

SOME EFFICIENT SEVENTH-ORDER DERIVATIVE-FREE FAMILIES IN ROOT-FINDING

Fazlollah Soleymani

Communicated by Zdzisław Jackiewicz

Abstract. The interest in efficient root-finding iterations is nowadays growing and influenced by the widespread use of high-speed computers. On the other hand, the calculation of derivatives is often hard, when the problems are formulated in terms of nonlinear equations and as a result, the importance of derivative-free methods emerges. For these reasons, some efficient three-step families of iterations for solving nonlinear equations are suggested, where the analytical proofs show their seventh-order error equations consuming only four function evaluations per iteration. We employ hard numerical test problems to illustrate the accuracy of the new methods from the families.

Keywords: numerical analysis, derivative-free families, order, iterative methods.

Mathematics Subject Classification: 65H05.

1. INTRODUCTION

There are many situations in which the calculation of derivatives of the functions is hard or they should be computed numerically. Due to this, the application of derivative-involved methods in solving nonlinear scalar equations, such as Newton's iteration, is confined. Our investigation here concerns the approximation of a zero of $f(x)$ or equivalently, the estimation of a root of the equation $f(x) = 0$ by derivative-free methods. Herein, we restrict ourselves to $f(x)$ which are real single-valued functions of a real variable possessing a smooth condition in the domain D including the simple root. Moreover, using computers meant that many high-order algorithms, formerly of academic interest only, become feasible for calculation and they can, in fact, be used repeatedly in many establishments for a wide variety of problems.

In the fundamental book [12], iterations for solving nonlinear equations are divided into two main categories of “one-point” and “multi-point” iterations where each of them

are sub-classified into “with” or “without memory” iterative methods. The interest in multi-point methods has renewed due to overcoming the drawbacks of one-point iterations. In fact, they possess better convergence order and efficiency indices when solving single-valued nonlinear equations. Note that the efficiency of a method is measured by the concept of efficiency index (if both function and derivative evaluations have the same computational cost) as $\sqrt[p]{p}$, where p is the convergence (speed) rate and n denotes the whole number of evaluations per full iteration [12]. Moreover, we define the order and asymptotic error constant of an iteration function as follows. Let $x_1, x_2, \dots, x_i, \dots$ be a sequence converging to α , namely, the simple root of a nonlinear equation. Let furthermore $e_i = x_i - \alpha$ be the error equation in the i th step. If there exists a real number p and a nonzero constant C such that $\lim_{i \rightarrow \infty} |e_{i+1}|/|e_i|^p = C$, then p is called the convergence (speed) order of the sequence and C is called the asymptotic error constant.

In this paper, we look for high-order methods in which there is no need for a derivative calculation per full cycle and can be easily applied to hard problems, when the evaluation of the derivatives is impossible.

In what follows, we provide a short literature review on the available famous derivative-free methods in Section 2. Next, in Section 3, we furnish some families of efficient three-step without memory iterations for simple roots, which comprise four function evaluations per cycle only. The analytical proofs of the main contributions are also given therein. Section 4 gives a thorough numerical comparison between the existing derivative-free methods and our novel proposed derivative-free methods from the families. This section also contains a seventh-order algorithm with globally convergence. Finally, in Section 5, some concluding remarks are given.

2. LITERATURE REVIEW

The main objective of this section is to give a short review on the most important derivative-free methods in the literature. However, note that the overview of the published articles during the last ten years (2000–2010) concerning numerical methods for solving nonlinear equations is given in Part 1 of the well-written standard text-book [1]. The first derivative-free method was provided in [11] as follows (in the backward form)

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f(x_n) - f(x_n - f(x_n))}, \quad (2.1)$$

wherein the iteration uses two functional evaluations per cycle, as the Newton’s iteration does, to reach convergence order two. Note that, the forward form of Steffensen’s method; $f(x_n + f(x_n)) - f(x_n)$ in the denominator of (2.1), also possess the same order of convergence.

