A NOTE
ON BLASIUS TYPE BOUNDARY VALUE PROBLEMS

Grzegorz Andrzejczak, Magdalena Nockowska-Rosiak, and Bogdan Przeradzki

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Abstract. The existence and uniqueness of a solution to a generalized Blasius equation with asymptotic boundary conditions is proved. A new numerical approximation method is proposed.

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1. INTRODUCTION

We study the BVP of the form:

\[ x''' + cx^p \cdot x'' = 0, \quad x(0) = 0 = x'(0), \quad \lim_{t \to \infty} x'(t) = \beta, \tag{1.1} \]

where \( p \geq 1, c \) and \( \beta \) are positive constants. The problem is motivated by the classical Blasius equation describing the velocity profile of a fluid in a boundary layer where \( c = \frac{1}{2}, p = \beta = 1 \). The Blasius equation is a basic equation in fluid mechanics which appears in the study of the flow of an incompressible viscous fluid over a semi infinite plane. Blasius ([2]) used a similarity transform technique to convert the partial differential equation into his famous ordinary differential equation

\[ x''' + \frac{1}{2}x \cdot x'' = 0, \quad x(0) = 0 = x'(0), \quad \lim_{t \to \infty} x'(t) = 1, \tag{1.2} \]

where \( x \) is the stream function \( x = \frac{\psi}{\sqrt{2U\nu y_1}} \), \( U \) is the fluid velocity, \( \nu \) is the fluid kinematic viscosity and \( t \) is the similarity variable defined as \( t = y_2 \sqrt{\frac{U}{2\nu y_1}} \), where \( y_1, y_2 \) are Cartesian coordinates with \( y_1 \) pointing along the free stream direction and \( y_2 \) perpendicular to \( y_1 \). We refer to [3,4] for an excellent introduction to the problem.
A series expansions method was used to solve (1.2) by Blasius. There has been many analytical and numerical methods handling this problem since Blasius’s work, [1, 6, 8, 9, 11–15] for instance.

The existence and the uniqueness for the Blasius problem was proved by Weyl in 1942 ([14]) by using an elegant method based on the symmetry of the equation (1.2). If \( \eta \to w(\eta) \) is a solution of (1.2) then \( \eta \to kw(k\eta) \) is as well for any \( k \). It enabled him to work with the initial condition

\[
x(0) = x'(0) = 0, \quad x''(0) = 1.
\]

This problem where the second-order derivative is given at 0 is equivalent to the fixed point problem with an operator possessing some monotonicity property. Weyl solved it as a limit of odd and even successive iterates. For our problem this method cannot work so simply and we shall consider the third initial condition \( x''(0) = a \) and the dependence of the solution with respect to \( a > 0 \). Moreover, our method gives estimates for errors. It enables us to propose a combined numerical method to solve the problem. For a given \( \varepsilon > 0 \), we can find an interval \( [0, T] \) such that outside of it the derivative of a solution cannot change more than \( \varepsilon \).

In the first part of this paper, the existence and uniqueness of (1.1) will be analytically proved by changing the boundary value problem to an initial problem. Using the obtained estimates we will be able to find the value of \( a = x''(0) \) which guarantees that the solution \( x_a \) on an initial problem

\[
x''' + cx' x'' = 0, \quad x(0) = 0 = x'(0), \quad x''(0) = a, \quad (1.3)
\]

is the solution of (1.1) we are looking for.

In the second part of the article, we shall propose a numerical approximation method and some examples that compare it with many others in the literature.

2. AUXILIARY LEMMAS

Let \( x_a \) stand for the unique solution of \( x''' + cx' x'' = 0 \) satisfying initial conditions

\[
x(0) = 0 = x'(0), \quad x''(0) = a.
\]

If \( p \in \mathbb{N} \) and \( a < 0 \), then \( x_a \) and \( x_a'' \) are negative for small \( t \)’s thus the solution is concave and negative for all arguments and it cannot solve (1.1). Using the analogous argument about negative values of \( x_a'' \) we get that the negative value of \( a \) for \( p > 1 \), \( p \notin \mathbb{N} \) contradicts the assumption of nonnegative \( x_a \). For \( a = 0 \) we have a trivial solution \( x_a \equiv 0 \) and the seeking solution can be obtained for \( a > 0 \).

