Opuscula Mathematica • Vol. 32 • No. 4 • 2012

http://dx.doi.org/10.7494/OpMath.2012.32.4.731

ON A CLASS OF NONHOMOGENOUS QUASILINEAR PROBLEMS IN ORLICZ-SOBOLEV SPACES

Asma Karoui Souayah

Abstract. We study the nonlinear boundary value problem $-\operatorname{div}\big((a_1(|\nabla u(x)|)+$ $+a_2(|\nabla u(x)|))\nabla u(x)\big) = \lambda|u|^{q(x)-2}u - \mu|u|^{\alpha(x)-2}u$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, λ, μ are positive real numbers, q and α are continuous functions and a_1, a_2 are two mappings such that $a_1(|t|)t$, $a_2(|t|)t$, are increasing homeomorphisms from $\mathbb{R}$ to $\mathbb{R}$. The problem is analysed in the context of Orlicz-Soboev spaces. First we show the existence of infinitely many weak solutions for any $\lambda, \mu > 0$. Second we prove that for any $\mu > 0$, there exists λ_* sufficiently small, and λ^* large enough such that for any $\lambda \in (0, \lambda_*) \cup (\lambda^*, \infty)$, the above nonhomogeneous quasilinear problem has a non-trivial weak solution.

Keywords: variable exponent Lebesgue space, Orlicz-Sobolev space, critical point, weak solution.

Mathematics Subject Classification: 35D05, 35J60, 35J70, 58E05, 68T40, 76A02.

1. INTRODUCTION AND PRELIMINARY RESULTS

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with smooth boundary $\partial\Omega$. In this paper we are concerned with the problem

$$\begin{cases} -\operatorname{div}\big((a_1(|\nabla u|) + a_2(|\nabla u|))\, \nabla u\big) = \lambda|u|^{q(x)-2}u - \mu|u|^{\alpha(x)-2}u & \text{for } x \in \Omega, \\ u \not\equiv 0 & \text{for } x \in \Omega, \\ u = 0 & \text{for } x \in \partial\Omega. \end{cases} \tag{1.1}$$

We assume that $a_i : (0, \infty) \to \mathbb{R}$, $i = 1, 2$, are two functions such that the mappings $\varphi_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, defined by

$$\varphi_i = \begin{cases} a_i(|t|)t & \text{for} \quad t \neq 0, \\ 0 & \text{for} \quad t = 0, \end{cases}$$

are odd, increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$, q, $\alpha : \overline{\Omega} \to (1, \infty)$ are continuous functions, and λ, μ are positive real numbers.

The study of this kind of problems has received more and more interest in the last few years. In fact the interest in studying such problems was stimulated by their application in mathematical physics see [17]. We refer especially to the results in the recent papers [7–10, 15, 18, 20, 22–24, 29].

Next, we introduce the functional space setting where problem (1.1) will be discussed. In fact, the operator in the divergence form is not homogeneous and thus, we introduce an Orlicz-Sobolev space setting for problems of this type.

We start by recalling some basic facts about Orlicz spaces. We refer to the books of Adams and Hedberg [1], Adams [2] and Rao and Ren [30] and the papers of Clement *et al.* [3, 4], Garciá-Huidobro *et al.* [5] and Gossez [6].

For $\varphi_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, which are odd, increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$, we define

$$\Phi_i(t) = \int_0^t \varphi_i(s)ds, \quad (\Phi_i)^*(t) = \int_0^t (\varphi_i)^{-1}(s)ds \qquad \text{for all} \qquad t \in \mathbb{R}, i = 1, 2.$$

We observe that Φ_i, $i = 1, 2$, are Young functions, that is, $\Phi_i(0) = 0$, Φ_i are convex, and $\lim_{x\to\infty} \Phi_i(x) = +\infty$. Furthermore, since $\Phi_i(x) = 0$ if and only if $x = 0$, $\lim_{x\to 0} \frac{\Phi_i(x)}{x} = 0$, and $\lim_{x\to\infty} \frac{\Phi_i(x)}{x} = +\infty$, then Φ_i are called N-functions. The functions $(\Phi_i)^*$, $i = 1, 2$, are called the *complementary* functions of Φ_i, $i = 1, 2$, and they satisfy

$$(\Phi_i)^*(t) = \sup\{st - \Phi_i(s) : s \geq 0\} \qquad \text{for all } t \geq 0.$$

