ON VERTEX \( b \)-CRITICAL TREES

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Abstract. A \( b \)-coloring is a proper coloring of the vertices of a graph such that each color class has a vertex that has neighbors of all other colors. The \( b \)-chromatic number of a graph \( G \) is the largest \( k \) such that \( G \) admits a \( b \)-coloring with \( k \) colors. A graph \( G \) is \( b \)-critical if the removal of any vertex of \( G \) decreases the \( b \)-chromatic number. We prove various properties of \( b \)-critical trees. In particular, we characterize \( b \)-critical trees.

Keywords: \( b \)-coloring, \( b \)-critical graphs, \( b \)-critical trees.

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1. INTRODUCTION

Let \( G \) be a simple graph with vertex-set \( V(G) \) and edge-set \( E(G) \). A coloring of the vertices of \( G \) is any mapping \( c : V(G) \to \mathbb{N} \). For every vertex \( v \) the integer \( c(v) \) is called the color of \( v \). A coloring is proper if any two adjacent vertices have different colors. The chromatic number \( \chi(G) \) of graph \( G \) is the smallest integer \( k \) such that \( G \) admits a proper coloring using \( k \) colors.

A \( b \)-coloring of \( G \) by \( k \) colors is a proper coloring of the vertices of \( G \) such that in each color class there exists a vertex that has neighbors in all the other \( k - 1 \) colors classes. We call any such vertex a \( b \)-vertex. The concept of \( b \)-coloring was introduced by Irving and Manlove [3,4]. The \( b \)-chromatic number \( b(G) \) of a graph \( G \) is the largest integer \( k \) such that \( G \) admits a \( b \)-coloring with \( k \) colors. It was proved in [3,4] that determining the \( b \)-chromatic number of a graph is an NP-complete problem.

A graph \( G \) is edge \( b \)-critical (resp. vertex \( b \)-critical) if the removal of any edge (resp. vertex) of \( G \) decreases the \( b \)-chromatic number. Ikhlef Eschouf [2] began the study of edge \( b \)-critical graphs. He characterized the edge \( b \)-critical \( P_4 \)-sparse graphs and edge \( b \)-critical quasi-line graphs. We propose here to study the effect of removing a vertex of a graph \( G \) on the \( b \)-chromatic number. From here on, “\( b \)-critical” will always mean vertex \( b \)-critical. We prove several properties of \( b \)-critical trees. In particular,
we show that if $T$ is a b-critical tree, then $\Delta(T) \leq b(T) \leq \Delta(T) + 1$, where $\Delta(T)$ is the maximum degree in $T$. Finally, we give a characterization of b-critical trees.

For notation and graph theory terminology we follow [1]. Consider a graph $G$. For any $A \subset V(G)$, let $G[A]$ denote the subgraph of $G$ induced by $A$, and let $G \setminus A$ be the subgraph induced by $V(G) \setminus A$. (If $x$ is a vertex, we may write $G \setminus x$ instead of $G \setminus \{x\}$). For any vertex $v$ of $G$, the neighborhood of $v$ is the set $N_G(v) = \{u \in V(G) \mid (u,v) \in E\}$ (or $N(v)$ if there is no confusion). Let $\omega(G)$ denote the size of a maximum clique of $G$. We let $P_k$ denote the path with $k$ vertices. A vertex of a path $P_k$ distinct from an end-vertex is said to be an internal vertex. The complete bipartite graph with classes of sizes $p$ and $q$ is denoted by $K_{p,q}$, and any graph $K_{1,q}$ is called a star.

A tree is a connected graph with no induced cycle. A rooted tree is a tree $T$ in which one vertex $x$ is distinguished and called the root. For every vertex $u$ of $T \setminus x$, the parent of $u$ is the neighbor of $u$ on the unique path from $u$ to $x$, while a child of $u$ is any other neighbor of $u$. A descendant of $u$ is defined (recursively) to be either any child of $u$ or any descendant of a child of $u$. We let $D(u)$ denote the set of descendants of $u$, and we write $D[u] = D(u) \cup \{u\}$. For any set $A$ let $D(A) = \bigcup_{u \in A} D(u)$. The subtree of $T$ induced by $D[u]$ is denoted by $T_u$. A vertex of degree one is called a leaf, and its neighbor is called a support vertex. If $v$ is a support vertex, then $L_v$ denotes the set of leaves adjacent to $v$.

