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VARIATIONAL CHARACTERIZATIONS FOR EIGENFUNCTIONS OF ANALYTIC SELF-ADJOINT OPERATOR FUNCTIONS

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Abstract. In this paper we consider Rellich's diagonalization theorem for analytic self-adjoint operator functions and investigate variational principles for their eigenfunctions and interlacing statements. As an application, we present a characterization for the eigenvalues of hyperbolic operator polynomials.

Keywords: operator functions, eigenfunctions, eigenvalues, variational principles.

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1. INTRODUCTION

Let an operator function $P(\lambda)$ defined on an interval $[a,b] \subset \mathbb{R}$, whose values are linear operators acting in a Hilbert space $\mathcal{H}$. Operator functions in general may be analytic, smooth or nonsmooth. Special classes include polynomial functions $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$, where A_j, $j = 0, \ldots, m$, are operators.

In this paper we are concerned with the development of variational theory for analytic self-adjoint operator functions $P(\lambda)$, i.e. $P(\lambda) = P^*(\lambda)$, of the spectral parameter $\lambda \in \mathbb{R}$ in a Hilbert space of finite dimension ($dim\mathcal{H} = n$) with domain $\mathcal{D}(P) = \mathcal{H}$. It is well known by Rellich's theorem [5, p.394] that for $\lambda \in \mathbb{R}$, $P(\lambda)$ is diagonalizable for all λ and precisely that there exists scalar analytic functions $\mu_1(\lambda), \ldots, \mu_n(\lambda)$ and a unitary operator function $U(\lambda)$ in $\mathcal{H}$, which possess the property

$$P(\lambda) = U(\lambda) diag\,(\mu_1(\lambda), \ldots, \mu_n(\lambda))\, U^*(\lambda). \tag{1.1}$$

In (1.1), the *eigenfunctions* $\mu_k(\lambda)$, $k = 1, \ldots, n$, are the roots of the equation

$$det\,(I\mu - P(\lambda)) = \mu^n + p_1(\lambda)\mu^{n-1} + \ldots + p_{n-1}(\lambda)\mu + p_n(\lambda) = 0, \tag{1.2}$$

where the coefficients $p_k(\lambda)$ are functions of the real variable λ and the columns $u_k(\lambda)$ of $U(\lambda) = \begin{bmatrix} u_1(\lambda) & \ldots & u_n(\lambda) \end{bmatrix}$ are eigenvectors of $P(\lambda)$ corresponding to $\mu_k(\lambda)$, $k = 1, \ldots, n$. Due to $P(\lambda)$ being self-adjoint, the analytic eigenfunctions $\mu_k(\lambda)$ are real and are written as power series of $\lambda - \lambda_0$ in a neighbourhood of λ_0:

$$\mu_k(\lambda) = a_{k,0} + a_{k,1}(\lambda - \lambda_0) + a_{k,2}(\lambda - \lambda_0)^2 + \ldots, \tag{1.3}$$

where $a_{k,i} \in \mathbb{R}$, $i = 0, 1, 2, \ldots$, $k = 1, \ldots, n$. In the case where $\mu_k(\lambda)$ are polynomials of degree 1 at most, the pencil $P(\lambda)$ has the property L (see [14]) and several results for this case are presented in [4, 12] and [13].

Our approach is to study variational principles for the eigenfunctions $\mu_k(\lambda)$ according to a suitable order for real analytic functions, which lead to corresponding properties of the spectrum $\sigma(P) = \{\lambda : P(\lambda) \text{ not invertible}\} = \{\lambda : \mu_k(\lambda) = 0 \text{ for some } k\}$ of $P(\lambda)$. These characterizations have not been presented in the subject's literature, despite the fact that Binding *et al.* [2] and more recently Eschwe-M. Langer [3] studied the roots $\lambda = \rho(x)$ of the functions $\pi_x(\lambda) = \langle P(\lambda)x, x\rangle$, when these are unique for each nonzero $x \in \mathcal{H}$, and led to the characterization of the eigenvalues of $P(\lambda)$ through min-max expressions. In our paper we consider forms $\langle P(\lambda)x(\lambda), x(\lambda)\rangle$, where $x(\lambda)$ is an analytic vector valued function of the real variable λ. It is clear that this approach is more general, since it does not include only constant vectors $x \in \mathcal{H}$. In particular, an eigenvector $u(\lambda)$ in Rellich's Theorem is independent of λ only in the trivial case where $\mu(\lambda) \equiv 0$. Moreover, if $u(\lambda)$ is a unit eigenvector of $P(\lambda)$ corresponding to eigenfunction $\mu(\lambda)$, and $\lambda_0 \in \sigma(P)$, then $\langle P(\lambda_0)u(\lambda_0), u(\lambda_0)\rangle = \mu(\lambda_0) = 0$, where upon according to the theory in [2, 3] we have $\pi_{x_0}(\lambda_0) = 0$ for $x_0 = u(\lambda_0)$, or equivalently that $\lambda_0 = u^{-1}(x_0) \equiv \rho(x_0)$. This gives an important motivation for consideration and study of the eigenfunctions, which we characterize through variational principles.

It is necessary to introduce an order for the eigenfunctions $\mu_k(\lambda)$. This can be attained via the lexicographic ordering of the infinite series of coefficients $\mu_k = (a_{k,0}, a_{k,1}, \ldots)$, $k = 1, \ldots, n$, in the analytic expressions (1.3) of $\mu_k(\lambda)$ in a neighbourhood of λ_0. More specifically we say:

$$\begin{aligned} \mu_i(\lambda) \prec \mu_j(\lambda) \ &\Leftrightarrow\ \mu_i \overset{l}{\prec} \mu_j \Leftrightarrow \\ &\Leftrightarrow\ \text{there exists } \sigma \in \mathbb{N} \text{ such that for all } \ell \in \{0, 1, \ldots, \sigma - 1\} \\ &\qquad \text{we have } a_{i,\ell} = a_{j,\ell} \text{ and } a_{i,\sigma} < a_{j,\sigma}. \end{aligned} \tag{1.4}$$

At this point it should be stressed that a clear distinction between the symbols $\preceq$ and $\leq$ should be made. The relation $\mu_i(\lambda) \preceq \mu_j(\lambda)$ holds independently of λ and does not imply $\mu_i(\lambda) \leq \mu_j(\lambda)$ for arbitrary λ. For example, the eigenfunctions $\mu_1(\lambda) = \lambda$ and $\mu_2(\lambda) = 3 - \lambda$ satisfy $\mu_1(\lambda) \preceq \mu_2(\lambda)$, but $\mu_1(\lambda) \leq \mu_2(\lambda)$ is not true for all λ.