**Lemma 2.1.** \( x_a \) is defined for all \( t \geq 0 \).

**Proof.** Notice that \( x_a \) is an analytic function. Moreover, \( x_a'' \) cannot vanish at any point. If there exists \( t_0 > 0 \) such that \( x_a''(t_0) = 0 \) then from the equation \( x''' + cx' x'' = 0 \) we get that \( x_a^{(k)}(t_0) = 0 \) for \( k \geq 2 \). So we get that the solution of the ODE is the form
\[
x_a(t) = c_1 t + c_2 \quad \text{which contradicts the initial conditions } x_a(0) = x'_a(0) = 0, \ x''_a(0) = a.
\]
Hence dividing the equation by \(x''_a\) and integrating on \([0, t]\) we have

\[
x''_a(t) = a \exp \left( -c \int_0^t x_a(s)^p \, ds \right), \quad (2.1)
\]

which implies

\[
x'_a(t) = a \int_0^t \exp \left( -c \int_0^s x_a(\tau)^p \, d\tau \right) \, ds. \quad (2.2)
\]

Integrating once more and applying the Fubini Theorem we get

\[
x_a(t) = a \int_0^t (t-s) \exp \left( -c \int_0^s x_a(\tau)^p \, d\tau \right) \, ds. \quad (2.3)
\]

By (2.1), (2.2) and (2.3) we have apriori estimates:

\[
0 < x_a(t) < \frac{1}{2} a t^2, \quad 0 < x'_a(t) < a t, \quad 0 < x''_a(t) < a
\]

for any \(t > 0\). It follows ([10, p. 146]) that \(x_a\) is extendable to \([0, \infty)\).

**Lemma 2.2.** For any \(a > 0\), \(\lim_{t \to \infty} x''_a(t) = 0\) and there exists a finite and positive limit

\[
h(a) := \lim_{t \to \infty} x'_a(t). \quad (2.4)
\]

**Proof.** Since \(x''_a > 0\) and \(x'_a > 0\), then \(\lim_{t \to \infty} x_a(t) = +\infty\). Moreover, since \(x''_a > 0\), \(x_a > 0\) and \(x'''_a = -c x''_a x'_a\), then \(x'''_a\) is a decreasing function so \(\lim_{t \to \infty} x'''_a(t) = g_a \in [0, a]\). Suppose \(g_a > 0\). From \(x'''_a = -c x''_a x'_a\) we get that

\[
\lim_{t \to +\infty} x'''_a(t) = -\infty. \quad (2.5)
\]

On the other hand

\[
\forall t \geq 0 \ \exists s_t \in (t, t+1) \quad x''_a(t+1) - x''_a(t) = x'''_a(s_t)
\]

and hence \(\lim_{t \to +\infty} x'''_a(t) = 0\), which contradicts (2.5).

Since \(x'''_a > 0\), then \(x'_a\) is an increasing function (so the limit defining \(h(a)\) exists, possibly infinite). From \(\lim_{t \to \infty} x_a(t) = +\infty\) we get there exists \(t_a > 0\) such that \(c x_a(t)^p > 1\) for \(t > t_a\). From (2.2), we obtain for \(t > t_a\),

\[
x'_a(t) \leq a \int_0^t \exp \left( -c \int_0^s x_a(\tau)^p \, d\tau \right) \exp \left( -\int_{t_a}^s d\tau \right) \, ds \leq
\]
\[ \int_0^t ds + a \int_{t_a}^t e^{-(s-t_a)} ds \leq at_a + a e^{t_a} \int_{t_a}^\infty e^{-s} ds = a(t_a + 1). \]

Thus
\[ h(a) \leq a(t_a + 1) \quad (2.6) \]
for any \( a > 0 \).