Some decades later, Kung and Traub in the fundamental paper [3] provided the following derivative-free family ($\beta \in \mathbb{R} \setminus \{0\}$) of methods by using inverse interpolation

$$\begin{cases} y_n = x_n + \beta f(x_n), \\ z_n = y_n - \beta \frac{f(x_n)f(y_n)}{f(y_n)-f(x_n)}, \\ x_{n+1} = z_n - \frac{f(x_n)f(y_n)}{f(y_n)-f(x_n)} \left[\frac{1}{f[y_n, x_n]} - \frac{1}{f[z_n, y_n]} \right], \end{cases} \quad (2.2)$$

where $f[y_n, x_n]$ and $f[z_n, y_n]$ are divided differences. Note that similar notations will be used throughout the paper. The scheme (2.2) comprises three evaluations of the function to reach convergence rate four. They moreover gave an n -step derivative-free family of one parameter with 2^n as its maximum convergence rate.

Later, in [6] the authors gave the following two-step iterative family using three function evaluations per full iteration

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n, w_n]}, \\ x_{n+1} = y_n - \frac{f(y_n)}{f[x_n, y_n] + f[y_n, w_n] - f[x_n, w_n] + b(y_n - x_n)(y_n - w_n)}, \end{cases} \quad (2.3)$$

where $w_n = x_n + f(x_n)$ and $b \in \mathbb{R}$. Note that the first step of (2.3) or any improvements of derivative-free without memory iterations is mostly the (backward or forward forms of) Steffensen’s method (2.1).

Recently, in [2], authors provided a three-step sixth-order family of derivative-free methods by four evaluations per cycle as comes next

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n, w_n]}, \\ z_n = y_n - \frac{f(y_n)}{f[x_n, w_n]} \left[1 + \frac{f(y_n)}{f(x_n)} + \frac{f(y_n)}{f(w_n)} + \alpha \left(\frac{f(y_n)}{f(x_n)} \right)^2 + \beta \left(\frac{f(y_n)}{f(w_n)} \right)^2 \right], \\ x_{n+1} = z_n - \frac{f(z_n)}{f[x_n, w_n]} \left[1 + \frac{f(z_n)}{f(x_n)} + \frac{f(z_n)}{f(w_n)} + \alpha \left(\frac{f(z_n)}{f(x_n)} \right)^2 + \beta \left(\frac{f(z_n)}{f(w_n)} \right)^2 + \eta \frac{f(z_n)}{f(y_n)} \right], \end{cases} \quad (2.4)$$

whence $w_n = x_n - \kappa f(x_n)$, $\kappa \in \mathbb{R} \setminus \{0\}$, α, β , and $\eta \in \mathbb{R}$. For further study on this topic or related matters, one may consult the papers [4,7,8,9,10].

3. NOVEL DERIVATIVE-FREE FAMILIES

This section contains the central contributions of this paper. As can be seen from the existing methods in Section 2, to give high-order derivative-free methods, first of all, we should consider a three-step cycle in which (2.3) with $b = 0$ is in the first and second steps of the cycle, as well as a suitable weight function at the third step alongside $f'(z_n) \approx f[x_n, z_n]$. Hence, we suggest the following three-step iteration

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n, w_n]}, \\ z_n = y_n - \frac{f(y_n)}{f[x_n, y_n] + f[y_n, w_n] - f[x_n, w_n]}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[x_n, z_n]} \left[1 + \frac{f(y_n)}{f(w_n)} + \frac{f(z_n)}{f(y_n)} + \left\{ \frac{2 + f[x_n, w_n]}{(1 + f[x_n, w_n])^2} \right\} \left(\frac{f(y_n)}{f(x_n)} \right)^2 + \right. \\ \left. + \gamma \frac{f(z_n)}{f(x_n)} + \delta \frac{f(z_n)}{f(w_n)} \right], \end{cases} \quad (3.1)$$

wherein $w_n = x_n + f(x_n)$, $\gamma, \delta \in \mathbb{R}$. The without memory iteration (3.1) includes four function evaluations per full cycle just like (2.4), but it arrives at seventh-order convergence according to Theorem 3.1. Moreover, γ and δ are two real valued free parameters. At this time a question one might raise is how was the weight function in the third step of (3.1) attained; this question will be answered after ending the proof of Theorem 3.1.