2. PRELIMINARY RESULTS

We will use several definitions and results due to Irving and Manlove [3]. Remark that if a graph $G$ admits a $b$-coloring with $k$ colors, then $G$ has at least $k$ vertices of degree at least $k - 1$. Irving and Manlove define the $m$-degree $m(G)$ of $G$ to be the largest integer $\ell$ such that $G$ has at least $\ell$ vertices of degree at least $\ell - 1$. Thus every graph $G$ satisfies $b(G) \leq m(G)$. The difference $m(G) - b(G)$ can be arbitrarily large: for example, $m(K_{p,p}) = p + 1$ while $b(K_{p,p}) = 2$. Irving and Manlove [3] proved that $b(T)$ can be computed easily for every tree, as follows. A vertex $v$ is dense if $d_G(v) \geq m(G) - 1$.

**Definition 2.1** ([3]). A tree $T$ is pivoted if $T$ has exactly $m(T)$ dense vertices and $T$ contains a vertex $v$ such that $v$ is not dense and every dense vertex is adjacent either to $v$ or to a neighbor of $v$ of degree $m - 1$.

**Theorem 2.2** ([3]). Let $T$ be a tree. If $T$ is a pivoted tree, then $b(T) = m(T) - 1$; else, $b(T) = m(T)$.

Now we prove a few general facts about b-critical trees.

**Lemma 2.3.** Let $T$ be a b-critical tree and $c$ be a $b$-coloring of $T$ with $b(T)$ colors. Let $B$ be the set of all $b$-vertices of $c$. Then:

(i) Every vertex of $V(T) \setminus B$ has a neighbor in $B$.

(ii) If $z$ is a support vertex, then $z$ is in $B$ and is the only $b$-vertex of color $c(z)$.

Moreover, $z$ does not have two neighbors of the same color such that one of them is a leaf.
Proof. (i) If a vertex $u$ in $V(T) \setminus B$ has no neighbor in $B$, then $c$ remains a b-coloring of $T \setminus u$ with $b(T)$ colors.

(ii) If any part of (ii) does not hold, then the removal of some leaf adjacent to $z$ does not decrease the b-chromatic number.

**Theorem 2.4.** Let $T$ be a b-critical tree, and let $c$ be a b-coloring of $T$ with $b(T)$ colors. Let $B$ be the set of all b-vertices of $c$. Then:

(i) $c$ does not have two b-vertices of the same color, i.e., $|B| = b(T)$.

(ii) Every vertex $u$ in $V(T) \setminus B$ satisfies $d_T(u) \leq b(T) - 1$.

(iii) Every vertex $x$ in $B$ satisfies $b(T) - 1 \leq d_T(x) \leq b(T)$.

**Proof.** Let $k = b(T)$. If $k = 2$, it is immediate to see that $T = P_2$ and the theorem holds. So we may assume that $k \geq 3$.

(i) Suppose that $c$ has two b-vertices $x$ and $y$ of the same color. If $x$ or $y$ is a support vertex, adjacent to a leaf $z$, then $c$ remains a b-coloring with $k$ colors of $T \setminus z$, a contradiction. So $x$ and $y$ are not support vertices. Let us root $T$ at vertex $x$. Let $u_1, \ldots, u_h$ be the neighbors of $x$. Since $x$ is a b-vertex, we have $h \geq k - 1$. For each $i$ in $\{1, \ldots, h\}$, let $T_i$ be the component of $T \setminus x$ that contains $u_i$. Since $x$ is not a support vertex, $T_i$ contains a support vertex $z_i$ of $T$. Lemma 2.3 (ii) implies that $z_i$ is the only b-vertex of color $c(z_i)$ in $T$; in particular, $c(z_i) \neq c(x)$. Therefore $T$ contains at least $k - 1$ support b-vertices $z_1, \ldots, z_h$ of distinct colors. If the number of support vertices is more than $k - 1$, then two of them have the same color, which contradicts Lemma 2.3 (ii). So it must be that $h = k - 1$ and each $T_i$ contains exactly one support vertex. If any vertex $u$ of $V(T) \setminus \{x, z_1, \ldots, z_{k-1}\}$ has degree at least 3, then the subgraph $T_i$ that contains $u$ contains two support vertices of $T$, a contradiction. So $d_T(u) \leq 2$. In particular, $d_T(y) \leq 2$. This implies $k = 3$, $d_T(x) = d_T(y) = 2$, and by Lemma 2.3 (ii) we also have $d_T(z_1) = d_T(z_2) = 2$. Hence $T$ is a path. Since $T$ contains at least four b-vertices such that two of them $(x$ and $y$) are non-support vertices of the same color, it follows that $T$ is a path of at least 7 vertices. But this is not b-critical, a contradiction. Thus (i) holds.

By (i), we have $B = \{b_1, b_2, \ldots, b_k\}$, where $b_i$ is the unique b-vertex of $c$ of color $i$, for each $i$ in $\{1, \ldots, k\}$. By Lemma 2.3 (i), we have $V(T) = N[b_1] \cup \cdots \cup N[b_k]$.