Notice that the above mentioned ordering of the coefficients yields a *total* order on the set of analytic functions. Indeed, suppose that $f(\lambda) = \sum a_k(\lambda - \lambda_0)^k$ is a nonzero analytic function with $a_p > 0$ being the first nonzero coefficient in the series. Apparently, as $\lambda \to \lambda_0^+$, the limit of $f(\lambda)/(\lambda - \lambda_0)^p$ is positive and f is positive in some right neighbourhood of λ_0. Therefore for two distinct real analytic functions $f(\lambda)$ and $g(\lambda)$ of a real variable λ the relation $f \prec g$ in the lexicographic sense for their power series means that f is below g in a right open neighbourhood of λ_0, i.e. that $f(\lambda) < g(\lambda)$ in (λ_0, ϵ) for some $\epsilon > 0$. If two analytic functions f and g coincide on any interval, then they must coincide over the whole real axis. So, given two real analytic functions that do not coincide, one is greater than the other on a right neighbourhood of λ_0. Hence, by (1.4) we may have an order of eigenfunctions $\mu_k(\lambda)$, $k = 1, \ldots, n$, of the operator function $P(\lambda)$ in a neighbourhood of λ_0 and let

$$\mu_1(\lambda) \preceq \mu_2(\lambda) \preceq \ldots \preceq \mu_n(\lambda). \tag{1.5}$$

In the next section we provide the necessary theoretical background on the spectral analysis of operator functions and, more specifically, polynomial functions in a finite dimensional space $\mathcal{H}$. The main aim of this paper is to generalize in Section 3 the variational principles for the analytic eigenfunctions of self-adjoint operator functions, according to the lexicographic order. Then we may reform known interlacing inequalities for eigenvalues of self-adjoint operators in [1,10]. This is attained showing a relation of the lexicographic order to the convexity and a characteristic expression of eigenfunctions as sup or inf of the quantity $\langle P(\lambda)x(\lambda), x(\lambda)\rangle$ for suitable unit vectors $x(\lambda)$. The variational principles for eigenfunctions are then connected with the classical Courant-Fischer principle for eigenvalues of self-adjoint operators and are applied to prove variational formulae for the eigenvalues of hyperbolic polynomial operators.

In Section 4 an interaction of the eigenfunctions of $P(\lambda)$ and those of its restriction on a (closed) subspace is presented, as well as some relations between the eigenfunctions of operator functions $P_1(\lambda)$ and $P_2(\lambda)$ and those of their difference $R(\lambda) = P_1(\lambda) - P_2(\lambda)$.

2. SOME PRELIMINARIES ON THE SPECTRAL ANALYSIS OF OPERATOR POLYNOMIALS

Let the operator polynomial of the form $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$, where A_j, $j = 0, \ldots, m$, are operators in $\mathcal{H}$ and $\lambda \in \mathbb{R}$. A scalar $z_0 \in \mathbb{R}$ is said to be an *eigenvalue* of $P(\lambda)$ if $P(z_0)x_0 = 0$ for some nonzero $x_0 \in \mathcal{H}$. This vector x_0 is called *right eigenvector* of $P(\lambda)$ corresponding to z_0. The set of all eigenvalues of the operator function $P(\lambda)$ is the *spectrum* $\sigma(P)$, i.e. $\sigma(P) = \{\lambda \in \mathbb{R} : 0 \in \sigma(P(\lambda))\}$, where $\sigma(P(\lambda))$ denotes the spectrum of the matrix $P(\lambda)$ for the value λ. In the finite dimensional case we are concerned with, the above definition is equivalent to $\sigma(P) = \{\lambda \in \mathbb{R} : detP(\lambda) = 0\}$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_r \in \sigma(P)$ be the eigenvalues of $P(\lambda)$. Suppose also that for a $\lambda_i \in \sigma(P)$ there exist vectors $x_{i,0}, x_{i,1}, \ldots, x_{i,s_i-1} \in \mathcal{H}$ with $x_{i,0} \neq 0$ that satisfy

$$\begin{aligned}
&P(\lambda_i)x_{i,0} = 0,\\
&\frac{P'(\lambda_i)}{1!}x_{i,0} + P(\lambda_i)x_{i,1} = 0,\\
&\quad\vdots\\
&\frac{P^{(s_i-1)}(\lambda_i)}{(s_i-1)!}x_{i,0} + \frac{P^{(s_i-2)}(\lambda_i)}{(s_i-2)!}x_{i,1} + \ldots + \frac{P'(\lambda_i)}{1!}x_{i,(s_i-2)} + P(\lambda_i)x_{i,(s_i-1)} = 0,
\end{aligned} \tag{2.1}$$

where the indices denote the derivatives of $P(\lambda)$ and s_i is less than or equal to the algebraic multiplicity of λ_i. Then the vector $x_{i,0}$ is an eigenvector of λ_i and $x_{i,1}, x_{i,2}, \ldots, x_{i,(s_i-1)}$ are the *generalized eigenvectors* and constitute a *Jordan chain of length* s_i of $P(\lambda)$ corresponding to λ_i (see [5]).

Hyperbolic polynomials form a widely studied class of self-adjoint polynomial functions (see [11]). These are defined by the conditions that the leading coefficient satisfies $A_m > 0$ and that the scalar polynomial $\pi_x(\lambda) := \langle P(\lambda)x, x\rangle$ defined for any nonzero $x \in \mathcal{H}$ has m real and distinct roots. Denote by $\{\rho_j(x)\}_{j=1}^m$ the roots of the polynomial $\pi_x(\lambda)$ indexed in nondecreasing order. The sets $\Delta_j := \{\rho_j(x) : x \in \mathcal{H}\setminus\{0\}\}$, $j = 1, \ldots, m$, are called *root zones*. Clearly each Δ_j is just the range of the functional $\rho_j(x)$ and is a nonempty interval. In this context, the notion of "eigenvalue types" is fundamental. A real number z_0 is said to have *definite (positive or negative) type* if the quadratic form $\pi_x^{'}(z_0) = \left\langle P^{'}(z_0)x, x\right\rangle$ is definite (positive or negative definite, respectively) on the kernel $KerP(\lambda_0)$. Equivalently, z_0 is of positive or negative type, if the function $\pi_x(\lambda)$ increases or decreases through z_0 respectively.

It is well known [11] that the root zones of hyperbolic polynomials are disjoint, i.e. $\Delta_i \cap \Delta_j = \emptyset$ for $i \neq j \in \{1, 2, \ldots, m\}$. Therefore, there are some real eigenvalues $a_1 \leq b_1 < a_2 \leq b_2 < \ldots < a_m \leq b_m$ of $P(\lambda)$ such that each interval $\Delta_j = [a_j, b_j]$ contains exactly n eigenvalues of $P(\lambda)$ (including multiplicities) all of which are of the same (positive or negative) type. The eigenvalues in adjacent zones Δ_j, Δ_{j+1} $(j = 1, \ldots, m-1)$ are of opposite type [9].

3. VARIATIONAL PRINCIPLES FOR EIGENFUNCTIONS

In the following we consider the eigenfunctions $\{\mu_j(\lambda)\}_{j=1}^n$ ordered lexicographically according to their expansion around $\lambda_0 = 0$ in (1.3) and in nondecreasing order as in (1.5). We begin with a Lemma related to the convexity of a finite set of eigenfunctions, with respect to the lexicographic order. Denoting by $co\{\ldots\}$ the convex hull of a set, we state the following lemma.

Lemma 3.1. *Let the eigenfunctions* $\mu_k(\lambda)$ *in* (1.5) *and* $\mu(\lambda) \in co\{\mu_i(\lambda), \ldots, \mu_j(\lambda)\}$ *for* $1 \leq i < j \leq n$. *Then* $\mu_i(\lambda) \preceq \mu(\lambda) \preceq \mu_j(\lambda)$.