Theorem 3.1. *Let $\alpha \in D$ be a simple zero of a sufficiently differentiable function $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and let that $c_j = f^{(j)}(\alpha)/j!$, $j \geq 1$. If x_0 is sufficiently close to α , then,*

- (i) *the order of convergence of the solution by the family defined in (3.1) is seven, and*
- (ii) *this solution reads the error equation*

$$e_{n+1} = -((1 + c_1)^2 c_2^2 (-c_2^2 + c_1 c_3) (c_1 (1 + c_1) c_3 (-2 + \gamma + c_1 \gamma + \delta) - c_2^2 (-6 + \gamma + \delta + c_1 (-6 + c_1 (-1 + \gamma) + 2\gamma + \delta))) e_n^7 / (c_1^6) + O(e_n^8)). \quad (3.2)$$

Proof. We expand any terms of (3.1) around the simple root α in the n th iterate. Thus, we write $f(x_n) = c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + \dots + O(e_n^8)$. Accordingly, we attain

$$y_n = \alpha + \left(1 + \frac{1}{c_1}\right) c_2 e_n^2 + O(e_n^3). \quad (3.3)$$

Note that for simplicity, we just included the first or second terms of the obtained Taylor expansion for any expression in the proof. Now we should expand $f(y_n)$ around the simple root by using (3.3). We have

$$f(y_n) = (1 + c_1) c_2 e_n^2 + O(e_n^3). \quad (3.4)$$

Using (3.4) and the second step of (3.1), we attain

$$\frac{f(y_n)}{f[x_n, y_n] + f[y_n, w_n] - f[x_n, w_n]} = \left(1 + \frac{1}{c_1}\right) c_2 e_n^2 + O(e_n^3). \quad (3.5)$$

Additionally, the Taylor's series expansion of the second step of (3.1), using (3.5) gives us

$$z_n - \alpha = (1/c_1^3) (1 + c_1)^2 c_2 (c_2^2 - c_1 c_3) e_n^4 + O(e_n^5). \quad (3.6)$$

Now, the Taylor expansion of $f(z_n)$ around the simple root is needed. Therefore, we find that

$$f(z_n) = \frac{(1 + c_1)^2 c_2 (c_2^2 - c_1 c_3)}{c_1^2} e_n^4 + O(e_n^5). \quad (3.7)$$

In the last step of (3.1), we attain

$$1 + \frac{f(y_n)}{f(w_n)} + \frac{f(z_n)}{f(y_n)} + \left\{ \frac{2 + f[x_n, w_n]}{(1 + f[x_n, w_n])^2} \right\} \left(\frac{f(y_n)}{f(x_n)} \right)^2 + \gamma \frac{f(z_n)}{f(x_n)} + \delta \frac{f(z_n)}{f(w_n)} = 1 + \frac{c_2 e_n^1}{c_1} + \frac{c_3 e_n^2}{c_1} + O(e_n^3). \quad (3.8)$$

Finally, using (3.6)-(3.8) in the last step of (3.1) and simplifying ends in

$$e_{n+1} = -((1 + c_1)^2 c_2^2 (-c_2^2 + c_1 c_3) (c_1 (1 + c_1) c_3 (-2 + \gamma + c_1 \gamma + \delta) - c_2^2 (-6 + \gamma + \delta + c_1 (-6 + c_1 (-1 + \gamma) + 2\gamma + \delta))) e_n^7 / (c_1^6) + O(e_n^8). \tag{3.9}$$

This shows that (3.1) is a seventh-order bi-parametric family of derivative-free without memory iterations consuming only four function evaluations per full cycle. This completes the proof. \square