(ii) Let $u$ be any vertex in $V(T) \setminus B$ and suppose that $d_T(u) \geq k$. Since $V(T) = N[b_1] \cup \cdots \cup N[b_k]$, we may assume that $u \in N(b_1)$. Vertex $u$ is adjacent to at most $k - 2$ b-vertices, for otherwise either $u$ is a b-vertex or there is no available color for $u$. Thus we may assume that $N(u) \cap B = \{b_1, \ldots, b_r\}$ with $1 \leq r \leq k - 2$. Since $T$ is a tree, $u$ has at most one neighbor in $N[b_i]$ for each $i$ in $\{1, \ldots, k\}$. Hence $d_T(u) = k$, vertex $u$ has a neighbor $u_j$ in $N(b_j)$ for every $j$ in $\{r + 1, \ldots, k\}$ and $u_{r+1}, \ldots, u_k$ are pairwise distinct. We may assume that $c(u) = r + 1$. For each $j$ in $\{r + 1, \ldots, k - 1\}$, let $v_j$ be a vertex of color $j + 1$ in $N(b_j)$ (possibly $v_j = u_j$). We define a coloring $\pi$ of $T$ with $k$ colors obtained from $c$ as follows. For each $j$ in $\{r + 1, \ldots, k - 1\}$, if $v_j \neq u_j$, then interchange the colors of $u_j$ and $v_j$. All other vertices keep their color. We obtain that $\pi$ is a b-coloring with $k$ colors such that $u$ and $b_{r+1}$ are b-vertices of the same color, which contradicts Theorem 2.4 (i) for $\pi$. 

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(iii) Let \( x \) be a b-vertex and \( p = d_T(x) \). Clearly, \( p \geq k - 1 \) since \( x \) is a b-vertex. Let \( T_1, T_2, \ldots, T_p \) be the components of \( T \setminus x \). Suppose that \( p \geq k + 1 \). Then \( N(x) \) contains a leaf, for otherwise Theorem 2.4 (i) and Lemma 2.3 (ii) imply that \( |B| \geq d_T(x) + |\{x\}| \geq k + 2 \), a contradiction. For each \( r \in \{1, \ldots, k\} \) let \( N^r(x) \) be the set of neighbors of \( x \) of color \( r \). Let \( u \) be a leaf adjacent to \( x \), and let \( \ell \) be the color of \( u \). Then \( |N^\ell(x)| = 1 \), for otherwise, \( c \) remains a b-coloring of \( T \setminus u \) with \( k \) colors, a contradiction. Since \( p \geq k + 1 \), there is a color \( t \neq \ell \) such that \( |N^t(x)| \geq 2 \). We distinguish among two cases.

\textbf{Case 1}. \( |N^t(x)| \geq 3 \). Let \( x_1, x_2, x_3 \) be three vertices in \( N^t(x) \). We may assume that \( x_i \in T_i \), for \( i = 1, 2, 3 \). Lemma 2.3 (ii) and Theorem 2.4 (i) imply that one of \( T_1, T_2, T_3 \), say \( T_1 \), does not contain any b-vertex of color \( t \) or \( \ell \). We recolor the vertices of \( V(T_1) \cup \{u\} \) by exchanging colors \( t \) and \( \ell \). We obtain a b-coloring where the color of \( u \) appears on another vertex of \( N(x) \). Hence, \( c \) remains a b-coloring of \( T \setminus u \) with \( k \) colors, a contradiction.

\textbf{Case 2}. For every \( r \) in \( \{1, \ldots, k\} \), \( |N^r(x)| \leq 2 \). Since \( d_T(x) \geq k + 1 \), there are two colors that appear exactly twice in \( N(x) \). Without loss of generality, we may suppose that \( x_1, x_2 \in N^t(x) \) and \( x_3, x_4 \in N^h(x) \), with \( h \neq t, \ell \). Also we may suppose that \( x_i \in T_i \) for each \( i \in \{1, 2, 3, 4\} \). By Theorem 2.4 (i) and the pigeonhole principle, there exists a component \( T_s \) with \( 1 \leq s \leq 4 \) that contains no b-vertex with color in \( \{t, \ell\} \) (or in \( \{h, \ell\} \)). Without loss of generality, we may suppose that \( T_1 \) contains no b-vertex colored \( t \) or \( \ell \). We recolor the vertices \( V(T_1) \cup \{u\} \) by exchanging colors \( t \) and \( \ell \) and obtain a contradiction as at the end of Case 1. This completes the proof of the theorem.