**Lemma 2.3.** For any \( a > 0 \), there exists a finite and positive limit
\[ \mu(a) := \lim_{t \to \infty} (h(a)t - x_a(t)), \quad (2.7) \]
where the constant \( h(a) \) is defined by (2.4). It means that the graph of \( x_a \) has a slant asymptote and the following estimates hold:
\[ \max(0, h(a)t - \mu(a)) \leq x_a(t) \leq h(a)t. \quad (2.8) \]

**Proof.** The function \( t \mapsto h(a)t - x_a(t) \) is increasing, hence the limit from the assertion exists but it can be infinite. Suppose it equals \( +\infty \). By the arguments from the proof of Lemma 2.2, we have \( x_a''(t) \leq -x_a'(t) \) for \( t \geq t_a \). Integrating this inequality from \( s \) to \( +\infty \) and using the fact \( x_a'(+\infty) = 0 \), we get
\[ -x_a''(s) \leq -h(a) + x_a'(s). \]

Next integration from \( t_a \) to \( t \) leads to the following inequality
\[ x_a'(t_a) - x_a'(t) \leq -h(a)(t - t_a) + x_a(t) - x_a(t_a) \]
or equivalently
\[ h(a)t - x_a(t) \leq h(a)t_a - x_a(t_a) + h(a) - x_a'(t_a). \]

Thus
\[ 0 < \mu(a) \leq h(a)t_a - x_a(t_a) + h(a) - x_a'(t_a). \quad (2.9) \]
The last part of the assertion is a simple consequence. \[ \square \]

For an upper bound on \( h(a) \), \( \mu(a) \) depending explicitly on \( a \) we use \( x_a(t) \leq at^2/2 \) to (2.3). Hence,
\[ x_a(t) \geq a \int_0^t (t-s) \exp \left( -c a^p \frac{\tau^{2p} d\tau}{2p} d\tau \right) ds = a \int_0^t (t-s) \exp \left( -c a^p \frac{\tau^{2p} s^{2p+1}}{(2p+1)2p} \right) ds. \]

One can easily show that the function
\[ \varphi(t) := \int_0^t (t-s) \exp (-k s^\alpha) ds, \quad k = \frac{ca^p}{2p(2p+1)}, \quad \alpha = 2p + 1 \]
has a similar behaviour as $x_a$ in the sense that its graph has an asymptote $x = h^*t - \mu^*$, where

$$h^* = \int_0^\infty \exp(-ks^\alpha) \, ds = \frac{\Gamma(1/\alpha)}{ak^{1/\alpha}}, \quad \mu^* = \int_0^\infty s \exp(-ks^\alpha) \, ds = \frac{\Gamma(2/\alpha)}{ak^{2/\alpha}}$$

(2.10)

and its graph sits above this line. Hence,

$$x_a(\tau) \geq a^\frac{p+1}{2p+1} \cdot \tau - a^\frac{1}{2p+1} \cdot \frac{\Gamma(1/(p+1))}{c^{1/(p+1)} \cdot (c^p/2)^{(2/p) \cdot (2p+1)}} \cdot \tau - a^\frac{1}{2p+1} \cdot \frac{\Gamma(2/(2p+1))}{(2p+1) \cdot (2p+1) \cdot \tau \cdot (2p+1)}.$$  

(2.11)

where

$$c_2 := \Gamma\left(\frac{1}{2p+1}\right) \left(\frac{2p}{c(2p+1)^{2p}}\right)^{1/2p+1}, \quad c_3 := \Gamma\left(\frac{2}{2p+1}\right) (2p+1)^{1/2p+1} \cdot \left(\frac{2p}{c}\right)^{2p+1}. \quad (2.12)$$

Now, we are able to get appropriate estimates for $h(a)$.