To discuss on how the weight function $1 + \frac{f(y_n)}{f(w_n)} + \frac{f(z_n)}{f(y_n)} + \left\{ \frac{2+f[x_n, w_n]}{(1+f[x_n, w_n])^2} \right\} \left(\frac{f(y_n)}{f(x_n)} \right)^2 + \gamma \frac{f(z_n)}{f(x_n)} + \delta \frac{f(z_n)}{f(w_n)}$ was obtained in the last step of (3.1), we should remark that according to the approach of the weight function, in fact we generally consider the following

$$x_{n+1} = z_n - \frac{f(z_n)}{f[x_n, z_n]} [G(A) + H(B) + K(\Gamma) + L(\Delta) + P(E)], \tag{3.10}$$

wherein $G(A), H(B), K(\Gamma), L(\Delta)$ and $P(E)$ are five real-valued weight functions with $A = \frac{f(y_n)}{f(w_n)}, B = \frac{f(z_n)}{f(y_n)}, \Gamma = \frac{f(y_n)}{f(x_n)}, \Delta = \frac{f(z_n)}{f(x_n)}$ and $E = \frac{f(z_n)}{f(w_n)}$. The weight function should be chosen such that order of convergence seven is attained. That is to say, by Taylor's series expanding around the simple root, we will find that the conditions on the weight functions, which lead the order to seven are as follows $G(0) = G'(0) = 1, G''(0) = 0, |G^{(3)}(0)| < \infty, H(0) = 0, H'(0) = 1, K(0) = K'(0) = 0, K''(0) = 2 \left(\frac{2+f[x_n, w_n]}{(1+f[x_n, w_n])^2} \right), |K^{(3)}(0)| < \infty, L(0) = 0, |L'(0)| < \infty,$ and $P(0) = 0, |P'(0)| < \infty$. Therefore, in general, we can give a class of three-step derivative-free seventh-order methods which reads in the follow-up error equation and (3.1) is just a especial case of this class

$$e_{n+1} = \frac{1}{6c_1^6} (1 + c_1)^2 c_2^2 (-c_2^2 + c_1 c_3) (-6c_1 (1 + c_1) c_3 (-2 + (1 + c_1) L'(0) + P'(0)) + c_2^2 (-36 + 6L'(0) + 6P'(0) + 6c_1 (-6 + c_1 (-1 + L'(0)) + 2L'(0) + P'(0)) + G^{(3)}(0) + K^{(3)}(0) + c_1 (3 + c_1 (3 + c_1)) K^{(3)}(0)) e_n^7 + O(e_n^8). \tag{3.11}$$

From the viewpoint of applications, the iterations such as (3.1) are much better than derivative-involved methods. Since the derivatives cannot always be calculated. Moreover, some hard nonlinear functions are not differentiable (smooth) in the neighborhood of the simple zero (D).

In addition to (3.1), we would like to present more new seventh-order families. In (3.1), we used the finite difference approximation $f[x_n, z_n]$ in the denominator of the fraction which its numerator is $f(z_n)$. In what follows, we change this *finite difference approximation* and try to suggest some other new seventh-order families like (3.1). Toward this end, we again consider a three-step cycle, like (3.1), but with a different

third step as comes next (i.e. a different procedure in constructing the weight function to make the order seven based on a new finite difference approximation)

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n, w_n]}, \\ z_n = y_n - \frac{f(y_n)}{f[x_n, y_n] + f[y_n, w_n] - f[x_n, w_n]}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[w_n, z_n]} \left[1 + \frac{f(z_n)}{f(y_n)} + \frac{f(y_n)}{f(x_n)} + \right. \\ \left. + \{2 + f[x_n, w_n](3 + f[x_n, w_n])\} \left(\frac{f(y_n)}{f(w_n)} \right)^2 + \omega \frac{f(z_n)}{f(x_n)} + \varphi \frac{f(z_n)}{f(w_n)} \right], \end{cases} \quad (3.12)$$

wherein $w_n = x_n + f(x_n)$, $\omega, \varphi \in \mathbb{R}$. The without memory iteration (3.12) includes four function evaluations per full cycle just like (2.4) and (3.1). Theorem 3.2 illustrates that (3.12) arrives at seventh order of convergence. Moreover, ω and φ are two real valued free parameters. The discussion on how the weight function in (3.12) was attained is quite similar to the previous case given above, thus it is omitted.