An immediate consequence of Theorem 2.4 is the following.

\textbf{Corollary 2.5.} \textit{If} \( T \) \textit{is a b-critical tree, then} \( \Delta(T) \leq b(T) \leq \Delta(T) + 1 \).

3. CHARACTERIZATION OF b-CRITICAL TREES

In this section, we give a characterization of b-critical trees. By Corollary 2.5, this amounts to characterizing the b-critical trees having a b-chromatic number equal to \( \Delta(T) \) or \( \Delta(T) + 1 \).

3.1. b-CRITICAL TREES WITH \( b(T) = \Delta(T) \)

In order to characterize the b-critical trees \( T \) with \( b(T) = \Delta(T) \), we define a family \( \mathcal{T}_1 \) as follows:

\textbf{Definition 3.1 (Class \( \mathcal{T}_1 \))}. A tree \( T \) is in \( \text{class } \mathcal{T}_1 \) if, and only if, for some integers \( k \) and \( p \) with \( k \geq 4 \) and \( 2 \leq p \leq k - 2 \), the vertex-set of \( T \) can be partitioned into four sets \( \{v\}, D_1, D_2, X \) with the following properties:

- \( |D_1| = p \), and every vertex of \( D_1 \) is adjacent to \( v \);
- \( |D_2| = k - p \), and every vertex of \( D_2 \) has a neighbor in \( D_1 \);
- Every vertex of \( X \) has a neighbor in \( D_1 \cup D_2 \);
There is a vertex \( w \in D_1 \) such that \( w \) has a neighbor in \( D_2 \), \( w \) has degree \( k \), and every vertex of \( D_1 \cup D_2 \setminus \{w\} \) has degree \( k - 1 \).

Note that there is no other edge than those mentioned in the definition, because \( T \) is a tree. The definition implies easily that \( |X| = k^2 - 3k + p + 1 \). So \( T \) has \( k^2 - 2k + p + 2 \) vertices. Also, \( \Delta(T) = k \), \( m(T) = k \), the dense vertices are the vertices in \( D_1 \cup D_2 \), and \( b(T) = k \).

Class \( T_1 \) may contain several non-isomorphic graphs with the same value of \( k \) and \( p \), depending on the adjacency between \( D_1 \) and \( D_2 \).

**Lemma 3.2.** If \( T \in T_1 \), then \( T \) is \( b \)-critical.

**Proof.** As observed above, we have \( b(T) = k \) and \( m(T) = k \). Let \( Y = D_1 \cup D_2 \setminus \{w\} \). Consider any vertex \( x \) of \( T \). If \( x \in N[Y] \cup \{w\} \), then \( b(T \setminus x) \leq m(T \setminus x) \leq m(T) - 1 = k - 1 \). If \( x \in N(w) \), then \( T \setminus x \) is a pivoted tree. By Theorem 2.2, \( b(T \setminus x) = m(T) - 1 = k - 1 \). Thus \( T \) is \( b \)-critical. \( \square \)

**Theorem 3.3.** Let \( T \) be a tree with \( b(T) = \Delta(T) \). Then \( T \) is \( b \)-critical if and only if \( T \in T_1 \).

**Proof.** If \( T \in T_1 \), then by Lemma 3.2, \( T \) is \( b \)-critical. Now let us prove the converse. Let \( T \) be a \( b \)-critical tree with \( b(T) = \Delta(T) \). Let \( k = b(T) \). Let \( c \) be a \( b \)-coloring of \( T \) with \( k \) colors and let \( B \) be the set of all \( b \)-vertices of \( c \). By Theorem 2.4, there is a unique \( b \)-vertex \( b_i \) of color \( i \) for each \( i \in \{1, \ldots, k\} \), and so \( B = \{b_1, \ldots, b_k\} \).

Pick a vertex \( x \) of maximum degree, and root \( T \) at \( x \). Let \( L_x \) be the set of leaves adjacent to \( x \), let \( B_x = B \cap N(x) \) and \( Y_x = N(x) \setminus (B_x \cup L_x) \). Put \( Y_x = \{y_1, \ldots, y_q\} \).

By Theorem 2.4, \( x \) is a \( b \)-vertex. Since \( d_T(x) = k \), there are two vertices of the same color in \( N(x) \). On the other hand, since \( x \) is a \( b \)-vertex of degree \( b(T) \), all neighbors of \( x \) except these two must have different colors. We call these two the repeating pair. By Lemma 2.3 (ii), these two vertices are not in \( L_x \), and by Theorem 2.4 (i), one of them is not in \( B \). So one of them is in \( Y_x \), and so \( q \geq 1 \).

