(t, \cdot), \cdot) - f(t, y(t, \cdot), \cdot) \|_{L^p(\Omega)} = \\
\left( \int_{\Omega} |f(t, x(t, \omega), \omega) - f(t, y(t, \omega), \omega)|^p dP(\omega) \right)^{1/p} \leq \\
\left( \int_{\Omega_a} |L(\omega)|^p |x(s, \omega) - y(s, \omega)|^p dP(\omega) \right)^{1/p} \leq a \| X(t) - Y(t) \|_{L^p(\Omega)}.
\]

\[\square\]

**Remark 3.2.** It follows from Lemma 3.1 that \( F(t, X(t)) \in L^p(\Omega) \) for all \( t \in T_0 \) and \( X \in C_{\beta} \). Moreover, (3.2) implies that the function \( t \mapsto F(t, X(t)) \) belong to \( C_{\beta} \) for all \( X \in C_{\beta} \).

**Lemma 3.3.** Let us consider the integral operator \( T : C_c \rightarrow C_c \) defined by

\[
(\mathcal{T}X)(t) = \int_0^t K(t, s)X(s)\Delta s, \quad t \in T_0.
\]

If (h1) holds, then \( \mathcal{T}(C_{\beta}) \subset C_{\beta} \).

**Proof.** Let \( X \in C_{\beta} \). We have that

\[
\| (\mathcal{T}X)(t) \|_{L^p(\Omega)} \leq \int_0^t \| K(t, s)X(s) \|_{L^p(\Omega)} \Delta s \leq \int_0^t \| K(t, s) \|_{L^\infty(\Omega)} \| X(s) \|_{L^p(\Omega)} \Delta s = \\
= \int_0^t \| K(t, s) \|_{L^\infty(\Omega)} \frac{\| X(s) \|_{L^p(\Omega)}}{e^{-\beta(s, 0)}} e^{-\beta(s, 0)}\Delta s \leq \\
\leq \| X \|_{C_{\beta}} \int_0^t \| K(t, s) \|_{L^\infty(\Omega)} e^{-\beta(s, 0)}\Delta s.
\]
Take into account (h1), we infer that
\[
\int_0^t \| K(t,s) \|_{L^\infty(\Omega)} e^{-\beta(s,0)} \Delta s = \int_0^t e^{-\alpha(t,\sigma(s))} e^{-\beta(s,0)} \Delta s = \frac{k_0}{\alpha - \beta} [e^{-\beta(t,0)} - e^{-\alpha(t,0)}].
\]
Since \(-\alpha, -\beta \in \mathbb{R}^+\) and \(-\alpha < -\beta\), then (see [6, Corollary 2.10]) we have that
\[e^{-\beta(t,0)} > e^{-\alpha(t,0)}, \quad t \in T_0,\]
and it follows that
\[
\int_0^t \| K(t,s) \|_{L^\infty(\Omega)} e^{-\beta(s,0)} \Delta s \leq \frac{k_0}{\alpha - \beta} e^{-\beta(t,0)}, \quad t \in T_0.
\]
Consequently,
\[
\| (TX)(t) \|_{L^p(\Omega)} \leq \frac{k_0}{\alpha - \beta} \| X \|_{C_\beta} e^{-\beta(t,0)}, \quad t \in T_0,
\]
and thus \(TX \in C_\beta\) for every \(X \in C_\beta\), that is, \(T(C_\beta) \subset C_\beta\).

\textbf{Remark 3.4.} Since, by Lemma 3.3, the pair \((C_\beta, C_\beta)\) is admissible with respect to the linear operator \(T: C_c \to C_c\) then, by Remark 2.5, it follows that \(T\) is a continuous operator from \(C_\beta\) to \(C_\beta\). Therefore, there exists a \(M > 0\) such that
\[
\| TX \|_{C_\beta} \leq M \| X \|_{C_\beta}, \quad X \in C_\beta.
\]
In fact, it easy to see that \(M = \frac{k_0}{\alpha - \beta}\) is the norm of \(T\) as a linear operator from \(C_\beta\) into \(C_\beta\).

A solution \(X \in C_\beta\) of the integral equation \((3.1)\) is called \textbf{asymptotically exponentially stable} if there exists a \(\rho > 0\) and a \(\beta > 0\) such that \(-\beta \in \mathbb{R}^+\) and
\[
\| X(t) \|_{L^p(\Omega)} \leq \rho e^{-\beta(t,0)}, \quad t \in T_0.
\]

\textbf{Remark 3.5.} The admissibility concept is related to stability in various senses (see [17]). Let \(T: C_c \to C_c\) be a linear operator. Roughly speaking we say that the pair of function spaces \(B, D \subset C_c\) is admissible with respect to the equation
\[
X = H + TX,
\]
if this equation has its solution in the space \(D\), for each \(H \in D\). Therefore, if we choose \(D = C_\beta\) and if \(X \in C_\beta\) is a solution of the equation \((3.6)\), then there exists a \(\rho > 0\) such that \(\| X \|_{C_\beta} \leq \rho\). Using \((2.1)\) we infer that
\[
\| X(t) \|_{L^p(\Omega)} \leq \rho e^{-\beta(t,0)}
\]
for all \(t \in T_0\), that is, the solution of the equation \((3.6)\) is asymptotically exponentially stable. For several results concerning the admissibility theory for Volterra integral equations see [11].
These preliminaries being completed, we shall state the following result.