Theorem 3.2. *Let $\alpha \in D$ be a simple zero of a sufficiently differentiable function $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and let that $c_j = f^{(j)}(\alpha)/j!$, $j \geq 1$. If x_0 is sufficiently close to α , then the order of convergence of the solution by the family defined in (3.12) is seven.*

Proof. We assume the symbolic computation as done in the proof of Theorem 3.1, and then we obtain (3.3)–(3.7) again. For the weight function in the third step of (3.12), we have

$$\begin{aligned} & 1 + \frac{f(z_n)}{f(y_n)} + \frac{f(y_n)}{f(x_n)} + \{2 + f[x_n, w_n](3 + f[x_n, w_n])\} \left(\frac{f(y_n)}{f(w_n)} \right)^2 + \omega \frac{f(z_n)}{f(x_n)} + \\ & + \varphi \frac{f(z_n)}{f(w_n)} = 1 + \left(1 + \frac{1}{c_1} \right) c_2 e_n^1 + \frac{(c_2^2 + (1 + c_1)^2 c_3) e_n^2}{c_1} + O(e_n^3). \end{aligned} \quad (3.13)$$

Finally using (3.7) and (3.13) and the last step of (3.12) results in

$$\begin{aligned} e_{n+1} = & (1/(c_1^6))(1 + c_1)^3 c_2^2 (c_2^2 - c_1 c_3) (c_2^2 (6 + c_1(6 + c_1 - \omega) - \omega - \varphi) + \\ & + c_1 c_3 ((1 + c_1)(-2 + \omega) + \varphi)) e_n^7 + O(e_n^8). \end{aligned} \quad (3.14)$$

This shows the seventh order of convergence of (3.12) by using only four function evaluations per full iteration. The proof is complete. \square

More similar families to (3.1) and (3.12) could be given by changing w_n (i.e. the backward finite difference approximation of $f'(x_n)$ instead of forward) and the weight function in the last step of a three-step cycle. To propose another novel three-step without memory iteration with the same number of evaluations as in (3.1) and (3.12), we suggest the following scheme in which $w_n = x_n - f(x_n)$ and $\rho, \tau \in \mathbb{R}$

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n, w_n]}, \\ z_n = y_n - \frac{f(y_n)}{f[x_n, y_n] + f[y_n, w_n] - f[x_n, w_n]}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[x_n, z_n]} \left[1 + \frac{f(y_n)}{f(w_n)} + \frac{f(z_n)}{f(y_n)} + \left\{ \frac{2 - f[x_n, w_n]}{(f[x_n, w_n] - 1)^2} \right\} \left(\frac{f(y_n)}{f(x_n)} \right)^2 + \right. \\ \left. + \rho \frac{f(z_n)}{f(x_n)} + \tau \frac{f(z_n)}{f(w_n)} \right]. \end{cases} \quad (3.15)$$

Theorem 3.3. Let $\alpha \in D$ be a simple zero of a sufficiently differentiable function $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and let that $c_j = f^{(j)}(\alpha)/j!$, $j \geq 1$. If x_0 is sufficiently close to α , then, the order of convergence of the solution by the family defined in (3.15) is seven when $w_n = x_n - f(x_n)$ and it reads the following error equation

$$e_{n+1} = -(1/(c_1^6))(-1 + c_1)^2 c_2^2 (-c_2^2 + c_1 c_3) ((-1 + c_1) c_1 c_3 (2 + (-1 + c_1) \rho - \tau) - c_2^2 (-6 + \rho + c_1 (6 + c_1 (-1 + \rho) - 2\rho - \tau) + \tau)) e_n^7 + O(e_n^8). \quad (3.16)$$

Proof. The proof of this theorem is similar to the Proofs of Theorem 3.1 and Theorem 3.2. Hence, it is omitted. \square

We can further develop the following scheme (without any free parameter to reduce the computational load) by choosing $w_n = x_n - f(x_n)$ and a different weight function which also possesses seventh-order convergence according to Theorem 3.4.