For each \( i \in \{1, \ldots, q\} \), let \( T_i \) be the component of \( T \setminus x \) that contains \( y_i \), and let \( B_i = B \cap V(T_i) \). Let \( B'_i = B \cap D(B_i) \). The definition of \( L_x \) and \( Y_x \) implies that \( T_i \) contains a support vertex of \( T \), and Lemma 2.3 implies that such a vertex is a \( b \)-vertex. Hence \( |B_i| \geq 1 \) for all \( i \in \{1, \ldots, q\} \). So \( |B| \geq q + 1 + |B_x| \). If \( L_x = \emptyset \), this inequality implies \( |B| \geq d_T(x) + 1 \), a contradiction. Therefore we have \( L_x \neq \emptyset \). For each \( i \in \{1, \ldots, q\} \), let \( L_i = \{v \in L_x \mid b_{c(v)} \in B_i\} \) and \( L' = \{v \in L_x \mid b_{c(v)} \in B'_i\} \).

Note that for any vertex \( z \) in \( L_x \), the color \( c(z) \) does not appear in \( N(x) \setminus z \), by Lemma 2.3. So \( L_x = L_1 \cup \cdots \cup L_q \cup L' \).

We observe that the following fact holds:

Let \( b_i, b_j \in B \) and \( y \in N(x) \). Suppose that \( c(x) \neq i, j \) and \( b_i \) and \( b_j \) are either both in \( D(y) \) or not in \( D(y) \). Then interchanging colors \( i \) and \( j \) in \( G[D(y)] \) produces a \( b \)-coloring of \( T \) with \( k \) colors.

Indeed, after the interchange the coloring is proper (because \( c(x) \neq i, j \)), every \( b \)-vertex \( b_h \) with \( h \not\in i, j \) is still a \( b \)-vertex of color \( h \), and \( b_i \) and \( b_j \) are either unchanged or \( b \)-vertices of color \( j \) and \( i \) respectively. Thus (3.1) holds.

All colors that appear in \( Y_x \) are different. \( \square \)
Suppose on the contrary that two vertices $y_1, y_2$ in $Y_x$ have the same color $h$ (so they form the unique repeating pair). Up to symmetry, we may assume that $b_h \notin B_1$. Recall that $L_x \neq \emptyset$. Pick any vertex $z \in L_x$ and let $\ell = c(z)$. By Lemma 2.3, we have $\ell \neq h$.

If $b_\ell$ is not in $T_1$, then we interchange colors $h$ and $\ell$ in $T_1$. By (3.1), this produces a $b$-coloring $\pi$ of $T$ with $k$ colors such that $z$ is a leaf of a repeated color in $N(x)$. Hence, $\pi$ remains a $b$-coloring of $T \setminus z$ with $k$ colors, a contradiction. Therefore every color that appears in $L_x$ has its $b$-vertex in $T_1$, i.e., $L_x = L_1$, and $L_2 = \cdots = L_q = L' = \emptyset$.

Then $b_h \in T_2$, for otherwise we should also have $L_x = L_2$. The set $B_3 \cup \cdots \cup B_q$ (if $q \geq 3$) must contain $q - 2$ $b$-vertices (because $B_j \neq \emptyset$ for all $j \in \{1, \ldots, q\}$), and these must be the $b$-vertices whose colors are in $\{y_3, \ldots, y_q\}$ (because all other colors have their $b$-vertices in $B_2 \cup B_1 \cup \{b_h\}$). By the pigeonhole principle, we have $|B_i| = 1$ for all $i$ in $\{3, \ldots, q\}$ and also $|B_2| = 1$, i.e., $B_2 = \{b_h\}$. If $T_2$ has a vertex of degree at least 3 other than $b_h$, then there are at least two support vertices in $T_2$, and these are $b$-vertices, a contradiction. Therefore $T_2$ consists of a path between $y_2$ and $b_h$ plus leaves attached to $b_h$. It is easy to recolor the vertices of $T_2$ in such a way that the coloring is proper, $y_2$ gets color $\ell$ and $b_h$ remains a $b$-vertex of color $h$. This produces a $b$-coloring of $T$ with $k$ colors such that $z$ is a leaf of a repeated color in $N(x)$. Hence, $\pi$ remains a $b$-coloring of $T \setminus z$ with $k$ colors, a contradiction. Thus (3.2) holds.