**Theorem 3.6.** If the assumptions (h1)–(h3) hold and \( H \in C_\beta \), then the integral equation (3.1) has a unique asymptotically exponentially stable solution, provided that \(|\lambda| a M < 1\), where \( M > 0 \) is the norm of the operator \( T \).

**Proof.** Let us consider the operator \( \mathcal{V} : C_\beta \to C_\epsilon \) defined by

\[
(\mathcal{V}X)(t) = H(t) + \lambda \int_0^t K(t,s)F(s,X(s))\Delta s, \quad t \in T_0.
\]

(3.7)

Then we can rewrite the operator \( \mathcal{V} \) as

\[
(\mathcal{V}X)(t) = H(t) + \lambda (TG)(t), \quad t \in T_0,
\]

(3.8)

where \( G(t) := F(t,X(t)), \ t \in T_0 \) and \( T \) is the operator given by (3.4). Since by Remark 3.2 and Lemma 3.1 we have that \( \|G\|_{C_\beta} \leq a \|X\|_{C_\beta} + r \), then

\[
\|(TG)(t)\|_{L^p(\Omega)} \leq b M e^{-\beta(t,0)}, \quad t \in T_0,
\]

(3.9)

where \( b := a \|X\|_{C_\beta} + r \). From (3.8) and (3.9) we obtain that

\[
\|(\mathcal{V}X)(t)\|_{L^p(\Omega)} \leq \|H(t)\|_{L^p(\Omega)} + b |\lambda| M e^{-\beta(t,0)},
\]

for all \( t \in T_0 \). Dividing both sides of the last inequality by \( e^{-\beta(t,0)} > 0 \) and taking the supremum with respect to \( t \in T_0 \), it follows that

\[
\|\mathcal{V}X\|_{C_\beta} \leq \|H\|_{C_\beta} + b |\lambda| M,
\]

(3.10)

and so \( \mathcal{V}X \in C_\beta \) for all \( X \in C_\beta \). Further, we show that the operator \( \mathcal{V} \) is a contraction on \( C_\beta \). Indeed, using (3.3) and (3.5), we have

\[
\|(\mathcal{V}X)(t) - (\mathcal{V}Y)(t)\|_{L^p(\Omega)} \leq |\lambda| \int_0^t \|K(t,s)[F(s,X(s)) - F(s,Y(s))]\|_{L^p(\Omega)} \Delta s \leq
\]

\[
\leq |\lambda| \int_0^t \|K(t,s)\|_{L^\infty(\Omega)} \|F(s,X(s)) - F(s,Y(s))\|_{L^p(\Omega)} \Delta s \leq
\]

\[
\leq a |\lambda| \int_0^t \|K(t,s)\|_{L^\infty(\Omega)} \frac{\|X(s) - Y(s)\|_{L^p(\Omega)} e^{-\beta(s,0)}}{e^{-\beta(s,0)}} \Delta s \leq
\]

\[
\leq a |\lambda| \|X - Y\|_{C_\beta} \int_0^t \|K(t,s)\|_{L^\infty(\Omega)} e^{-\beta(s,0)} \Delta s \leq
\]

\[
\leq a |\lambda| \frac{k_0}{\alpha - \beta} \|X - Y\|_{C_\beta} e^{-\beta(t,0)} =
\]

\[
= a |\lambda| M \|X - Y\|_{C_\beta} e^{-\beta(t,0)}.
\]
Thus
\[ \| (\mathcal{V}X)(t) - (\mathcal{V}Y)(t) \|_{L^p(\Omega)} \leq a |\lambda| M \| X - Y \|_{C_\beta} \]
for all \( t \in T_0 \), and so
\[ \| \mathcal{V}X - \mathcal{V}Y \|_{C_\beta} \leq a |\lambda| M \| X - Y \|_{C_\beta}, \]
with \( a |\lambda| M < 1 \), that is, \( \mathcal{V} \) is a contraction on \( C_\beta \). From Banach’s Fixed Point Theorem, it follows that there exist a unique solution \( X \in C_\beta \) of the integral equation (3.1). From Remark 3.5, we infer that the solution is asymptotically exponentially stable.