We also observe that $(\Phi_i)^*$, $i = 1, 2$, are also N-functions and Young's inequality holds true

$$st \leq \Phi_i(s) + (\Phi_i)^*(t) \qquad \text{for all } s, t \geq 0.$$

The Orlicz spaces $L_{\Phi_i}(\Omega)$, $i = 1, 2$, defined by the N-functions Φ_i (see [1–3]) are spaces of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L_{\Phi_i}} := \sup \left\{ \int_\Omega uv dx : \int_\Omega (\Phi_i)^*(|v|)dx \leq 1 \right\} < \infty.$$

Then $(L_{\Phi_i}(\Omega), \|.\|_{L_{\Phi_i}})$, $i = 1, 2$, are Banach spaces whose norm is equivalent to the Luxemburg norm

$$\|u\|_{\Phi_i} := \inf \left\{ k > 0 : \int_\Omega \Phi_i\left(\frac{u(x)}{k}\right)dx \leq 1 \right\}.$$

For Orlicz spaces Hölder's inequality reads as follows (see [30, Inequality 4, p.79]):

$$\int_\Omega uv dx \leq 2\|u\|_{L_{\Phi_i}} \|v\|_{L_{(\Phi_i)^*}} \text{ for all } u \in L_{\Phi_i}(\Omega) \text{ and } v \in L_{(\Phi_i)^*}(\Omega), \ i = 1, 2. \tag{1.2}$$

Next, we introduce the Orlicz-Sobolev space. We denote by $W^1L_{\Phi_i}(\Omega)$, $i = 1, 2$ the Orlicz-Sobolev spaces defined by:

$$W^1L_{\Phi_i}(\Omega) := \Big\{ u \in L_{\Phi_i}(\Omega) : \frac{\partial u}{\partial x_i} \in L_{\Phi_i}(\Omega), i = 1, \dots, N \Big\}.$$

These are Banach spaces with respect to the norms

$$\|u\|_{1,\Phi_i} := \|u\|_{\Phi_i} + \| \, |\nabla u| \, \|_{\Phi_i}, \quad i = 1, 2.$$

We also define the Orlicz-Sobolev spaces $W_0^1L_{\Phi_i}(\Omega), i = 1, 2$, as the closure of $C_0^\infty(\Omega)$. By Lemma 5.7 in [6] we obtain that on $W_0^1L_{\Phi_i}(\Omega), i = 1, 2$, we may consider some equivalent norms:

$$\|u\|_i := \| \, |\nabla u| \, \|_{\Phi_i}.$$

For an easier manipulation of the spaces defined above, we define

$$(\varphi_i)_0 := \inf_{t>0} \frac{t\varphi_i(t)}{\Phi_i(t)} \quad \text{and} \quad (\varphi_i)^0 := \sup_{t>0} \frac{t\varphi_i(t)}{\Phi_i(t)}, \quad \text{and} \quad i \in \{1, 2\}. \tag{1.3}$$

In this paper we assume that for each $i \in \{1, 2\}$ we have

$$1 < (\varphi_i)_0 \le \frac{t\varphi_i(t)}{\Phi_i(t)} \le (\varphi_i)^0 < \infty, \quad \forall t \ge 0.$$

The above relation implies that each $\Phi_i, i \in \{1, 2\}$, satisfies the Δ_2-condition, i.e.

$$\Phi_i(2t) \le K_i \Phi_i(t), \quad \forall t \ge 0, \tag{1.4}$$

where K_i, $i \in \{1, 2\}$, are positive constants ($K_i \ge 2$) (see [25, Proposition 2.3]). On the other hand, the following relations hold true:

$$\|u\|_i^{(\varphi_i)^0} \le \int_\Omega \Phi_i(|\nabla u|)dx \le \|u\|_i^{(\varphi_i)_0}, \forall u \in W_0^1L_{\Phi_i}(\Omega) \text{ with } \|u\|_i < 1, \quad i = 1, 2, \tag{1.5}$$

$$\|u\|_i^{(\varphi_i)_0} \le \int_\Omega \Phi_i(|\nabla u|)dx \le \|u\|_i^{(\varphi_i)^0}, \forall u \in W_0^1L_{\Phi_i}(\Omega) \text{ with } \|u\|_i > 1, \quad i = 1, 2. \tag{1.6}$$

Furthermore, in this paper we assume that for each $i \in \{1, 2\}$ the function Φ_i satisfies the following condition:

$$[0, \infty) \ni t \to \Phi_i(\sqrt{t}) \tag{1.7}$$

is convex.