Claim (3.2) implies that the repeating pair can be written as $\{y_1, b_h\}$ with $y_1 \in Y_x$ and $b_h \in B_x$. Moreover we claim that

$$Y_x = \{y_1\}. \tag{3.3}$$

Suppose on the contrary that $|Y_x| \geq 2$. Then $D(y_2)$ contains a support vertex of $T$, and by Lemma 2.3, such a vertex is a $b$-vertex $b_r$. Note that $r \neq t$ and $T_1$ does not contain a $b$-vertex of color $t$ or $r$. We interchange colors $t$ and $r$ in $G[T_1]$. By (3.1), this produces a $b$-coloring $\pi$ with $k$ colors. Vertex $x$ has a neighbor $x'$ of color $r$, and we have $x' \in L_x \cup Y_x$. Suppose that $x' \in Y_x$. Then $x' \neq y_1$, and $Y_x$ contains two vertices of color $r$ (in $\pi$), a contradiction to (3.2). So $x' \in L_x$. Then $x'$ is a vertex with a repeated color in $N(x)$. Thus $\pi$ remains a $b$-coloring of $T \setminus x'$ with $k$ colors, a contradiction. Thus (3.3) holds.

Every child of a vertex in $B_x$ is a leaf. \quad \tag{3.4}

Suppose the contrary. Then, for some vertex $\beta \in B_x$ the set $D(\beta)$ contains a support vertex of $T$, and by Lemma 2.3 such a vertex is a $b$-vertex $b_r$. Clearly $r \neq t$. Now $T_1$ contains no $b$-vertex colored $t$ or $r$. Since $x$ is a $b$-vertex, and by (3.3), $L_x$ contains a vertex $x'$ of color $r$. We interchange colors $t$ and $r$ in $T_1$. By (3.1), this produces a $b$-coloring $\pi$ of $T$ with $k$ colors such that $x'$ is a leaf of repeated color in $N(x)$. Hence, $\pi$ remains a $b$-coloring of $T \setminus x'$ with $k$ colors, a contradiction. Thus (3.4) holds.

Every child of $y_1$ is a $b$-vertex. \quad \tag{3.5}

Suppose that some child $u$ of $y_1$ is not a $b$-vertex. By Lemma 2.3 (i), $u$ is adjacent to a $b$-vertex $b_r$. Clearly, $r \neq t$ and $c(u) \neq r, t$. Since $x$ is a $b$-vertex, and by (3.3), $L_x$ contains a vertex $x'$ of color $r$. Note that $D(y_1) \setminus D[u]$ contains no $b$-vertex of
color \( r \) or \( t \). We interchange colors \( t \) and \( r \) in \( G[T_1 \setminus D[u]] \). By (3.1), this produces a b-coloring \( \pi \) of \( T \) with \( k \) colors such that \( x^r \) is vertex with a repeated color in \( N(x) \). Thus \( \pi \) remains a b-coloring of \( T \setminus x^r \) with \( k \) colors, a contradiction. So (3.5) holds.

\[
L_x = L_1. \tag{3.6}
\]

Pick any vertex \( z \in L_x \) and let \( \ell = c(z) \). Suppose that \( b_\ell \) is not in \( T_1 \). Recall that \( b_\ell \in B_x \). So \( b_t \) and \( b_\ell \) are not in \( T_1 \). We interchange colors \( t \) and \( \ell \) in \( T_1 \). By (3.1), this produces a b-coloring \( \pi \) of \( T \) with \( k \) colors such that \( z \) is a leaf of a repeated color in \( N(x) \). Hence, \( \pi \) remains a b-coloring of \( T \setminus z \) with \( k \) colors, a contradiction. Therefore every color that appears in \( L_x \) has its b-vertex in \( T_1 \). Thus (3.6) holds.

Note that the preceding claims imply that \( B = \{ x \} \cup B_x \cup B_1 \).

The distance between two vertices \( x \) and \( y \), denoted by \( \text{dist}(x, y) \), is the length of a shortest path from \( x \) to \( y \) in \( T \).

Every b-vertex \( b_r \in B_1 \) satisfies \( \text{dist}(y_1, b_r) \leq 2 \). \tag{3.7}

Suppose there exists a b-vertex \( b_r \in B_1 \) such that \( \text{dist}(y_1, b_r) \geq 3 \). Without loss of generality, we may suppose that \( \text{dist}(y_1, b_r) = \max \{ \text{dist}(y_1, v) \mid v \in B_1 \} \). This and Lemma 2.3 (ii) imply that \( b_r \) is a support vertex. Since \( x \) is a b-vertex, and by (3.3), \( L_x \) contains a vertex \( u \) of color \( r \). Let \( z_0 \) be the parent of \( b_r \) and \( z_1 \) be the parent of \( z_0 \). Note that \( z_0 \) is not adjacent to \( y_1 \) (in particular, \( z_1 \neq y_1 \)), for otherwise \( \text{dist}(y_1, b_r) < 3 \). Then there are two cases to consider.