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n, w_n]}, \\ z_n = y_n - \frac{f(y_n)}{f[x_n, y_n] + f[y_n, w_n] - f[x_n, w_n]}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[w_n, z_n]} \left[1 + \frac{f(z_n)}{f(y_n)} + \frac{f(y_n)}{f(x_n)} + \{2 + f[x_n, w_n](-3 + f[x_n, w_n])\} \left(\frac{f(y_n)}{f(w_n)} \right)^2 \right], \end{cases} \quad (3.17)$$

Theorem 3.4. Let $\alpha \in D$ be a simple zero of a sufficiently differentiable function $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ and let that $c_j = f^{(j)}(\alpha)/j!$, $j \geq 1$. If x_0 is sufficiently close to α , then, the order of convergence of the solution by the method defined in (3.17) is seven when $w_n = x_n - f(x_n)$ and it reads the following simple error equation

$$e_{n+1} = \frac{(-1 + c_1)^3 c_2^3 (-c_2^2 + c_1 c_3) ((6 + (-6 + c_1) c_1) c_2^2 + 2(-1 + c_1) c_1 c_3) e_n^7}{c_1^6} + O(e_n^8). \quad (3.18)$$

Proof. The proof of this theorem is similar to the proofs of Theorem 3.1 and Theorem 3.2. Hence, it is omitted. \square

In terms of the computational point of view and efficiency index (defined in Section 1), each derivative-free method of the families (3.1), (3.12), (3.15) and (3.17) arrives at $\sqrt[4]{7} \approx 1.6265$, which is greater than that of (2.1) i.e. $\sqrt[2]{2} \approx 1.4142$, that of the cubical methods $\sqrt[3]{3} \approx 1.4422$ such as the methods in [13], that of (2.2) and (2.3) with $\sqrt[3]{4} \approx 1.5874$ and that of (2.4) with $\sqrt[4]{6} \approx 1.5650$.

4. NUMERICAL REPORTS

Computational tests for the reported families of derivative-free without memory methods were done by means of MATLAB 7.6 using 500 digits floating point arithmetic (VPA: 500). The level of approximation of the solution (root) is directly tied to the

precision $\varepsilon = 10^{-500}$ of the computer. The adopted stopping criterion for the computer programs were selected as $|f(x_n)| \leq \varepsilon$, in the n th iterate.

The nonlinear hard test functions are displayed in Table 1 with their simple roots and the initial approximations for each test function. The methods considered in numerical comparisons are the quadratic method of Steffensen (2.1), the quartic method of Ren et al. with $b = 2$, the hexical method of Khattri and Argyros with $\kappa = \eta = 1$ and $\alpha, \beta = 0$, the seventh-order family (3.1)-(PM1) with $\gamma = \delta = 0$, and the seventh-order family (3.15)-(PM2) with $\rho = \tau = 0$. Family (3.12) and method (3.17) produce somehow similar results to (3.1) and (3.15), hence, we omit them in numerical comparisons.

The results of comparisons are displayed in Table 2, where “IT”, “TNE”, “CPU time” (in seconds) show the number of iterations, the total number of evaluations, the elapsed time taken by the MATLAB m-file of the method to run, respectively. In Table 2, $|f|$ shows the absolute value of the function after the specified number of iterations. The computer specifications are Microsoft Windows XP, Intel(R) Pentium(R) 4, CPU 3.20 GHz, with 4 GB of RAM.

We should remark that CPU run time is not unique and depends on the specifications of the computers. However, we list the CPU-time by a mean over 15 performances of the MATLAB Codes to reveal its validness.

As can be seen from Table 2, the numerical results support the theory developed in Section 3. However, in viewpoint of e-time our methods from the developed derivative-free families (or classes) require a little more time, which dose not limit the applicability of our suggested methods.

The only drawback of multi-point iterations is finding a proper starting point near to the sought zero. Mostly, a high-order iterative method does not guarantee convergence and this emerges the importance of the attraction basin (the set of all guesses which lead to find the solution). However, in such cases the best way is to combine the high-order method with the low-order schemes which guarantee the convergence, such as Bisection or Regula Falsi methods [5]. In what follows, we present a combination between our derivative-free seventh-order family (3.1) with the bisection method to reach global convergence. Note that any of the schemes (3.12), (3.15) and (3.17) could also be applied instead of (3.1) in the following algorithms.

Algorithm 1. A seventh-order globally convergent iterative algorithm free from derivative for $f(x) = 0$.