Condition (1.4) and (1.7) assure that for each $i \in \{1, 2\}$ the Orlicz spaces $L_{\Phi_i(\Omega)}$ are uniformly convex spaces and thus, reflexive Banach spaces (see [25, Proposition 2.2]). That fact implies that also the Orlicz-Sobolev spaces $W_0^1L_{\Phi_i}(\Omega)$, $i \in \{1, 2\}$, are reflexive Banach spaces.

Remark 1.1. If Φ_i, $i \in \{1,2\}$, are N-functions we deduce that $\Phi(t) = \sup\{\Phi_1(t), \Phi_2(t)\}$ is an N-function and Φ has a right derivate denoted by $\Phi_d^{'}(t) = \varphi(t)$ and $\Phi(t) = \int_0^t \Phi_d^{'}(x)dx = \int_0^t \varphi(x)dx$ for all $t \geq 0$.

The right- derivative $\Phi_d^{'}(x)$ is non-decreasing and right-continuous (see [14, p. 51]). On the other hand, since Φ_i satisfies the Δ_2-condition for $i \in \{1,2\}$ we can deduce that Φ satisfies the Δ_2-condition i.e.

$$\Phi(2t) \leq K\Phi(t), \quad \forall t \geq 0, \tag{1.8}$$

where K is a positive constant ($K \geq 2$).

We define

$$\varphi_0 := \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)} \quad \text{and} \quad \varphi^0 := \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)}, \tag{1.9}$$

and we assume that

$$1 < \varphi_0 \leq \frac{t\varphi(t)}{\Phi(t)} \leq \varphi^0 < \infty, \quad \forall t \geq 0.$$

Thus, the following relations hold true

$$\|u\|^{\varphi^0} \leq \int_\Omega \Phi(|\nabla u|)dx \leq \|u\|^{\varphi_0},\ \forall u \in W_0^1 L_\Phi(\Omega) \text{ with } \|u\| < 1, \tag{1.10}$$

$$\|u\|^{\varphi_0} \leq \int_\Omega \Phi(|\nabla u|)dx \leq \|u\|^{\varphi^0},\ \forall u \in W_0^1 L_\Phi(\Omega) \text{ with } \|u\| > 1. \tag{1.11}$$

Since the function $[0,\infty) \ni t \to \Phi_i(\sqrt{t})$ $i \in \{1,2\}$ is convex, we can deduce that

$$[0,\infty) \ni t \to \Phi(\sqrt{t}) \tag{1.12}$$

is convex.

Condition (1.8) and (1.12) assure that the Orlicz spaces $L_\Phi(\Omega)$ are uniformly convex spaces and thus, reflexive Banach spaces. This fact implies that also the Orlicz-Sobolev spaces $W_0^1 L_\Phi(\Omega)$, are reflexive Banach spaces.

Remark 1.2. Since $\Phi(t) = \max\{\Phi_1(t), \Phi_2(t)\}$ for any $t \geq 0$, we deduce that $W_0^1 L_\Phi(\Omega)$ is continuously embedded in $W_0^1 L_{\Phi_i}(\Omega)$, $i \in \{1,2\}$ (see condition (7) in [25]). By relation (1.9), $W_0^1 L_\Phi(\Omega)$ is continuously embedded in $W_0^{1,\varphi_0}(\Omega)$. On the other hand, it is known that $W_0^{1,\varphi_0}(\Omega)$ is compactly embedded in $L^{r(x)}(\Omega)$ for any $r(x) \in C(\bar{\Omega})$ with $1 < r^- \leq r^+ < \frac{N\varphi_0}{N-\varphi_0}$. Thus, we deduce that $W_0^1 L_\Phi(\Omega)$ is compactly embedded in $L^{r(x)}(\Omega)$ for any $r(x) \in C(\bar{\Omega})$ with $1 < r(x) < \frac{N\varphi_0}{N-\varphi_0}$ for all $x \in \bar{\Omega}$.