**Lemma 2.4.** For any $a > 0$,

$$c_2 \cdot a^{(p+1)/(2p+1)} \leq h(a) \leq c_1 \cdot a^{(p+1)/(2p+1)},$$  

(2.13)

where

$$c_1 := \frac{c_3}{c_2} + \frac{\Gamma(1/(p+1))}{c^{1/(p+1)} \cdot (c^p/2)^{(p/(p+1))}}.$$  

**Proof.** For a lower bound we apply the estimate $x_a(\tau) \leq \frac{1}{2}a\tau^2$ to the equality

$$h(a) = a \int_0^\infty \exp\left(-c \int_0^s x_a(\tau)^p \, d\tau\right) \, ds. \quad (2.14)$$

This leads to the inequality

$$h(a) \geq a \int_0^\infty \exp\left(-\frac{ca^p}{2p(2p+1)^{2p+1}} \cdot s^{2p+1}\right) \, ds.$$

(2.10) for $k = \frac{ca^p}{2p(2p+1)}$, $\alpha = 2p + 1$ gives the lower bound on $h$. 


For an upper bound on $h$ we use the lower estimate of $x_a - (2.11)$ to the equality (2.14) and we get

$$h(a) \leq a \int_0^{c_3/c_2 a^{-p/(2p+1)}} ds + a \int_{c_3/c_2 a^{-p/(2p+1)}}^{\infty} \exp \left( -c \int_{c_3/c_2 a^{-p/(2p+1)}}^{s} \left( c_2 a^{p+1} \tau - c_3 a^{1/(2p+1)} \right)^p d\tau \right) ds \leq$$

$$\leq \frac{c_3}{c_2} a^{p+1} + \frac{1}{c_2} a^{p+1} \int_0^{\infty} \exp \left( -\frac{c}{(p+1)c_2} a^{-\frac{p+1}{2p+1}} \right) dt,$$

where we used linear substitutions twice. At last, using (2.10) for $k = \frac{c}{(p+1)c_2} a^{-\frac{p+1}{2p+1}}$, $\alpha = p + 1$ we have for any $a > 0$,

$$h(a) \leq \left( \frac{c_3}{c_2} + \frac{\Gamma(1/(p+1))}{c^{1/(p+1)} \cdot (c_2(p+1))^{p/(p+1)}} \right) a^{p+1}.
$$

The next lemma presents estimates for $\mu(a)$.

**Lemma 2.5.** For any $a > 0$, the constant $\mu(a)$, which is defined by (2.7), satisfies the following estimates:

$$c_4 \cdot a^{1/(2p+1)} \leq \mu(a) \leq c_5 \cdot a^{1/(2p+1)},$$

where

$$c_4 := \frac{2^{2p/(2p+1)} \Gamma(2/(2p+1))}{(2p+1)^{(2p-1)/(2p+1)} c_2^{2/(2p+1)}},$$

$$c_5 := \frac{1}{c^2} \left[ \frac{1}{2} + \left( \frac{2}{2p+1} \right)^{(2p+1)/(p+1)} \left( \Gamma\left( \frac{p}{p+1} \right) \Gamma\left( \frac{2}{p+1} \right) + c_3 \left( \frac{c_2}{(p+1)c_2} \right)^{1/(p+1)} \right) \right].$$

**Proof.** From (2.3) and (2.14) we get

$$\mu(a) = a \int_0^{\infty} s \exp \left( -c \int_0^s x_a(\tau)^p d\tau \right) ds. \tag{2.15}$$