Proof. We begin by proving that $\mu(\lambda) \preceq \mu_j(\lambda)$ for every $1 < j \leq n$. By induction, for $j = 2$ we have $\mu(\lambda) = t\mu_1(\lambda) + (1-t)\mu_2(\lambda)$, for $t \in [0, 1]$. Then by (1.3) we obtain

$$\mu(\lambda) = (ta_{1,0} + (1-t)a_{2,0}) + \lambda\,(ta_{1,1} + (1-t)a_{2,1}) + \ldots + \lambda^\tau\,(ta_{1,\tau} + (1-t)a_{2,\tau}) + \ldots.$$

If $\mu_1(\lambda) = \mu_2(\lambda)$ there is nothing to prove, so we may assume that $\mu_1(\lambda) \prec \mu_2(\lambda)$. Then by definition there exists an index $p \in \mathbb{N}$ such that $a_{1,p} < a_{2,p}$ and $a_{1,j} = a_{2,j}$ $(j = 1, \ldots, p-1)$, so obviously $a_{1,j} = ta_{1,j} + (1-t)a_{2,j} = a_{2,j}$ $(j = 1, \ldots, p-1)$ and also

$$a_{1,p} \;<\; ta_{1,p} + (1-t)a_{2,p} \;<\; a_{2,p}.$$

Thus, $\mu_1(\lambda) \preceq \mu(\lambda) \preceq \mu_2(\lambda)$. Following, we assume that for every $2 \leq j-1 < n$, the relation

$$\sum_{k=1}^{j-1} t_k \mu_k(\lambda) \preceq \mu_{j-1}(\lambda), \tag{3.1}$$

where $\sum_{k=1}^{j-1} t_k = 1$, $t_k \in [0,1]$ holds true. If $\mu(\lambda) = \sum_{k=1}^{j} s_k \mu_k(\lambda)$ with $s_1, \ldots, s_j \in [0,1]$ and $\sum_{k=1}^{j} s_k = 1$, letting $t_k = s_k$ $(k = 1, \ldots, j-2)$ and $t_{j-1} = s_{j-1} + s_j$, by (3.1), we have

$$\sum_{k=1}^{j-1} s_k \mu_k(\lambda) \preceq (1 - s_j)\mu_{j-1}(\lambda) \preceq (1 - s_j)\mu_j(\lambda).$$

Therefore, we receive $\mu(\lambda) = \sum_{k=1}^{j} s_k \mu_k(\lambda) \preceq \mu_j(\lambda)$.

Similarly, we conclude that $\mu(\lambda) \succeq \mu_i(\lambda)$, which completes the proof. □

Since any unit vector $x(\lambda) \in \mathcal{H}$ is expressed as $x(\lambda) = U(\lambda)[x_1 \ldots x_n]^T$, where $U(\lambda)$ is the unitary matrix with columns the eigenvectors of $P(\lambda)$, then $\langle P(\lambda)x(\lambda), x(\lambda)\rangle = \sum_{k=1}^{n} |x_k(\lambda)|^2 \mu_k(\lambda)$ holds and clearly for the quantity $\langle P(\lambda)x(\lambda), x(\lambda)\rangle$ we have

$$\mu_1(\lambda) \preceq \langle P(\lambda)x(\lambda), x(\lambda)\rangle \preceq \mu_n(\lambda),$$

i.e. the set $\{\langle P(\lambda)x(\lambda), x(\lambda)\rangle : x(\lambda) \in \mathcal{H}, \|x(\lambda)\|_2 = 1\}$ is bounded according to the lexicographic order.

The ordering for eigenfunctions $\mu_k(\lambda)$ and the remark above lead to the clarification of $\mu_k(\lambda)$ as *sup-inf* expressions, generalizing thus the variational principles for the eigenvalues of self-adjoint operators [1, 8].

Theorem 3.2. *Let $P(\lambda)$ be an analytic self-adjoint operator function in Hilbert space $\mathcal{H}$ with $dim\mathcal{H} = n$ and let $\mu_k(\lambda)$ $(k = 1, \ldots, n)$ be its eigenfunctions arranged in nondecreasing order as in* (1.5) *according to their expansion in a neighbourhood of $\lambda_0 = 0$ in* (1.3). *Then*

$$\begin{aligned}\mu_k(\lambda) &= \inf_{\substack{\mathcal{S}(\lambda)\subset\mathcal{H} \\ dim\mathcal{S}(\lambda)=k}} \sup_{\substack{x(\lambda)\in\mathcal{S}(\lambda) \\ \|x(\lambda)\|_2=1}} \langle P(\lambda)x(\lambda), x(\lambda)\rangle = \\ &= \sup_{\substack{\mathcal{T}(\lambda)\subset\mathcal{H} \\ dim\mathcal{T}(\lambda)=n-k+1}} \inf_{\substack{x(\lambda)\in\mathcal{T}(\lambda) \\ \|x(\lambda)\|_2=1}} \langle P(\lambda)x(\lambda), x(\lambda)\rangle .\end{aligned} \tag{3.2}$$

Proof. We follow analogue ideas as in the Courant-Fischer theorem. Let $\mathcal{J}$ be a subspace of $\mathcal{H}$ of dimension k and $\mathcal{T}_k(\lambda) \equiv span\{u_k(\lambda), \ldots, u_n(\lambda)\}$, where $u_j(\lambda)$ are the orthonormal eigenvectors of $P(\lambda)$ corresponding to the eigenfunctions $\mu_j(\lambda)$, $(j = k, \ldots, n)$. Since $\mathcal{J} \cap \mathcal{T}_k(\lambda) \neq \{0\}$ for every λ, let $x(\lambda) \in \mathcal{J} \cap \mathcal{T}_k(\lambda)$, with $\|x(\lambda)\|_2 = 1$. Hence, $x(\lambda)$ may be expressed as

$$x(\lambda) = \sum_{j=k}^{n} c_j u_j(\lambda) \quad \text{with} \quad \sum_{j=k}^{n} |c_j|^2 = 1$$

and then

$$\begin{aligned}
&\langle P(\lambda)x(\lambda), x(\lambda)\rangle = \\
&= [\bar{c}_k \ \ldots \ \bar{c}_n] \begin{bmatrix} u_k^*(\lambda) \\ \vdots \\ u_n^*(\lambda) \end{bmatrix} P(\lambda)[u_k(\lambda) \ \ldots \ u_n(\lambda)] \begin{bmatrix} c_k \\ \vdots \\ c_n \end{bmatrix} = \\
&= [\bar{c}_k \ \ldots \ \bar{c}_n] \begin{bmatrix} u_k^*(\lambda) \\ \vdots \\ u_n^*(\lambda) \end{bmatrix} U(\lambda) diag\,(\mu_1(\lambda), \ldots, \mu_n(\lambda))\, U^*(\lambda)\big[u_k(\lambda) \ \ldots \ u_n(\lambda)\big] \begin{bmatrix} c_k \\ \vdots \\ c_n \end{bmatrix} = \\
&= [\bar{c}_k \ \ldots \ \bar{c}_n] \big[0_{n-k+1,k-1} \quad I_{n-k+1}\big]\, diag\,(\mu_1(\lambda), \ldots, \mu_n(\lambda)) \begin{bmatrix} 0_{k-1,n-k+1} \\ I_{n-k+1} \end{bmatrix} \begin{bmatrix} c_k \\ \vdots \\ c_n \end{bmatrix} = \\
&= \sum_{j=k}^{n} |c_j|^2 \mu_j(\lambda) \ \in \ co\,\{\mu_k(\lambda), \ldots, \mu_n(\lambda)\}\,.
\end{aligned} \tag{3.3}$$