Remark 1.3. We point out certain examples of functions $\varphi : \mathbb{R} \to \mathbb{R}$ which are odd, increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$ and satisfy conditions (1.3) and (1.7) (see [26, Remark 1]). For more details the reader can consult [13, Examples 1–3, p. 243].

— Let

$$\varphi(t) = p\,|t|^{p-2}\,t, \quad \forall t \in \mathbb{R},\ (\text{with } p > 1).$$

For this function it can be proved that

$$(\varphi)_0 = (\varphi)^0 = p.$$

Furthermore, in this particular case the corresponding Orlicz space $L_\Phi(\Omega)$ is the classical Lebesgue space $L^p(\Omega)$ while the Orlicz-Sobolev spaces $W_0^1 L_\Phi(\Omega)$ is the classical Sobolev space $W_0^{1,p}(\Omega)$. We will use the classical notation to denote the Orlicz-Sobolev spaces in this particular case.

— Consider

$$\varphi(t) = log(1 + |t|^s)\,|t|^{p-2}\,t, \quad \forall t \in \mathbb{R},\ (\text{with } p, s > 1).$$

In this case it can be proved that

$$(\varphi)_0 = p, \quad (\varphi)^0 = p + s.$$

— Let

$$\varphi(t) = \frac{|t|^{p-2}\,t}{log(1 + |t|)},\ \text{if } t \neq 0, \varphi(0) = 0,\ \text{with } p > 2.$$

In this case we have

$$(\varphi)_0 = p - 1, \quad (\varphi)^0 = p.$$

Next, we recall some background facts concerning the variable exponent Lebesgue spaces. For more details we refer to the book by Musielak [27] and the paper by Kováčik and Rákosník [21], Mihăilescu and Rădulescu [22]. For relevant applications and related results we refer to the recent books by Ghergu and Rădulescu [16] and Kristály, Rădulescu and Varga [19].

Set

$$C_+(\overline{\Omega}) = \{h\colon h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h^- = \inf_{x\in\Omega} h(x).$$

For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{u\colon u \text{ is a Borel real-valued function on } \Omega \text{ and } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\right\}.$$

We define on $L^{p(x)}$, the so-called *Luxemburg norm*, by the formula

$$|u|_{p(x)} := \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many aspects: they are separable and Banach spaces [21, Theorem 2.5, Corollary 2.7] and the

Hölder inequality holds [21, Theorem 2.1]. The inclusions between Lebesgue spaces are also naturally generalized [21, Theorem 2.8]: if $0 < |\Omega| < \infty$ and r_1, r_2 are variable exponents so that $r_1(x) \leq r_2(x)$ almost everywhere in Ω then there exists the continuous embedding $L^{r_2(x)}(\Omega) \hookrightarrow L^{r_1(x)}(\Omega)$.

We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x)+1/p'(x)=1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder type inequality

$$\left| \int\limits_{\Omega} uv\, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(x)} |v|_{p'(x)}, \tag{1.13}$$

is held.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int\limits_{\Omega} |u|^{p(x)}\, dx.$$

The space $W^{1,p(x)}(\Omega)$ is equipped by the following norm:

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

We recall that if (u_n), $u \in W^{1,p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations hold:

$$\min(|u|^{p^-}_{p(x)}, |u|^{p^+}_{p(x)}) \leq \rho_{p(x)}(u) \leq \max(|u|^{p^-}_{p(x)}, |u|^{p^+}_{p(x)}), \tag{1.14}$$

$$\min(|\nabla u|^{p^-}_{p(x)}, |\nabla u|^{p^+}_{p(x)}) \leq \rho_{p(x)}(|\nabla u|) \leq \max(|\nabla u|^{p^-}_{p(x)}, |\nabla u|^{p^+}_{p(x)}), \tag{1.15}$$

$$\begin{gathered} |u|_{p(x)} \to 0 \;\Leftrightarrow\; \rho_{p(x)}(u) \to 0,\ \lim_{n\to\infty} |u_n - u|_{p(x)} = 0 \;\Leftrightarrow\; \lim_{n\to\infty} \rho_{p(x)}(u_n - u) = 0, \\ |u_n|_{p(x)} \to \infty \;\Leftrightarrow\; \rho_{p(x)}(u_n) \to \infty. \end{gathered} \tag{1.16}$$

2. MAIN RESULTS

In what follows, we consider problem (1.1). Since $\Phi(t) = \max\{\Phi_1(t), \Phi_2(t)\}$ for any $t \geq 0$, we deduce that $W_0^1 L_\Phi(\Omega)$ is continuously embedded in $W_0^1 L_{\Phi_i}(\Omega)$, $i \in \{1, 2\}$ (see remark 1.2). Thus, problem (1.1) will be analyzed in the space $W_0^1 L_\Phi(\Omega)$.