Using the estimate $x_a(\tau) \leq \frac{1}{2} a \tau^2$ we get

$$\mu(a) \geq a \int_0^{\infty} s \cdot \exp \left( -\frac{ca^p}{2^p(2p+1)} s^{2p+1} \right) ds.$$
Using (2.10) for \( k = \frac{ca^p}{2p(2p+1)} \), \( \alpha = 2p + 1 \) we obtain the lower bound. On the other hand, from (2.11), (2.15) we have

\[
\mu(a) \leq a \int_0^\infty s \, ds + a \int_{c_3/c_2}^\infty s \exp \left( -c \int_{c_3/c_2}^s \left( c_2 a^{-\frac{1}{2p+1}} \tau - c_3 a^{-\frac{1}{2p+1}} \right)^p d\tau \right) \, ds \leq \frac{c_3^2}{2c_2^2} a^{-\frac{1}{2p+1}} + \frac{1}{c_2} a^{-\frac{1}{2p+1}} \int_0^\infty t \exp \left( -\frac{c}{(p+1)c_2} a^{-\frac{p+1}{2p+1}} t^{p+1} \right) dt + \frac{c_3}{c_2} \int_0^\infty dt
\]

where we used linear substitutions twice. At last, using (2.10) for \( k = \frac{c}{(p+1)c_2} a^{-\frac{p+1}{2p+1}} \), \( \alpha = p + 1 \) we have for any \( a > 0 \) the upper bound.

3. MAIN RESULTS

Now, we are able to prove the existence of a solution to (1.1).

**Theorem 3.1.** Let \( p \geq 1 \), \( c > 0 \). The BVP

\[ x''' + cx^p \cdot x' = 0, \quad x(0) = 0 = x'(0), \quad \lim_{t \to \infty} x'(t) = \beta, \]

has a solution for any \( \beta \geq 0 \).

**Proof.** The function \( h : [0, \infty) \to \mathbb{R} \) is continuous on \((0, +\infty)\) by the continuous dependence of solutions of ODEs on initial conditions and locally uniform convergence of the integral

\[
\int_0^\infty \exp \left( -c \int_0^s x_a(\tau)^p d\tau \right) \, ds.
\]

By the estimates from Lemma 2.4, we have

\[
\lim_{a \to 0^+} h(a) = 0, \quad \lim_{a \to \infty} h(a) = +\infty.
\]

Thus, for any \( \beta > 0 \), there exists \( a > 0 \) such that \( h(a) = \beta \). For \( \beta = 0 \), it is obvious. \( \square \)

Finally, the uniqueness of the solution of (1.1) will be proved by using ideas from [5]. For any \( a > 0 \) consider the one-to-one function \( v_a : [0, h(a)^2] \to [0, \infty) \)
increasing functions and it belongs to $C^2(0, h(a)^2)$. Substituting $y = x'(t)^2$, we shall find an ODE satisfied by $v$ (we omit subscript $a$ for simplicity).

$$x(t) = v(y), \quad x'(t) = v'(y)2x'(t)x''(t)$$

hence,

$$x''(t) = \frac{1}{2v'(y)}, \quad x'''(t) = -\frac{v''(y)}{2v'(y)^2} \cdot 2x'(t)x''(t) = -\frac{v''(y)\sqrt{y}}{2v'(y)^3}.\] Put $x$ and its derivatives in our ODE and find

$$v''(y) = \frac{cv(y)^p v'(y)^2}{\sqrt{y}}.\] (3.1)

From boundary conditions on $x$ we get

$$v(0) = 0, \quad v'(0) = \frac{1}{2a}, \quad \lim_{y \to h(a)^2-} v(y) = +\infty. \] (3.2)

Now, we are in a position to prove the following theorem.

**Theorem 3.2.** Let $p \geq 1, c > 0$. For any $\beta \geq 0$ the solution of

$$x'''' + cx^p \cdot x''' = 0, \quad x(0) = x'(0), \quad \lim_{t \to \infty} x'(t) = \beta,$$

is unique.

**Proof.** We need to show that the function $h$ is one-to-one. Suppose that $h(a_1) = h(a_2)$, $a_2 > a_1$ and take $v_1$ and $v_2$ obtained by $x_{a_1}$ and $x_{a_2}$, respectively, that is $v_i$ satisfies (3.1) with boundary conditions (3.2) (for $a = a_i, i = 1, 2$). Put $w = v_1 - v_2$. Then $w(0) = 0, w'(0) = \frac{a_2 - a_1}{2a_1 a_2} > 0$. Notice that $w' > 0$ on the whole interval $(0, h(a_1)^2)$

both functions are defined on the same interval.