0. Let the initial small interval $[a_0, b_0]$ with $f(a_0)f(b_0) < 0$ and the initial guess $x_0 \in [a_0, b_0]$ be available. The precision $\varepsilon = 10^{-500}$ is chosen. If $|f(x_0)| < \varepsilon$, then terminate and output x_0 at the approximate zero of f . Then, do $n := 0$;

1. If $|a_n - b_n| < \varepsilon$, then terminate and output x_n at the approximate zero of f ;
2. Compute y_n by the first step of (3.1). If $|f(y_n)| < \varepsilon$, then terminate and output y_n at the approximate zero of f ;
3. Compute z_n by the second step of (3.1). If $|f(z_n)| < \varepsilon$, then terminate and output z_n at the approximate zero of f ;
4. Compute x_{n+1} by the third step of (3.1). If $|f(x_{n+1})| < \varepsilon$, then terminate and output x_{n+1} at the approximate zero of f ;
5. Compute $k_{n+1} = \frac{a_n - b_n}{2}$ by the Bisection method. If $|f(k_{n+1})| < \varepsilon$, then terminate and output k_{n+1} at the approximate zero of f ;

6. If $x_{n+1} \in [a_n, b_n]$, then
 - If $f(x_{n+1})f(k_{n+1}) < 0$, then $a_{n+1} := \min(x_{n+1}, k_{n+1}), b_{n+1} := \max(x_{n+1}, k_{n+1})$, do $n := n+1$, and go to 1.
 - If $f(x_{n+1})f(k_{n+1}) > 0$, and $f(x_{n+1})f(a_n) < 0$, then $b_{n+1} := \min(x_{n+1}, k_{n+1}), a_{n+1} := a_n, x_{n+1} := b_{n+1}$, do $n := n+1$, and go to 1.
 - If $f(x_{n+1})f(k_{n+1}) > 0$, and $f(x_{n+1})f(a_n) > 0$, then $a_{n+1} := \max(x_{n+1}, k_{n+1}), b_{n+1} := b_n, x_{n+1} := a_{n+1}$, do $n := n+1$, and go to 1.
 - else if x_{n+1} is not in $[a_n, b_n]$, then
 - If $f(a_n)f(k_{n+1}) < 0$, then $b_{n+1} := k_{n+1}, a_{n+1} := a_n, x_{n+1} = \operatorname{argmin}\{|f(a_n)|, |f(k_{n+1})|\}$, do $n := n+1$, and go to 1;
 - If $f(b_n)f(k_{n+1}) < 0$, then $a_{n+1} := k_{n+1}, b_{n+1} := b_n, x_{n+1} = \operatorname{argmin}\{|f(b_n)|, |f(k_{n+1})|\}$, do $n := n+1$, and go to 1.

Table 1. Test nonlinear function, their roots and the initial guesses

Test Functions	Roots	Initial guesses
$f_1(x) = x^5 - x^2 + 7x - 41$	1.9878112719284...	1.97
$f_2(x) = \sqrt{\cos(x^2)} - \ln(x\sqrt{x})$	1.217890626801...	1.24
$f_3(x) = \tan(\sin(x^2)) \times \sin(x) - x^3 + 17$	2.581711667829...	2.8
$f_4(x) = \cos(x) + \ln(x) \times \sqrt{x^3 + 7} - 10$	3.845238953520...	5
$f_5(x) = \arcsin(x)(x^2 + x - 3) + x^5 - x + 1$	0.969472186368...	1
$f_6(x) = \arcsin(x)(x^2 + x - 3) + x^5 - x + 1$	-1.130365453500...	-1.2