We say that $u \in W_0^1 L_\Phi(\Omega)$ is a *weak solution* of (1.1) if

$$\int\limits_{\Omega} \Big((a_1(|\nabla u|) + a_2(|\nabla u|))\, \nabla u \nabla v - \lambda |u|^{q(x)-2} uv + \mu |u|^{\alpha(x)-2} uv \Big)\, dx = 0,$$

for any $v \in W_0^1 L_\Phi(\Omega)$.

We will prove the following two results.

Theorem 2.1. *For any* $\lambda, \mu > 0$ *problem* (1.1) *has infinitely many weak solutions provided that*

$$q^- > \max(\varphi^0, (\varphi_1)^0, (\varphi_2)^0, \alpha^+) \quad \textit{and} \quad q^+ < \frac{N\varphi^0}{N - \varphi^0}.$$

Theorem 2.2. (i) *For any* $\mu > 0$ *there exists* $\lambda_* > 0$ *under which problem* (1.1) *has a nontrivial weak solution, provided that* $q^- < \min(\varphi_0, (\varphi_1)_0, (\varphi_2)_0, \alpha^-)$ *and* $\max(\alpha^+, q^+) < \frac{N\varphi_0}{N-\varphi_0}$.

(ii) *If* $q^+ < \alpha^-$ *and* $\alpha^+ < \frac{N\varphi_0}{N-\varphi_0}$, *then for any* $\mu > 0$, *there exists also a critical value* $\lambda^* > 0$ *such that for any* $\lambda \geq \lambda^*$, *problem* (1.1) *has a nontrivial weak solution.*

3. PROOF OF THEOREM 2.1

The proof of Theorem 2.1 is based on a $\mathbb{Z}_2$-symmetric version for even functionals of the mountain pass theorem (see Theorem 9.12 in [28]).

Let E denote the generalized Sobolev space $W_0^1 L_\Phi(\Omega)$ and $\|\cdot\|$ denote the norm $\||\nabla \cdot |\|_\Phi$. Let λ and μ be arbitrary but fixed. The energy functional corresponding to the problem (1.1) is defined as $J_{\lambda,\mu} : E \to \mathbb{R}$,

$$J_{\lambda,\mu}(u) := \int_\Omega \Phi_1(|\nabla u|)dx + \int_\Omega \Phi_2(|\nabla u|)dx - \lambda \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx + \mu \int_\Omega \frac{1}{\alpha(x)}|u|^{\alpha(x)}dx.$$

The functional $J_{\lambda,\mu}$ is well-defined on E and $J_{\lambda,\mu} \in C^1(E, \mathbb{R})$. A simple calculation shows that $J_{\lambda,\mu}$ is well-defined on E and $J_{\lambda,\mu} \in C^1(E, \mathbb{R})$ with the derivative given by

$$\langle dJ_{\lambda,\mu}(u), v\rangle = \int_\Omega \big(a_1(|\nabla u|) + a_2(|\nabla u|)\big)\nabla u \nabla v dx - \\ - \lambda \int_\Omega |u|^{q(x)-2}uv dx + \mu \int_\Omega |u|^{\alpha(x)-2}uv dx, \ \forall\, v \in E.$$

In order to use the mountain pass theorem, we need the following lemmas.

Lemma 3.1. *For any* $\lambda, \mu > 0$ *there exists* $r, a > 0$ *such that* $J_{\lambda,\mu}(u) \geq a > 0$ *for any* $u \in E$ *with* $\|u\| = r$.