Thus, by Lemma 3.1, we obtain $\mu_k(\lambda) \preceq \langle P(\lambda)x(\lambda), x(\lambda)\rangle$ and then

$$\mu_k(\lambda) \preceq \sup_{\substack{x(\lambda)\in\mathcal{J} \\ \|x(\lambda)\|_2=1}} \langle P(\lambda)x(\lambda), x(\lambda)\rangle\,,$$

whereupon, due to the subspace $\mathcal{J}$ $(dim\mathcal{J} = k)$ being arbitrary,

$$\mu_k(\lambda) \preceq \inf_{\substack{\mathcal{J}\subset\mathcal{H} \\ dim\mathcal{J}=k}} \ \sup_{\substack{x(\lambda)\in\mathcal{J} \\ \|x(\lambda)\|_2=1}} \langle P(\lambda)x(\lambda), x(\lambda)\rangle\,. \tag{3.4}$$

A k-dimensional subspace is also $\mathcal{S}_k(\lambda) \equiv span\,\{u_1(\lambda), \ldots, u_k(\lambda)\}$, i.e. we may have $\mathcal{J} \equiv \mathcal{S}_k(\lambda)$. Then for any unit vector $x(\lambda) \in \mathcal{S}_k(\lambda)$ as before holds $\langle P(\lambda)x(\lambda), x(\lambda)\rangle = \sum_{j=1}^{k} |c_j|^2 \mu_j(\lambda) \in co\,\{\mu_1(\lambda), \ldots, \mu_k(\lambda)\}$. Thus, Lemma 3.1 implies $\langle P(\lambda)x(\lambda), x(\lambda)\rangle \preceq \mu_k(\lambda)$ and then

$$\sup_{\substack{x(\lambda)\in\mathcal{S}_k(\lambda) \\ \|x(\lambda)\|_2=1}} \langle P(\lambda)x(\lambda), x(\lambda)\rangle \preceq \mu_k(\lambda).$$

Choosing $x(\lambda) = u_k(\lambda)$, clearly we deduce that

$$\mu_k(\lambda) = \sup_{\substack{x(\lambda)\in\mathcal{S}_k(\lambda) \\ \|x(\lambda)\|_2=1}} \langle P(\lambda)x(\lambda), x(\lambda)\rangle\,.$$

For this particular choice of subspace $\mathcal{S}_k$, we get the equality in (3.4), i.e.

$$\mu_k(\lambda) = \inf_{\substack{\mathcal{S}(\lambda)\subset\mathcal{H} \\ dim\mathcal{S}(\lambda)=k}} \ \sup_{\substack{x(\lambda)\in\mathcal{S}(\lambda) \\ \|x(\lambda)\|_2=1}} \langle P(\lambda)x(\lambda), x(\lambda)\rangle\,.$$

We proceed in a similar way for the sup-inf characterization of $\mu_k(\lambda)$. □

Notice that the above proof shows that for the subspaces $\mathcal{S}_k(\lambda) = span\{u_1(\lambda), \ldots, u_k(\lambda)\}$ and $\mathcal{T}_k(\lambda) = span\{u_k(\lambda), \ldots, u_n(\lambda)\}$, where $1 \le k \le n$, actually holds

$$\mu_k(\lambda) = \sup_{\substack{x(\lambda)\in\mathcal{S}_k(\lambda)\\ \|x(\lambda)\|_2=1}} \langle P(\lambda)x(\lambda), x(\lambda)\rangle = \inf_{\substack{x(\lambda)\in\mathcal{T}_k(\lambda)\\ \|x(\lambda)\|_2=1}} \langle P(\lambda)x(\lambda), x(\lambda)\rangle . \tag{3.5}$$

Remark 3.3. It is clear that Theorem 3.2 remains valid for the ordering of the eigenfunctions according to their expansion in a neighbourhood of any $\lambda_0 \in \mathbb{R}$.

The intersections of the graphs of the eigenfunctions $\mu_1(\lambda), \ldots, \mu_n(\lambda)$ with the line $\lambda = \lambda_0$ define the eigenvalues $\{\mu_j(\lambda_0)\}_{j=1}^n$ of the self-adjoint operator $P(\lambda_0)$. In this case, by the lexicographic order of the eigenfunctions according to their power series expressions around λ_0

$$\mu_j(\lambda) = a_{j,0} + a_{j,1}(\lambda - \lambda_0) + a_{j,2}(\lambda - \lambda_0)^2 + \ldots, \quad j = 1, \ldots, n$$

we get $\mu_j(\lambda_0) = \alpha_{j,0}$ and clearly the lexicographic order $\mu_1(\lambda) \preceq \mu_2(\lambda) \preceq \ldots \preceq \mu_n(\lambda)$ is compatible with the order $\mu_1(\lambda_0) \le \mu_2(\lambda_0) \le \ldots \le \mu_n(\lambda_0)$ of the eigenvalues of $P(\lambda_0)$. Hence, setting $\lambda = \lambda_0$ and substituting *min* for *inf* and *max* for *sup* in the variational principles of Theorem 3.2, turns these lexicographic equalities into arithmetic ones, i.e. to the classical variational principles for the eigenvalues of the self-adjoint operator $P(\lambda_0)$.

In the case when $P(\lambda) = \sum_{j=0}^m A_j\lambda^j$ is a selfadjoint operator polynomial with $\lambda \in \mathbb{R}$, an alternate description of the spectrum in terms of the eigenfunctions is

$$\sigma(P) = \{\lambda \in \mathbb{R} : \text{there exists } j \in \{1, 2, \ldots, n\} \text{ such that } \mu_j(\lambda) = 0\},$$

since all eigenvalues of $P(\lambda)$ are defined as the intersection of the eigenfunctions $\mu_1(\lambda)$, $\mu_2(\lambda)$, ..., $\mu_n(\lambda)$ with the real axis. With respect to the eigenvector $u_k(\lambda)$ corresponding to the eigenfunction $\mu_k(\lambda)$ according to the analytic property in $\mathbb{R}$ (Rellich's theorem), we may consider the power series expansion around λ_0:

$$u_k(\lambda) = u_{k,0} + u_{k,1}(\lambda - \lambda_0) + u_{k,2}(\lambda - \lambda_0)^2 + \ldots . \tag{3.6}$$

We recall that a vector-valued function $x(\lambda)$ which is analytic in a neighbourhood of λ_0 should be called [6] *generating function* for $P(\lambda)$ of order p at $\lambda = \lambda_0$ if $P(\lambda)x(\lambda) = O(|\lambda - \lambda_0|^p)$.