**Corollary 3.7.** If all the hypotheses of Theorem 3.6 hold for \( \beta = 0 \), then the integral equation (3.1) has a unique solution \( X \in C \).

**Corollary 3.8.** If all the hypotheses of Theorem 3.6, then the solution of the integral equation (3.1) is asymptotically stable in mean, that is,
\[ E[|X(t)|] \to 0 \text{ as } t \to \infty. \]

**Proof.** Since \( -\beta < 0 \), then \( e^{-\beta t}(t,0) \) decreases monotonically towards zero as \( t \to \infty \), and therefore \( \| X(t) \|_{L^p(\Omega)} \to 0 \text{ as } t \to \infty. \) Since \( E[|X(t)|^p] = \| X(t) \|_{L^p(\Omega)}^p \) then, using the Jensen’s inequality, we infer that \( E[|X(t)|] \to 0 \text{ as } t \to \infty. \)

**Remark 3.9.** Let \( T_0 = [0, \infty) \). Then, for \( g(t) = q(t) = e^{-\beta t}, t \geq 0 \), we obtain Theorem 2.2 from [7]. For \( p = 2 \) and \( f(t,x,\omega) = f(t,x) \), we obtain Theorem 3.1 from [26]. Let \( T_0 = \mathbb{N} \). Then, for \( p = 2 \) and \( f(t,x,\omega) = f(t,x) \), we obtain Theorem 5.3.1 from [27].

In what follows, using the concept of admissibility, we prove a general result of the existence and uniqueness for the integral equation (3.1). From this result it is possible to derive many existence results, by particularizing the spaces \( B \) and \( D \).

Let us consider the integral equation (3.1) under the following conditions:

(\( \tilde{h}1 \)) \( K(t,s) \in L^\infty(\Omega) \) for all \( (t,s) \in \Gamma, K(\cdot,\cdot) : \Gamma \to L^\infty(\Omega) \) continuous in its first variable and rd-continuous in its second variable.

(\( \tilde{h}2 \)) \( B, D \subset C_c \) are Banach spaces stronger than \( C_c \) such that the pair \( (B,D) \) is admissible with respect to the linear operator \( T : C_c \to C_c \) defined by (3.4).

(\( \tilde{h}3 \)) For each \( X \in D \), the function \( t \mapsto F(t,X(t)) \) belong to \( B \), and the operator \( \mathcal{G} : D \to B \), defined by \( (\mathcal{G}X)(t) = F(t,X(t)) \) for all \( t \in T_0 \), satisfies the Lipschitz condition
\[ \| \mathcal{G}X - \mathcal{G}Y \|_B \leq a \| X - Y \|_D \]
for all \( X,Y \in D \) and some \( a > 0 \).

**Theorem 3.10.** If the assumptions (\( \tilde{h}1 \))–(\( \tilde{h}3 \)) hold and \( H \in D \), then the integral equation (3.1) has a unique solution \( X \in D \), provided that \( |\lambda| aM < 1 \), where \( M > 0 \) is the norm of the operator \( T \).
Proof. Let us consider the operator $\mathcal{V} : D \to C$ defined by $\mathcal{V}X = H + \lambda TGX$. Since the pair $(B, D)$ is admissible with respect to the linear operator $T$, it follows from Remark 2.5 that there exists a $M > 0$ such that $\|T X\|_D \leq M \|X\|_B$ for all $X \in B$. Using (h3) and the fact that $H \in D$ it follows from Minkowski’s inequality that

$$\|\mathcal{V}X\|_D \leq \|H\|_D + |\lambda| M \|GX - G0\|_B \leq \|H\|_D + |\lambda| M \|G0\|_B < \infty,$$

that is, $\mathcal{V}X \in D$ for all $X \in D$. Next, all $X,Y \in D$ we have that $\mathcal{V}X - \mathcal{V}Y = \lambda T(GX - GY)$. Obviously, $GX - GY \in B$ and $\mathcal{V}X - \mathcal{V}Y \in D$. It follows that

$$\|\mathcal{V}X - \mathcal{V}Y\|_D \leq |\lambda| a M \|X - Y\|_D,$$

with $|\lambda| a M < 1$, that is, $\mathcal{V}$ is a contraction on $D$. From Banach’s Fixed Point Theorem, it follows that there exist a unique solution $X \in D$ of the integral equation (3.1).

Remark 3.11. If $\mathbb{T}_0 = [0, \infty)$, we obtain Theorem 2.4 from [7]. For $p = 2$ and $f(t, x, \omega) = f(t, x)$, we obtain Theorem 2.1.2 from [27]. If $\mathbb{T}_0 = \mathbb{N}$, then, for $p = 2$ and $f(t, x, \omega) = f(t, x)$, we obtain Theorem 5.1.2 from [27].

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