Proof. Since $\Phi(t) = \max\{\Phi_1(t), \Phi_2(t)\}$ for any $t \geq 0$ then

$$\Phi_1(|\nabla u|) + \Phi_2(|\nabla u|) \geq \Phi(|\nabla u|) \quad \forall x \in \overline{\Omega}. \tag{3.1}$$

On the other hand, using Remark 1.2, E is continuously embedded in $L^{q(x)}(\Omega)$. So there exists a positive constant C such that, for all $u \in E$,

$$|u|_{q(x)} \leq C\|u\|. \tag{3.2}$$

Suppose that $\|u\| < \min(1, \frac{1}{C})$, then for all $u \in E$ with $\|u\| = \rho$ we have

$$|u|_{q(x)} < 1.$$

Furthermore, relation (1.14) yields

$$\int_\Omega |u|^{q(x)} dx \leq |u|_{q(x)}^{q^-}$$

for all $u \in E$ with $\|u\| = \rho$. The above inequality and relation (3.2) imply that for all $u \in E$ with $\|u\| = \rho$, we have

$$\int_\Omega |u|^{q(x)} dx \leq C^{q^-} \|u\|^{q^-}. \tag{3.3}$$

On the other hand, we have

$$\int_\Omega \Phi(|\nabla u|) dx \geq \|u\|^{\varphi^0}. \tag{3.4}$$

Then using relations (3.1), (3.3) and (3.4), we deduce that, for any $u \in E$ with $\|u\| = \rho$, the following inequalities hold true:

$$\begin{aligned} J_{\lambda,\mu}(u) &\geq \int_\Omega \Phi(|\nabla u|) dx - \frac{\lambda}{q^-} \int_\Omega |u|^{q(x)} dx \geq \\ &\geq \|u\|^{\varphi^0} - \frac{\lambda}{q^-} C^{q^-} \|u\|^{q^-}. \end{aligned}$$

Let $h_\lambda(t) = t^{\varphi^0} - \frac{\lambda}{q^-} C^{q^-} t^{q^-}$, $t > 0$. It is easy to see that $h_\lambda(t) > 0$ for all $t \in (0, t_1)$, where $t_1 < \left(\frac{q^-}{\lambda C^{q^-}}\right)^{\frac{1}{q^- - \varphi^0}}$. So for any $\lambda, \mu > 0$ we can choose $r, a > 0$ such that $J_{\lambda,\mu}(u) \geq a > 0$ for all $u \in E$ with $\|u\| = r$. The proof of Lemma 3.1 is complete. □

Lemma 3.2. *If $E_1 \subset E$ is a finite dimensional subspace, the set $S = \{u \in E_1 : J_{\lambda,\mu}(u) \geq 0\}$ is bounded in E.*

Proof. We have

$$\int_\Omega \Phi_i(|\nabla u|) dx \leq K_i(\|u\|^{\varphi_0} + \|u\|^{\varphi^0}), \quad \forall u \in E, \quad i \in \{1,2\}, \tag{3.5}$$

where K_i $(i \in \{1,2\})$ are positive constants. Indeed, using relations (1.5) and (1.6) we have

$$\int_\Omega \Phi_i(|\nabla u|) dx \leq \|u\|_i^{((\varphi_i)_0} + \|u\|_i^{(\varphi_i)^0}, \quad \forall u \in E, \quad i \in \{1,2\}. \tag{3.6}$$

On the other hand, using Remark 1.2, there exists a positive constant C_i such that

$$\|u\|_i \leq C_i \|u\|, \quad \forall u \in E, \quad i \in \{1,2\}. \tag{3.7}$$

The last two inequality yield

$$\int_\Omega \Phi_i(|u|)dx \leq C_i^{(\varphi_i)_0} \|u\|^{(\varphi_i)_0} + C_i^{(\varphi_i)^0} \|u\|^{(\varphi_i)^0}, \quad \forall u \in E, \quad i \in \{1,2\}, \tag{3.8}$$

and thus (3.5) holds true. Also we have

$$\int_\Omega |u|^{\alpha(x)}dx \leq |u|_{\alpha(x)}^{\alpha^-} + |u|_{\alpha(x)}^{\alpha^+}, \quad \forall u \in E. \tag{3.9}$$