Case 1. \( c(z_0) \neq t \). By Theorem 2.4 (i), \( D(y_1) \setminus D[z_0] \) contains no b-vertex of color \( r \) and \( t \). Thus, interchanging colors \( t \) and \( r \) in \( G[T_1 \setminus D[z_0]] \) produces a b-coloring \( \pi \) of \( T \) with \( k \) colors such that \( u \) is a vertex with a repeated color in \( N(x) \). Thus, \( \pi \) remains a b-coloring of \( T \setminus u \) with \( k \) colors, a contradiction.

Case 2. \( c(z_0) = t \). Then \( z_0 \) is not a b-vertex. Also, \( D(z_0) \) contains no b-vertices of colors \( c(x) \) and \( t \). If \( c(z_1) \neq c(x) \), then we interchange the color \( t \) and \( c(z_1) \) in \( D(z_0) \). Hence, \( c(z_0) = c(x) \). Therefore, an exchange of colors as described in Case 1 produces a new b-coloring \( \pi \) of \( T \) with \( k \) colors such that \( u \) is vertex with a repeated color in \( N(x) \). Thus, \( \pi \) remains a b-coloring of \( T \setminus u \) with \( k \) colors, a contradiction. If \( c(z_1) = c(x) \), then \( z_1 \) is not a b-vertex. In this case, we can interchange colors \( t \) and \( r \) in \( T_1 \setminus D[z_1] \). This is possible since \( T_1 \setminus D[z_1] \) contains no b-vertices of color \( t \) and \( r \). We obtain a b-coloring \( \pi \) of \( T \) with \( k \) colors such that \( u \) is a vertex with a repeated color in \( N(x) \). Thus, \( \pi \) remains a b-coloring of \( T \setminus u \) with \( k \) colors, a contradiction. So (3.7) holds.

Every vertex \( v \in B \setminus \{ x \} \) satisfies \( d_T(v) = \Delta(T) - 1 \). \tag{3.8}

Recall that \( B = \{ x \} \cup B_x \cup B_1 \). First suppose that \( v \in B_x \). By (3.4), every child of \( v \) is a leaf. By Lemma 2.3 (ii), \( d_T(v) = \Delta(T) - 1 \). Now suppose that \( v \in B_1 \). If \( \text{dist}(v, y_1) = 2 \), then all children of \( v \) are leaves, for otherwise, by Lemma 2.3 (ii), \( D(y_1) \) contains a b-vertex that lies at distance at least three from \( y_1 \), which contradicts (3.7).

By Lemma 2.3 (ii), we have \( d_T(v) = \Delta(T) - 1 \). If \( \text{dist}(v, y_1) = 1 \) (i.e., \( v \) is a child of \( y_1 \)), then by Theorem 2.4 (iii), we have \( d_T(v) \geq \Delta(T) - 1 \). If \( d_T(v) = \Delta(T) \), then \( v \) can
serve the same role as \(x\) with \(Y(v) = \{y_1\}\). Then the analogue of (3.3) implies that \(B_v\) contains a \(b\)-vertex of color \(t\) different from \(b_i\). This contradicts Theorem 2.4 (i). So \(d_T(v) = \Delta(T) - 1\). Thus (3.8) holds.

Claims (3.3)–(3.8) imply that \(T \in T_1\) (where \(y_1\) plays the role of \(v\) and \(x\) plays the role of \(w\)). This completes the proof of Theorem 3.3.

\[\square\]

3.2. \(b\)-CRITICAL TREES WITH \(b(T) = \Delta(T) + 1\)

Let \(k = \Delta(T) + 1\). For the purpose of characterizing \(b\)-critical trees with \(b(T) = k\), we define a family \(T_2\) of trees as follows. A tree \(T\) is in \(T_2\) if there is a sequence \(T_1, T_2, \ldots, T_k\) of trees, with \(T = T_k\), where \(T_1\) is a star of order \(k\), and, for each \(i\) in \(\{1, \ldots, k - 1\}\), \(T_{i+1}\) can be obtained from \(T_i\) by one of the operations listed below.

- Operation \(O_1\): Identify the center of a star of order \(k - 1\) with one leaf of a support vertex of degree \(k - 1\) of \(T_i\).
- Operation \(O_2\): Attach a star of order \(k - 1\) of center \(x\) by joining \(x\) to any vertex of \(T_i\) such that \(1 \leq d_{T_i}(u) \leq k - 3\).
- Operation \(O_3\): Attach a star of order \(k\) by joining one of its leaves to any vertex of \(T_i\) such that \(1 \leq d_{T_i}(u) \leq k - 3\).