Proposition 3.4. *Let $P(\lambda) = \sum_{j=0}^m A_j\lambda^j$ be a self-adjoint operator polynomial with $\lambda \in \mathbb{R}$ and its eigenvalue $\lambda_0 \in \sigma(P)$ be a root of the eigenfunction $\mu_k(\lambda)$ for some $k \in \{1, 2, \ldots, n\}$ with algebraic multiplicity s. Then $u_k(\lambda)$ is a generating function of $P(\lambda)$ of order s at λ_0.*

Proof. A vector-valued function $x(\lambda) = \sum_{j=0}^\infty x_k(\lambda - \lambda_0)^j$ is a generating function for $P(\lambda)$ of order p at λ_0 [11, Lemma 11.3] if and only if $x_0, \ldots, x_{p-1}$ constitute a Jordan chain of $P(\lambda)$ corresponding to $\lambda = \lambda_0$. Therefore, it is enough to show that the coefficients $u_{k,0}, u_{k,1}, \ldots, u_{k,(s-1)}$ in (3.6) constitute a Jordan chain corresponding to the eigenvalue λ_0 of $P(\lambda)$. Differentiating $u_k(\lambda)$ in (3.6) at $\lambda = \lambda_0$ we get

$$u_k^{(t)}(\lambda_0) = t! u_{k,t}, \quad 0 \leq t \leq s-1. \tag{3.7}$$

Moreover, differentiating t times the equation $P(\lambda)u_k(\lambda) = \mu_k(\lambda)u_k(\lambda)$ at $\lambda = \lambda_0$ we have

$$\sum_{j=0}^{t} \binom{t}{j} P^{(t-j)}(\lambda_0) u_k^{(j)}(\lambda_0) = \sum_{j=0}^{t} \binom{t}{j} \mu^{(t-j)}(\lambda_0) u_k^{(j)}(\lambda_0) = 0,$$

since $\mu_k(\lambda_0) = \mu_k'(\lambda_0) = \ldots = \mu_k^{(s-1)}(\lambda_0) = 0$. A combination of this relation with (3.7) shows that

$$\sum_{j=0}^{t} \frac{t!}{(t-j)!} P^{(t-j)}(\lambda_0) u_{k,j} = 0, \quad 0 \leq t \leq s-1. \tag{3.8}$$

Recalling the formula (2.1) for generalized eigenvectors, clearly by (3.8) we conclude that $u_{k,t} = x_t$ $(t = 0, 1, \ldots, s-1)$, where $\{x_0, \ldots, x_{s-1}\}$ is a Jordan chain corresponding to the eigenvalue λ_0. □

Apparently by Proposition 3.4, if λ_i is a root of eigenfunctions $\mu_{i_1}(\lambda), \ldots, \mu_{i_k}(\lambda)$ with multiplicities $s_{i_1}, \ldots, s_{i_k}$, then the generalized eigenvectors $x_{i,0} \in$ *span* $\{u_{i_1,0}, \ldots, u_{i_k,0}\}, \ldots, x_{i,s_r} \in$ *span* $\{u_{i_1,s_r}, \ldots, u_{i_k,s_r}\}$ with $s_r = \min\{s_{i_1}, \ldots, s_{i_k}\}$.

We next turn our attention to hyperbolic operator polynomials $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$ with $\lambda \in \mathbb{R}$ and use Theorem 3.2 to derive variational principles for their eigenvalues in terms of the roots $\{\rho_j(x)\}_{j=1}^{m}$ of the polynomials $\pi_x(\lambda) = \langle P(\lambda)x, x\rangle$. The characterizations in Proposition 3.6 extend those of Theorem 2.1 in [3] to include eigenvalues of hyperbolic operator polynomials. Here the polynomial $\pi_x(\lambda)$ has m distinct real roots and does not fulfill the assumptions in [3], where the authors consider that $\pi_x(\lambda)$ has at most a unique root for each nonzero $x \in \mathcal{H}$ or none at all. We need the following lemma.

Lemma 3.5. *Let the hyperbolic operator polynomial* $P(\lambda) = \sum_{j=0}^{m} \lambda^j A_j$ *with root zones* $\{\Delta_j^{\pm}\}_{j=1}^{m}$, *where the sign denotes the type of the eigenvalues of* $P(\lambda)$ *contained in each zone. Then for* $\lambda \in \Delta_j^{+}$ (Δ_j^{-}) *we have*

$$\lambda > \rho_j(x) \Leftrightarrow \pi_x(\lambda) = \langle P(\lambda)x, x\rangle > (<) 0,$$
$$\lambda < \rho_j(x) \Leftrightarrow \pi_x(\lambda) = \langle P(\lambda)x, x\rangle < (>) 0,$$

for every nonzero $x \in \mathcal{H}$.

Proof. Since $P(\lambda)$ is a hyperbolic operator polynomial, the leading coefficient $\langle A_m x, x\rangle$ of the scalar polynomial $\pi_x(\lambda)$ is positive for every nonzero $x \in \mathcal{H}$. Therefore, $\lim_{\lambda\to-\infty} \pi_x(\lambda) = -\infty$, if m is odd and $\lim_{\lambda\to-\infty} \pi_x(\lambda) = \infty$, if m is even. Hence, in the case m is odd (even), $\pi_x(\lambda)$ is increasing (decreasing) at $\rho_1(x)$ and moreover Δ_1^{+} (Δ_1^{-}) contains eigenvalues of positive (negative) type. Since the eigenvalue types alternate, the general result follows in any case. □

We note that the above considerations allow us to specify the types of eigenvalues in adjacent root zones, i.e. if $m = 2k$, then Δ_j^- for $j = 2\ell + 1$ ($\ell = 0, 1, \ldots, k-1$) contain eigenvalues of negative type, while Δ_j^+ for $j = 2\ell$ ($\ell = 0, 1, \ldots, k$) contain eigenvalues of positive type. For $m = 2k+1$, the signs in the zones are interchanged. This characterization allows us to determine eigenvalues λ_i in each root zone $\Delta_j^\pm$ through min-max expressions.

Proposition 3.6. *Let the hyperbolic operator polynomial $P(\lambda) = \sum_{j=0}^m \lambda^j A_j$ with eigenvalues $\{\lambda_i\}_{i=1}^{mn}$ in nondecreasing order. Then for an eigenvalue $\lambda_i \in \Delta_j^\pm$ ($j \in \{1, \ldots, m\}$) we have*

$$\lambda_i = \max_{\substack{\mathcal{T} \subset \mathcal{H} \\ dim\mathcal{T} = n-k+1}} \min_{\substack{x \in \mathcal{T} \\ x \neq 0}} \rho_j(x) = \min_{\substack{\mathcal{S} \subset \mathcal{H} \\ dim\mathcal{S} = k}} \max_{\substack{x \in \mathcal{S} \\ x \neq 0}} \rho_j(x), \tag{3.9}$$

where $i \equiv k$ (mod n) and $\rho_j(x)$ is the root of the polynomial $\pi_x(\lambda)$ that defines the root zone $\Delta_j^\pm = \{\rho_j(x) : x \in \mathcal{H} \setminus \{0\}\}$.

Proof. For the characterization of λ_i in some root zone Δ_j^+ ($j \in \{1, \ldots, m\}$), consider the order of the eigenfunctions $\mu_1(\lambda) \preceq \mu_2(\lambda) \preceq \ldots \preceq \mu_n(\lambda)$ according to their analytic expressions around λ_i. Recall that this order coincides with that of the eigenvalues of $P(\lambda_i)$, that is $\mu_1(\lambda_i) \leq \ldots \leq \mu_n(\lambda_i)$. Since $i \equiv k$ (*mod n*), then the eigenvalues in nondecreasing order of the operator polynomial $P(\lambda)$ in Δ_j^+ that are not greater than $\lambda_i \equiv \lambda_{(j-1)n+k}$ (i.e. $\lambda_{(j-1)n+1} \leq \lambda_{(j-1)n+2} \leq \cdots \leq \lambda_{(j-1)n+k-1}$) are roots of the eigenfunctions $\{\mu_{n-k+2}(\lambda), \ldots, \mu_n(\lambda)\}$, since these are the only eigenfunctions that assume positive values at the point $\lambda = \lambda_i$. Clearly λ_i is root of $\mu_{n-k+1}(\lambda)$ and in particular

$$\mu_1(\lambda_i) \leq \mu_2(\lambda_i) \leq \ldots \leq \mu_{n-k}(\lambda_i) \leq \mu_{n-k+1}(\lambda_i) = 0 \leq \mu_{n-k+2}(\lambda_i) \leq \ldots \leq \mu_n(\lambda_i).$$