In fact, if it is not true, then there exists $s$ in this interval such that $w' > 0$ on $(0, s)$ and $w'(s) = 0$. Hence $w'(s) > w'(0) = 0$

$$w''(s) = \lim_{\xi \to 0^+} \frac{w'(s - \xi) - w'(s)}{-\xi} \leq 0.$$

On the other hand,

$$w''(s) = v_1''(s) - v_2''(s) = cs^{-1/2}(v_1(s)^p - v_2(s)^p)v_a'(s)^2,$$

since $w'(s) = 0$ implies $v_1'(s) = v_2'(s)$. But $v_1(s)^p > v_2(s)^p$ from $w(s) > 0$ and this gives $w''(s) > 0 - a$ contradiction. Thus, we have $w > 0$ and $w' > 0$ on $(0, h(a_1)^2)$.

Set $V_i = 1/v_i, i = 1, 2$ and $W = V_1 - V_2$. We have, for any $y \in (0, h(a_1)^2)$,

$$W'(y) = V_1'(y) - V_2'(y) = -\frac{v_1''(y)}{v_1'(y)^2} + \frac{v_2''(y)}{v_2'(y)^2} = c\frac{v_2(y)^p - v_1(y)^p}{\sqrt{y}} < 0$$
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from \( w(y) > 0 \). Hence,

\[
W(y) < W(0) = \frac{1}{v_1'(0)} - \frac{1}{v_2'(0)} = 2(a_1 - a_2)
\]

and

\[
\lim_{y \to h(a_1)^2} W(y) \leq 2(a_1 - a_2) < 0.
\]

On the other hand, \( V_i(x_{a_i}'(t)^2) = 2x_{a_i}''(t) \) implies

\[
\lim_{y \to h(a_1)^2} V_i(y) = \lim_{t \to \infty} V_i(x_{a_i}'(t)^2) = 2 \lim_{t \to \infty} x_{a_i}''(t) = 0
\]

and, therefore,

\[
\lim_{y \to h(a_1)^2} W(y) = 0,
\]

which contradicts the previous inequality. \( \square \)

4. NUMERICAL APPROACH

All numerical methods cannot work on the infinite interval \([0, \infty)\) and we do not know the exact value of \( a = x'''(0) \) for the solution. Our earlier results make it possible to find a finite interval \([0, T]\) for any positive value \( \epsilon \) of the error control tolerance such that

\[
x''(T) < \epsilon, \quad h(a) - x'(T) < \epsilon, \quad x(T) - (h(a)T - \mu(a)) < \epsilon.
\]

Since all these functions decrease, all three inequalities hold for any \( t > T \). First, by using estimates (2.13), we can find an interval \([a_{\text{min}}, a_{\text{max}}]\) such that \( h(a_{\text{min}}) < \beta < h(a_{\text{max}}) \). Next, by (2.1), we need

\[
a_{\text{max}} \exp \left( -c \int_0^T x_{a_{\text{min}}}''(\tau)^p \, d\tau \right) < \epsilon,
\]

by (2.14), we should have

\[
a_{\text{max}} \int_0^\infty \exp \left( -c \int_0^s x_{a_{\text{min}}}''(\tau)^p \, d\tau \right) \, ds < \epsilon,
\]

and by (2.3) and (2.15), we get

\[
-a_{\text{max}} \int_T^\infty (T - s) \exp \left( -c \int_0^s x_{a_{\text{min}}}''(\tau)^p \, d\tau \right) \, ds < \epsilon.
\]
We do not know the function \( x_{a_{\min}} \) but we can use estimate (2.11) to get