Let \(P\) be the class of pivoted trees.

\[\text{Lemma 3.4. If } T \in P, \text{ then } T \text{ is not } b\text{-critical.}\]

\[\text{Proof. Since } T \text{ is pivoted, we have } b(T) = m(T) - 1. \text{ Let } z \text{ be any leaf of } T. \text{ Then it is easy to see that } B(T \setminus z) = b(T). \text{ So } T \text{ is not } b\text{-critical.}\]

\[\text{Lemma 3.5. If } T \in T_2 \setminus P, \text{ then } T \text{ is } b\text{-critical with } b(T) = \Delta(T) + 1.\]

\[\text{Proof. Since } T \text{ is in } T_2, \text{ it is easy to check that } \Delta(T) = m(T) - 1. \text{ Since } T \text{ is not pivoted, we have } b(T) = m(T). \text{ Hence, } b(T) = \Delta(T) + 1. \text{ Consider any vertex } x \text{ in } T. \text{ By the definition of } T_2, \text{ we have } m(T \setminus w) \leq m(T) - 1, \text{ and consequently } b(T \setminus w) \leq \Delta(T). \text{ Thus, } T \text{ is } b\text{-critical.}\]

\[\text{Theorem 3.6. Let } T \text{ be a tree with } b(T) = \Delta(T) + 1. \text{ Then } T \text{ is } b\text{-critical if and only if } T \in T_2 \setminus P.\]

\[\text{Proof. Let } k = \Delta(T) + 1. \text{ Lemma 3.5 implies the sufficiency. To prove the necessity, let } T \text{ be a } b\text{-critical tree with } b(T) = k. \text{ We first show that } T \text{ belongs to } T_2. \text{ Since } b(T) = k = \Delta(T) + 1, \text{ Theorem 2.4 implies that } T \text{ has a unique } b\text{-vertex } b_i \text{ of color } i, \text{ for each } i \in \{1, \ldots, k\}, \text{ and that } d_T(b_i) = k - 1. \text{ Let } B = \{b_1, b_2, \ldots, b_k\}. \text{ For each } i, \text{ let } S_i = T[N[b_i]]. \text{ Then } S_i \text{ is a star of order } k. \text{ Root } T \text{ at } b_1 \text{ and assume without loss of generality that } \text{dist}(b_1, b_2) \leq \text{dist}(b_1, b_3) \leq \cdots \leq \text{dist}(b_1, b_k). \text{ Let } T_1 = S_1. \text{ For } i = 2, \ldots, k, \text{ let } T_i \text{ be the subgraph of } T \text{ induced by } V(S_1) \cup \cdots \cup V(S_i). \text{ Assume that } i \leq k - 1. \text{ Let } r \in \{1, \ldots, i\} \text{ be such that } \text{dist}(b_r, b_{i+1}) = \min\{\text{dist}(b_s, b_{i+1}) \mid 1 \leq s \leq i\}. \text{ Since } T \text{ is a tree, there is a unique path } P \text{ connecting } b_r \text{ to } b_{i+1}. \text{ The choice of } b_r \text{ implies that any internal vertex of } P \text{ is not a } b\text{-vertex.}\]
Suppose that the length of $P$ is at least 4. Let $u$ be a vertex of $P$ that is not adjacent to $b_r$ or $b_{i+1}$. The choice of $b_r$ and $b_{i+1}$ implies that $u$ has no neighbor in $B$, so $b(T \setminus u) \geq b(T)$, a contradiction.

Now suppose that $P$ has length 3. Let $P = b_r - u - v - b_{i+1}$. Then $u$ and $v$ are not $b$-vertices and we have $u \in V(T_i)$ and $v \notin V(T_i)$. Thus $T_{i+1}$ is obtained from $T_i$ by the third operation applied with star $S_{i+1}$.

Now suppose that $P$ has length 2. Let $P = b_r - u - b_{i+1}$. Then $u$ is not a $b$-vertex, and $u \in V(T_i)$. Thus $T_{i+1}$ is obtained from $T_i$ by the second operation applied to star $S_{i+1} \setminus \{u\}$.

Finally, suppose that $P$ has length 1, that is, $b_r$ is adjacent to $b_{i+1}$. Then $T_{i+1}$ is obtained from $T_i$ by the first operation applied to star $S_{i+1} \setminus \{b_r\}$.

At the end of the procedure, we have $T = T_k$, so $T$ is obtained after $k - 1$ steps by one of the three operations $O_1, O_2$ or $O_3$, from a star of order $k$. Thus $T \in T_2$. By Lemma 3.4, $T \notin P$. This completes the proof.

We can now summarize our results as follows.

**Theorem 3.7.** A tree is $b$-critical if and only if it belongs to $T_1 \cup T_2 \setminus P$.

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