As seen in the proof of Theorem 3.2, we have the expression

$$\mu_{n-k+1}(\lambda) = \sup_{\substack{x(\lambda) \in \mathcal{S}_{n-k+1}(\lambda) \\ \|x(\lambda)\|_2 = 1}} \langle P(\lambda)x(\lambda), x(\lambda) \rangle,$$

where $\mathcal{S}_{n-k+1}(\lambda) = span\{u_1(\lambda), \ldots, u_{n-k+1}(\lambda)\}$. Substituting $\lambda = \lambda_i$ yields

$$0 = \mu_{n-k+1}(\lambda_i) = \max_{\substack{x(\lambda_i) \in \mathcal{S}_{n-k+1}(\lambda_i) \\ \|x(\lambda_i)\|_2 = 1}} \langle P(\lambda_i)x(\lambda_i), x(\lambda_i) \rangle = \max_{\substack{x \in \mathcal{S}_{n-k+1}(\lambda_i) \\ \|x\|_2 = 1}} \pi_x(\lambda_i),$$

which implies that $0 \geq \pi_x(\lambda_i)$ for every $x \in \mathcal{S}_{n-k+1}(\lambda_i)$. Application of Lemma 3.5 shows that

$$\lambda_i \leq \rho_j(x) \text{ for every } x \in \mathcal{S}_{n-k+1}(\lambda_i) \;\Rightarrow\; \lambda_i \leq \min_{\substack{x \in \mathcal{S}_{n-k+1}(\lambda_i) \\ x \neq 0}} \rho_j(x)$$

and, consequently,

$$\lambda_i \leq \max_{\substack{\mathcal{T} \subset \mathcal{H} \\ dim\mathcal{T} = n-k+1}} \min_{\substack{x \in \mathcal{T} \\ x \neq 0}} \rho_j(x). \tag{3.10}$$

On the other hand, since for every $(n-k+1)$-dimensional subspace $\mathcal{T} \subset \mathcal{H}$ we have that $\mathcal{T} \cap \mathcal{T}_{n-k+1}(\lambda) \neq \{0\}$ for $\mathcal{T}_{n-k+1}(\lambda) = span\{u_{n-k+1}(\lambda), \ldots, u_n(\lambda)\}$ and there exists some unit vector $\tilde{x}(\lambda) \in \mathcal{T} \cap \mathcal{T}_{n-k+1}(\lambda)$ for which $\mu_{n-k+1}(\lambda) \preceq \langle P(\lambda)\tilde{x}(\lambda), \tilde{x}(\lambda)\rangle$ clearly holds. Hence, for $\lambda = \lambda_i$ we get

$$0 = \mu_{n-k+1}(\lambda_i) \leq \langle P(\lambda_i)\tilde{x}(\lambda_i), \tilde{x}(\lambda_i)\rangle \leq \max_{\substack{x \in \mathcal{T} \\ \|x\|_2 = 1}} \pi_x(\lambda_i).$$

If for $x_0 \in \mathcal{T}$, $\max_{\substack{x \in \mathcal{T} \\ \|x\|_2 = 1}} \pi_x(\lambda_i)$ is attained, then Lemma 3.5 implies that $\lambda_i \geq \rho_j(x_0)$, whence we reach the conclusion

$$\lambda_i \geq \min_{\substack{x \in \mathcal{T} \\ x \neq 0}} \rho_j(x) \ \Rightarrow \ \lambda_i \geq \max_{\substack{\mathcal{T} \subset \mathcal{H} \\ dim\mathcal{T} = n-k+1}} \min_{\substack{x \in \mathcal{T} \\ x \neq 0}} \rho_j(x). \tag{3.11}$$

Clearly, by (3.10) and (3.11), we have the first equality in (3.9).

We proceed in a similar fashion for the remaining assertions. □

Specialization of the previous Proposition 3.6 for hyperbolic linear polynomials $P(\lambda) = A - \lambda B$ (hence $B < 0$) yields the following corollary.

Corollary 3.7. *For the eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ of a hyperbolic pencil $P(\lambda) = A - \lambda B$ (where $\lambda \in \mathbb{R}$ and $B < 0$) hold*

$$\lambda_i = \max_{\substack{\mathcal{T} \subset \mathcal{H} \\ dim\mathcal{T} = n-i+1}} \min_{\substack{x \in \mathcal{T} \\ x \neq 0}} \frac{\langle Ax, x\rangle}{\langle Bx, x\rangle} = \min_{\substack{\mathcal{S} \subset \mathcal{H} \\ dim\mathcal{S} = i}} \max_{\substack{x \in \mathcal{S} \\ x \neq 0}} \frac{\langle Ax, x\rangle}{\langle Bx, x\rangle}, \quad i = 1, 2, \ldots, n,$$

independently of their type.

Similarly, for a linear polynomial $P(\lambda) = A - \lambda B$ on $\mathbb{R}$, with $B \geq 0$, A self-adjoint operators in the n-dimensional Hilbert space $\mathcal{H}$, the variational principles in Theorem 3.2 may be applied to yield the following Proposition. For a self-adjoint operator A and each interval I we denote

$$\mathcal{L}_I(A) = span\{x : x \text{ is an eigenvector of } A \text{ corresponding to } \lambda \in \sigma(A) \cap I\}.$$

Proposition 3.8. *For the eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_r$ of $P(\lambda) = A - \lambda B$, where A and $B \geq 0$ are self-adjoint operators in the n-dimensional Hilbert space $\mathcal{H}$, hold*

$$\lambda_i = \min_{\substack{\mathcal{S} \subset \mathcal{H} \\ dim\mathcal{S} = k_i}} \max_{\substack{x \in S \\ \langle Bx, x\rangle > 0}} \frac{\langle Ax, x\rangle}{\langle Bx, x\rangle} = \max_{\substack{\mathcal{T} \subset \mathcal{H} \\ dim\mathcal{T} = n-k_i+1}} \min_{\substack{x \in T \\ \langle Bx, x\rangle > 0}} \frac{\langle Ax, x\rangle}{\langle Bx, x\rangle}, \tag{3.12}$$

where $k_i = dim\mathcal{L}_{(-\infty,0]}(P(\lambda_i))$.