\[
a_{\max} \exp \left( - c \int_0^T \left( c_2 a_{\min}^{\frac{p+1}{p+2}} \tau - c_3 a_{\min}^{1\over p+1} \right)^p d\tau \right) < \epsilon, \quad (4.1)
\]

\[
a_{\max} \int_0^T \exp \left( - c \int_0^s \left( c_2 a_{\min}^{\frac{p+1}{p+2}} \tau - c_3 a_{\min}^{1\over p+1} \right)^p d\tau \right) ds < \epsilon, \quad (4.2)
\]

\[
-a_{\max} \int_0^T (T - s) \exp \left( - c \int_0^s \left( c_2 a_{\min}^{\frac{p+1}{p+2}} \tau - c_3 a_{\min}^{1\over p+1} \right)^p d\tau \right) ds < \epsilon. \quad (4.3)
\]

We start with a family of initial value problems

\[
x_a(t), t \in [0, T], x_a(0) = 0 = x'_a(0), x''_a(0) = a, a \in [a_{\min}, a_{\max}], \quad (4.4)
\]

If we approximate this solution in \([0, T]\) with an error less than \( \epsilon \), then the best approximation of \( x \) in \([T, \infty)\) is

\[
x(t) = \beta t + (x_a(T) - \beta T).
\]

The lower and upper bounds for the second derivative describe the shooting window - for each \( a \) the only solution in this direction exists at \( t = T \), and the computed value of \( x'(T) \) is more and more close to the expected limit \( \beta \). As long as \( \beta \) is contained between the computed values \( x'(T) \) of the best two shots, we apply the classical bisection method:

If \( y \) and \( z \) are solutions such that \( y''(0) < z''(0) \), and there is \( y'(T) < \beta < z'(T) \), then the next problem to solve is (4.4) with \( a = (y''(0) + z''(0))/2 \).

For any value \( \epsilon > 0 \) we determine \( T \) by solving the inequality (4.1). After that we check if inequalities (4.2), (4.3) hold for the obtained \( T \). Otherwise, we increase \( T \) to the moment where (4.2), (4.3) hold.

4.1. EXAMPLES

While solving the initial value problems we apply an adaptive Runge-Kutta-Fehlberg method RK45 [7], in which a tolerance parameter \( \epsilon \) controls the local error of the method. The values of \( \epsilon \) range from \( 10^{-8} \) to \( 10^{-14} \). For representing real values we use the standard 16–17-digits double data type.

Numerical results for the classical Blasius equation \( p = 1, c = 1/2, \) and \( \beta = 1 \).

Here \( a_{\min} = 0.2694860459, a_{\max} = 0.3420953216 \). For \( T = 14 \) we get the all three inequalities (4.1), (4.2) and (4.3) for \( \epsilon = 1.0e - 14 \). Below \( N \) stands for the number of steps in RK45 (average) (Tab. 1).
The last value of \( a \) differs in the two last digits from the one cited in [3]:

\[
a = 0.332 057 336 215 196 30.
\]

The last two columns of Table 1 have been computed for \( \epsilon = 10^{-14} \). As there is \( x''(T) = 7.68e - 13 \), for \( t > T \) the straight line approximation of the solution is the most effective.

**Numerical results for the equation with \( p = 7 \), \( c = 1/2 \) and \( \beta = 1 \).**

Here, \( a_{\text{min}} = 0.373 397 838 8 \), \( a_{\text{max}} = 0.380 548 242 7 \). As above for the tolerance \( \epsilon = 1.0e - 14 \), the interval \([0, 4] \) is sufficiently large and we get the following results by the RK45 method (Tab. 2).