Table 2. Comparison of some derivative-free methods

f		(2.1)	(2.3)	(2.4)	PM1	PM2
f_1	IT	10	4	4	3	3
	TNE	20	12	16	12	12
	$ f $	0.2e-73	0.6e-141	0.8e-199	0.2e-150	0.3e-250
	CPU-time	0.52	0.43	0.49	0.45	0.44
f_2	IT	8	4	4	3	3
	TNE	16	12	16	12	12
	$ f $	0.3e-24	0.1e-121	0.3e-80	0.2e-171	0.6e-90
	CPU-time	0.74	0.47	0.55	0.50	0.71
f_3	IT	8	5	4	3	3
	TNE	16	15	16	12	12
	$ f $	0.3e-8	0.4e-108	0.3e-110	0.3e-88	0.1e-137
	CPU-time	0.54	0.65	0.61	0.54	0.52
f_4	IT	8	5	3	3	3
	TNE	16	15	12	12	12
	$ f $	0.1e-43	0.6e-111	0.8e-61	0.1e-136	0.1e-146
	CPU-time	0.77	0.62	0.64	0.53	0.73
f_5	IT	8	4	3	3	3
	TNE	16	12	12	12	12
	$ f $	0.9e-112	0.1e-169	0.2e-139	0.3e-222	0.2e-259
	CPU-time	0.50	0.47	0.41	0.52	0.50
f_6	IT	9	4	4	3	3
	TNE	18	12	16	12	12
	$ f $	0.6e-8	0.2e-110	0.5e-54	0.1e-150	0.4e-93
	CPU-time	0.57	0.47	0.55	0.52	0.50

5. CONCLUDING REMARKS

There are many situations at which the application of derivative-involved methods (schemes with the direct use of a derivative to proceed) is limited. On the other hand, multi-point iteration overcome many disadvantages of one-point methods in the view-point of the convergence rate and efficiency index. Due to these and the availability of high-speed computers in computations, we have given some families of without memory iterations free from derivative, which include four function evaluations per full cycle to proceed.

The analytical proofs of the main contributions were written and they have shown the seventh-order convergence for the proposed methods of families. It was discussed that, in terms of the computational point of view, our schemes in this paper arrive at 1.6265 as their efficiency indices, which is better than many available derivative-free methods in the literature. Finally, we have employed a lot of numerical examples to reveal the fast convergence of our families by comparing with the famous methods in the literature.

REFERENCES

- [1] A. Iliev, N. Kyurkchiev, *Nontrivial Methods in Numerical Analysis (Selected Topics in Numerical Analysis)*, Lambert Acad. Publishing, 2010.
- [2] S.K. Khattri, I.K. Argyros, *Sixth order derivative free family of iterative methods*, Appl. Math. Comput. **217** (2011), 5500–5507.
- [3] H.T. Kung, J.F. Traub, *Optimal order of one-point and multipoint iteration*, Journal of ACM **21** (1974), 643–651.
- [4] W. Liu, L. Wang, X. Bai, H. Zhang, *Optimization calculation and analysis of moving load of the railgun by Newton method*, Modern Appl. Sci. **5** (2011), 212–216.
- [5] Y. Peng, H. Feng, Q. Li, X. Zhang, *A fourth-order derivative-free algorithm for nonlinear equations*, J. Comput. Appl. Math. **15** (2011), 2551–2559.
- [6] H. Ren, Q. Wu, W. Bi, *A class of two-step Steffensen type methods with fourth-order convergence*, Appl. Math. Comput. **209** (2009), 206–210.
- [7] M. Sharifi, D.K.R. Babajee, F. Soleymani, *Finding the solution of nonlinear equations by a class of optimal methods*, Comput. Math. Appl. **63** (2012), 764–774.
- [8] F. Soleymani, M. Sharifi, *On a cubically iterative scheme for solving nonlinear equations*, Far East J. Appl. Math. **43** (2010), 137–143.
- [9] F. Soleymani, S. Karimi Vanani, *Optimal Steffensen-type methods with eighth order of convergence*, Comput. Math. Appl. **62** (2011), 4619–4626.
- [10] F. Soleymani, S.K. Khattri, S. Karimi Vanani, *Two new classes of optimal Jarratt-type fourth-order methods*, Appl. Math. Lett. **25** (2012), 847–853.
- [11] J.F. Steffensen, *Remarks on iteration*, Skand. Aktuarietidskr **16** (1933), 64–72.

- [12] J.F. Traub, *Iterative Methods for the Solution of Equations*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1964.
- [13] Q. Ye, X. Xu, *A class of Newton-like methods with cubic convergence for nonlinear equations*, *Fixed Point Theory* **11** (2010), 161–168.

Fazlollah Soleymani
fazlollah.soleymani@gmail.com

Islamic Azad University
Department of Mathematics
Zahedan Branch, Zahedan, Iran

Received: March 6, 2012.

Revised: June 1, 2012.

Accepted: June 14, 2012.