Proof. For the eigenvalue λ_i $(i \in \{1, 2, \ldots, r\})$ of $P(\lambda) = A - \lambda B$, consider the order of the eigenfunctions $\mu_1(\lambda) \preceq \mu_2(\lambda) \preceq \ldots \preceq \mu_n(\lambda)$ according to their analytic expressions around λ_i and supposing that λ_i is a root of $\mu_k(\lambda)$ for some

$k \in \{1, 2, \ldots, n\}$. As before, for the subspace $\mathcal{S}_k(\lambda) = span\,\{u_1(\lambda), \ldots, u_k(\lambda)\}$ we get that $0 = \mu_k(\lambda_i) = \max_{\substack{x \in \mathcal{S}_k(\lambda_i) \\ \|x\|_2 = 1}} \pi_x(\lambda_i) = \max_{\substack{x \in \mathcal{S}_k(\lambda_i) \\ \|x\|_2 = 1}} [\langle Ax, x\rangle - \langle Bx, x\rangle\, \lambda_i]$, whereby

$$\lambda_i = \max_{\substack{x \in \mathcal{S}_k(\lambda_i) \\ \langle Bx, x\rangle > 0}} \frac{\langle Ax, x\rangle}{\langle Bx, x\rangle}. \tag{3.13}$$

Also, for every k-dimensional subspace $\mathcal{S} \subset \mathcal{H}$, $0 = \mu_k(\lambda_i) \leq \max_{\substack{x \in \mathcal{S} \\ \|x\|_2 = 1}} \pi_x(\lambda_i)$ and solving for λ_i,

$$\lambda_i \leq \max_{\substack{x \in \mathcal{S} \\ \langle Bx, x\rangle > 0}} \frac{\langle Ax, x\rangle}{\langle Bx, x\rangle} \Rightarrow \lambda_i \leq \min_{\substack{\mathcal{S} \subset \mathcal{H} \\ dim \mathcal{S} = k}} \max_{\substack{x \in \mathcal{S} \\ \langle Bx, x\rangle > 0}} \frac{\langle Ax, x\rangle}{\langle Bx, x\rangle}. \tag{3.14}$$

Now the equality (3.12) follows from (3.13) and (3.14).

Similarly for the other implication. □

Proposition 2 of [3] is an analogue for unbounded operators A, B in an infinite-dimensional Hilbert space $\mathcal{H}$ using a different proof.

4. RELATED RESULTS

Let $\mathcal{H}_0$ be an r-dimensional (closed) subspace of $\mathcal{H}$, with corresponding orthogonal projection $R = VV^*$, where $V : \mathcal{H}_0 \to \mathcal{H}$ is an isometry. For any operator function $P(\lambda)$ in $\mathcal{H}$, $Q(\lambda) = RP(\lambda)R$ is called the *orthogonal projection* of $P(\lambda)$ on $\mathcal{H}_0$. The subspace $\mathcal{H}_0$ is invariant under $Q(\lambda)$ and therefore we may speak of the eigenfunctions of $Q(\lambda)$ in $\mathcal{H}_0$, i.e. of the eigenfunctions of the part $Q_r(\lambda) = V^*P(\lambda)V$ of $Q(\lambda)$ in $\mathcal{H}_0$. If $P(\lambda)$ is analytic and self-adjoint, then obviously so are $Q(\lambda)$ and $Q_r(\lambda)$. The characterizations of Theorem 3.2 have as a consequence the following interlacing results for eigenfunctions, as in [1] and [10]. They are proved in an analogous way as for the eigenvalues in the self-adjoint case.

Proposition 4.1. *Let $\mathcal{H}_0$ be a (closed) subspace of $\mathcal{H}$ of dimension $dim\mathcal{H}_0 = r(\leq n)$ and $Q_r(\lambda)$ the part of the orthogonal projection $Q(\lambda)$ of the analytic and self-adjoint operator function $P(\lambda)$ on $H_0(\lambda)$, with $\lambda \in \mathbb{R}$. If*

$$t_1(\lambda) \preceq t_2(\lambda) \preceq \ldots \prec t_r(\lambda)$$

are the eigenfunctions of $Q_r(\lambda)$, then for $1 \leq k \leq r$,

$$\mu_k(\lambda) \preceq t_k(\lambda) \preceq \mu_{k+n-r}(\lambda).$$

Proof. By Rellich's theorem for $Q_r(\lambda)$, we have

$$Q_r(\lambda) = W(\lambda) diag\,(t_1(\lambda), \ldots, t_r(\lambda))\, W^*(\lambda), \tag{4.1}$$

where $W(\lambda)$ is a unitary operator function in $\mathcal{H}_0$, for every $\lambda \in \mathbb{R}$.

Let $1 \leq k \leq r$ and the k-dimensional subspace $\tilde{\mathcal{S}}(\lambda)$ of $\mathcal{H}$, with orthonormal basis the first k columns of the isometry $VW(\lambda) : \mathcal{H}_0 \to \mathcal{H}$. Denoting by $\{e_j\}_{j=1}^{r}$ the standard basis of $\mathbb{C}^r$, Theorem 3.2 yields

$$\begin{aligned}\mu_k(\lambda) &= \inf_{\substack{\mathcal{S}(\lambda)\subset\mathcal{H}\\ dim\mathcal{S}(\lambda)=k}} \sup_{\substack{x(\lambda)\in\mathcal{S}(\lambda)\\ \|x(\lambda)\|_2=1}} \langle P(\lambda)x(\lambda), x(\lambda)\rangle \preceq \sup_{\substack{x(\lambda)\in\tilde{\mathcal{S}}(\lambda)\\ \|x(\lambda)\|_2=1}} \langle P(\lambda)x(\lambda), x(\lambda)\rangle = \\ &= \sup_{\substack{\xi\in span\{e_1,\ldots,e_k\}\in\mathbb{C}^r\\ \|\xi\|_2=1}} \langle P(\lambda)VW(\lambda)\xi, VW(\lambda)\xi\rangle = \\ &= \sup_{\substack{\xi=(\xi_1,\ldots,\xi_k,0,\ldots,0)\in\mathbb{C}^r\\ \|\xi\|_2=1}} \langle diag\,(t_1(\lambda),\ldots,t_r(\lambda))\,\xi,\xi\rangle = t_k(\lambda).\end{aligned}$$

For the second inequality we use the sup-inf characterization of μ_{k+n-r} in Theorem 3.2, considering the subspace $\tilde{T}$ spanned by the last $r-k+1$ columns of the isometry $VW(\lambda)$ and proceeding in a similar way. □

Proposition 4.2. *Let the self-adjoint operator functions $P_1(\lambda)$, $P_2(\lambda)$ in a Hilbert space $\mathcal{H}$ with $dim\mathcal{H} = n$ and $R(\lambda) = P_1(\lambda) - P_2(\lambda)$. Denoting by $(\mu_j(\lambda), u_j(\lambda))$, $(t_j(\lambda), v_j(\lambda))$ and $(s_j(\lambda), w_j(\lambda))$, $j = 1,\ldots,n$, the corresponding eigenpairs of $P_1(\lambda)$, $P_2(\lambda)$ and $R(\lambda)$ and considering that each set of eigenfunctions is arranged in increasing order, then*

$$\begin{aligned} s_k(\lambda) &\succeq \mu_i(\lambda) - t_n(\lambda) \quad \textit{for} \quad i \leq k,\\ s_k(\lambda) &\preceq \mu_i(\lambda) - t_1(\lambda) \quad \textit{for} \quad i \geq k.\end{aligned}$$

More specifically for $i = k$,

$$\mu_k(\lambda) - t_n(\lambda) \preceq s_k(\lambda) \preceq \mu_k(\lambda) - t_1(\lambda).$$