**Table 2.**

| \( \epsilon \) | \( N \) | \( a \) | \( |x'(0) - a| \) | \( |1 - x'(T)| \) | \( x(T) \) |
|---|---|---|---|---|---|
| 1.0e − 08 | 191 | 0.379 398 164 451 122 | 2.47e − 08 | 3.5e − 08 | 2.673 055 448 977 |
| 1.0e − 09 | 316 | 0.379 398 187 063 634 | 2.04e − 09 | 2.9e − 09 | 2.673 055 570 853 |
| 1.0e − 10 | 549 | 0.379 398 189 005 442 | 1.03e − 10 | 1.4e − 10 | 2.673 055 581 319 |
| 1.0e − 11 | 961 | 0.379 398 189 038 642 | 2.20e − 11 | 3.1e − 11 | 2.673 055 581 757 |
| 1.0e − 12 | 1688 | 0.379 398 189 106 960 | 1.69e − 12 | 2.3e − 12 | 2.673 055 581 866 |
| 1.0e − 13 | 2827 | 0.379 398 189 108 438 | 1.62e − 13 | 1.9e − 13 | 2.673 055 581 874 |
| 1.0e − 14 | 4634 | 0.379 398 189 108 571 | 2.91e − 14 | 0.0e − 14 | 2.673 055 581 875 |

Remarks – as above. Here \( x''(T) = 9.03e - 18 \). The value of \( a = 0.3793981891086 \) – here, all digits are true.

**Numerical experiment for the equation with \( p = 0.1 \), \( c = 1/2 \), \( \beta = 1 \).**

Taking \( \epsilon = 10^{-14} \) we get \( T = 50 \). The proof of the existence and uniqueness result for \( p < 1 \) fails, since we cannot claim that the initial value problem (1.3) has a unique solution and that it depends continuously on \( a \). Hence, function \( h \) can be multivalued. If one will prove the uniqueness, then, due to [10, p. 172], \( h \) will be continuous and all results of this paper will be true also for \( p < 1 \). The stability of numerical experiments cited below suggests it is the fact. Numerical results presented below in Table 3 for this case suggest that the method works also without the uniqueness of solution to initial value problems.
Table 3.

| $\epsilon$  | $N$ | $a$          | $|x''(0) - a|$ | $|1 - x'(T)|$ | $x(T)$ |
|--------------|-----|--------------|----------------|----------------|--------|
| $1.0e-08$    | 142 | 0.443643205958427 | 2.16e-07 | 4.5e-07 | 48.05426086324 |
| $1.0e-09$    | 256 | 0.443643403844908 | 1.78e-08 | 3.7e-08 | 48.05425805120 |
| $1.0e-10$    | 466 | 0.443643420192529 | 1.49e-09 | 3.1e-09 | 48.054221034 |
| $1.0e-11$    | 839 | 0.443643421402885 | 2.80e-10 | 5.8e-10 | 48.054233096 |
| $1.0e-12$    | 1505 | 0.443643421660499 | 2.25e-11 | 4.7e-11 | 48.054235664 |
| $1.0e-13$    | 2669 | 0.443643421681506 | 1.49e-12 | 3.6e-12 | 48.054235873 |
| $1.0e-14$    | 4922 | 0.443643421683245 | 2.45e-13 | 2.2e-16 | 48.054223589 |

Remarks – as above. Here $x''(T) = 1.02e - 15$. The value of $a = 0.443643421683$ – here, all digits are true.

5. CONCLUSIONS

The authors know that our computation of the value of the second derivative of the solution are not more exact than others. However, the proposed method gives a possibility of controlling errors and it is very simple. We hope a similar approach can be applied for more general equations as $x''' + f(x) \cdot g(x'') = 0$ with qualitative assumptions on functions $f$ and $g$.

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Grzegorz Andrzejczak
grzegorz.andrzejczak@p.lodz.pl

Lodz University of Technology
Institute of Mathematics
Wolczanska 215, 90-924 Lodz, Poland

Magdalena Nockowska-Rosiak
magdalena.nockowska@p.lodz.pl

Lodz University of Technology
Institute of Mathematics
Wolczanska 215, 90-924 Lodz, Poland

Bogdan Przeradzki
bogdan.przeradzki@p.lodz.pl

Lodz University of Technology
Institute of Mathematics
Wolczanska 215, 90-924 Lodz, Poland

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