Proof. For $k \geq i$ we consider the subspaces $\mathcal{J}_1(\lambda) = span\,\{u_i(\lambda),\ldots,u_n(\lambda)\}$, $\mathcal{J}_2(\lambda) = span\,\{v_{k-i+1}(\lambda),\ldots,v_n(\lambda)\}$, $\mathcal{J}_3(\lambda) = span\,\{w_1(\lambda),\ldots,w_k(\lambda)\}$ and the unit vector $y(\lambda) \in \mathcal{J}_1(\lambda) \cap \mathcal{J}_2(\lambda) \cap \mathcal{J}_3(\lambda)$, since the intersection of these subspaces is nontrivial. Then since $y(\lambda) \in \mathcal{J}_1(\lambda)$, as in the proof of Theorem 3.2, we have that $\mu_i(\lambda) \preceq \langle P_1(\lambda)y(\lambda), y(\lambda)\rangle \preceq \mu_n(\lambda)$ and since $y(\lambda) \in \mathcal{J}_2(\lambda)$ then $t_{k-i+1}(\lambda) \preceq \langle P_2(\lambda)y(\lambda), y(\lambda)\rangle \preceq t_n(\lambda)$. Hence, we obtain

$$\begin{aligned} s_k(\lambda) &= \sup_{x(\lambda)\in\mathcal{J}_3(\lambda)} \langle R(\lambda)x(\lambda), x(\lambda)\rangle \succeq \langle R(\lambda)y(\lambda), y(\lambda)\rangle = \\ &= \langle P_1(\lambda)y(\lambda), y(\lambda)\rangle - \langle P_2(\lambda)y(\lambda), y(\lambda)\rangle \succeq \mu_i(\lambda) - t_n(\lambda).\end{aligned}$$

For $k \leq i$, let $\tilde{\mathcal{J}}_1(\lambda) = span\,\{u_1(\lambda),\ldots,u_i(\lambda)\}$, $\tilde{\mathcal{J}}_2(\lambda) = span\,\{v_1(\lambda),\ldots,v_{n-i+k}(\lambda)\}$ and $\tilde{\mathcal{J}}_3(\lambda) = span\,\{w_k(\lambda),\ldots,w_n(\lambda)\}$. Similarly for the unit vector $y(\lambda) \in \tilde{\mathcal{J}}_1(\lambda) \cap \tilde{\mathcal{J}}_2(\lambda) \cap \tilde{\mathcal{J}}_3(\lambda)$ we have

$$s_k(\lambda) = \inf_{x(\lambda)\in\tilde{\mathcal{J}}_3(\lambda)} \langle R(\lambda)x(\lambda), x(\lambda)\rangle \preceq \langle R(\lambda)y(\lambda), y(\lambda)\rangle \preceq \mu_i(\lambda) - t_1(\lambda). \qquad \square$$

Using the same notation as in Proposition 4.2 we obtain the following proposition.

Proposition 4.3. *Let $\mu_k(\lambda)$, $t_k(\lambda)$ $(k = 1, \ldots, n)$ be the ordered eigenfunctions of the self-adjoint operator functions $P_1(\lambda)$ and $P_2(\lambda)$ respectively. If for the smallest eigenfunction of the operator $R(\lambda) = P_2(\lambda) - P_1(\lambda)$ holds $s_1(\lambda) \succeq 0$, then $\mu_k(\lambda) \preceq t_k(\lambda)$, for $k = 1, \ldots, n$.*

Proof. By the eigenvectors $v_j(\lambda)$ of $P_2(\lambda)$ we consider the subspace $\mathcal{S}_k(\lambda) = span\, \{v_1(\lambda), \ldots, v_k(\lambda)\}$. As in Theorem 3.2, for every k-dimensional subspace $\mathcal{S}(\lambda)$ the set

$$\{x^*(\lambda)P_1(\lambda)x(\lambda) : x(\lambda) \in \mathcal{S}(\lambda), \|x(\lambda)\|_2 = 1\}$$

is bounded according to the lexicographic order. Hence, let the unit vector $y(\lambda) \in \mathcal{S}_k(\lambda)$ such that

$$\langle P_1(\lambda)y(\lambda), y(\lambda)\rangle = \sup_{x(\lambda)\in\mathcal{S}_k(\lambda)} \langle P_1(\lambda)x(\lambda), x(\lambda)\rangle .$$

Then by Theorem 3.2, we have

$$\begin{aligned}\mu_k(\lambda) &= \inf_{\substack{\mathcal{S}(\lambda)\subset\mathcal{H} \\ dim\mathcal{S}(\lambda)=k}} \sup_{\substack{x(\lambda)\in\mathcal{S}(\lambda) \\ \|x(\lambda)\|_2=1}} \langle P_1(\lambda)x(\lambda), x(\lambda)\rangle \preceq \\ &\preceq \sup_{\substack{x(\lambda)\in\mathcal{S}_k(\lambda) \\ \|x(\lambda)\|_2=1}} \langle P_1(\lambda)x(\lambda), x(\lambda)\rangle = \langle P_1(\lambda)y(\lambda), y(\lambda)\rangle\end{aligned}$$

and also by (3.5),

$$t_k(\lambda) = \sup_{\substack{x(\lambda)\in\mathcal{S}_k(\lambda) \\ \|x(\lambda)\|_2=1}} \langle P_2(\lambda)x(\lambda), x(\lambda)\rangle \succeq \langle P_2(\lambda)y(\lambda), y(\lambda)\rangle .$$

Since $s_1(\lambda) \succeq 0$ and $\langle R(\lambda)x(\lambda), x(\lambda)\rangle \in co\,\{s_1(\lambda), s_n(\lambda)\}$ for every unit vector $x(\lambda) \in \mathcal{H}$, clearly $\langle R(\lambda)x(\lambda), x(\lambda)\rangle \succeq 0$, which implies in particular that $\langle P_2(\lambda)y(\lambda), y(\lambda)\rangle \succeq \langle P_1(\lambda)y(\lambda), y(\lambda)\rangle$. Consequently, $\mu_k(\lambda) \preceq t_k(\lambda)$. □

Let

$$s_j(\lambda) = s_{j,0} + \lambda s_{j,1} + \lambda^2 s_{j,2} + \ldots, \quad j = 1, \ldots, n,$$

be the analytic expressions of the eigenfunctions of the self-adjoint operator function $R(\lambda) = P_2(\lambda) - P_1(\lambda)$ in a neighbourhood of $\lambda_0 = 0$. Obviously for $\lambda = 0$, the coefficients $s_{j,0}$, $(j = 1, \ldots, n)$ are the eigenvalues of the self-adjoint operator $R(0)$ in nondecreasing order, i.e.

$$-\|R(0)\|_2 \leq s_{1,0} \leq s_{2,0} \leq \ldots \leq s_{n,0} \leq \|R(0)\|_2 .$$

Using Proposition 4.3 we formulate the next corollary.

Corollary 4.4. *Let the self-adjoint operator functions $P_1(\lambda)$ and $P_2(\lambda)$ and $R(\lambda) = P_2(\lambda) - P_1(\lambda)$. If $R(\lambda)$ has eigenfunctions*

$$-d \preceq s_1(\lambda) \preceq \ldots \preceq s_n(\lambda) \preceq d,$$

where $d = \|R(0)\|_2$, *then*

$$\mu_k(\lambda) - d \preceq t_k(\lambda) \preceq \mu_k(\lambda) + d.$$

Proof. Clearly $P_2(\lambda) - P_1(\lambda) + dI$ has eigenfunctions $s_j(\lambda) + d$ and $0 \preceq s_1(\lambda) + d$. Then, by Proposition 4.3 we have $\mu_k(\lambda) - d \preceq t_k(\lambda)$.

Similarly, the eigenfunctions of $P_1(\lambda) - P_2(\lambda) + dI$ are $-s_j(\lambda) + d$ and for the smallest of which, $-s_n(\lambda) + d$, holds $0 \preceq -s_n(\lambda) + d$. Thus, by Proposition 4.3 the asserted relation is obtained. □

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