The fact that E is continuously embedded in $L^\alpha(\Omega)$ assures the existence of a positive constant C_3 such that

$$|u|_{\alpha(x)} \leq C_3 \|u\|, \quad \forall u \in E. \tag{3.10}$$

The last two inequalities show that there exists a positive constant $K_3(\mu)$ such that

$$\begin{aligned} \mu \int_\Omega \frac{1}{\alpha(x)}|u|^{\alpha(x)}dx &\leq \frac{\mu}{\alpha^-}\left(C_3^{\alpha^-}\|u\|^{\alpha^-} + C_3^{\alpha^+}\|u\|^{\alpha^+}\right) \leq \\ &\leq K_3(\mu)\left(\|u\|^{\alpha^-} + \|u\|^{\alpha^+}\right), \quad \forall u \in E. \end{aligned} \tag{3.11}$$

By inequality (3.5) and (3.11), we get

$$\begin{aligned} J_{\lambda,\mu}(u) \leq{}& K_1(\|u\|^{(\varphi_1)_0} + \|u\|^{(\varphi_1)^0}) + K_2(\|u\|^{(\varphi_2)_0} + \|u\|^{(\varphi_2)^0}) + \\ &+ K_3(\mu)\left(\|u\|^{\alpha^-} + \|u\|^{\alpha^+}\right) - \frac{\lambda}{q^+}\int_\Omega |u|^{q(x)}dx \end{aligned} \tag{3.12}$$

for all $u \in E$.
Let $u \in E$ be arbitrary but fixed. We define

$$\Omega_< = \{x \in \Omega; |u(x)| < 1\}, \quad \Omega_\geq = \Omega \backslash \Omega_<.$$

Then we have

$$
\begin{aligned}
&J_{\lambda,\mu}(u) \le \\
&\le K_1(\|u\|^{(\varphi_1)_0} + \|u\|^{(\varphi_1)^0}) + K_2(\|u\|^{(\varphi_2)_0} + \|u\|^{(\varphi_2)^0}) + K_3(\mu)\left(\|u\|^{\alpha^-} + \|u\|^{\alpha^+}\right) - \\
&\quad - \frac{\lambda}{q^+}\int\limits_{\Omega} |u|^{q(x)} dx \le \\
&\le K_1(\|u\|^{(\varphi_1)_0} + \|u\|^{(\varphi_1)^0}) + K_2(\|u\|^{(\varphi_2)_0} + \|u\|^{(\varphi_2)^0}) + K_3(\mu)\left(\|u\|^{\alpha^-} + \|u\|^{\alpha^+}\right) - \\
&\quad - \frac{\lambda}{q^+}\int\limits_{\Omega_\ge} |u|^{q(x)} dx \le \\
&\le K_1(\|u\|^{(\varphi_1)_0} + \|u\|^{(\varphi_1)^0}) + K_2(\|u\|^{(\varphi_2)_0} + \|u\|^{(\varphi_2)^0}) + K_3(\mu)\left(\|u\|^{\alpha^-} + \|u\|^{\alpha^+}\right) - \\
&\quad - \frac{\lambda}{q^+}\int\limits_{\Omega_\ge} |u|^{q^-} dx \le \\
&\le K_1(\|u\|^{(\varphi_1)_0} + \|u\|^{(\varphi_1)^0}) + K_2(\|u\|^{(\varphi_2)_0} + \|u\|^{(\varphi_2)^0}) + K_3(\mu)\left(\|u\|^{\alpha^-} + \|u\|^{\alpha^+}\right) - \\
&\quad - \frac{\lambda}{q^+}\int\limits_{\Omega} |u|^{q^-} dx + \frac{\lambda}{q^+}\int\limits_{\Omega_<} |u|^{q^-} dx.
\end{aligned}
$$

But for each $\lambda > 0$ there exists positive constant $K_4(\lambda)$ such that

$$\frac{\lambda}{q^+}\int\limits_{\Omega_<} |u|^{q^-} dx \le K_4(\lambda), \quad \forall u \in E.$$

The functional $|\cdot|_{q^-} : E \to \mathbb{R}$ defined by

$$|u|_{q^-} = \left(\int\limits_{\Omega} |u|^{q^-} dx\right)^{1/q^-},$$

is a norm in E. In the finite dimensional subspace E_1 the norm $|u|_{q^-}$ and $\|u\|$ are equivalent, so there exists a positive constant $K = K(E_1)$ such that

$$\|u\| \le K|u|_{q^-}, \quad \forall u \in E_1.$$

So that there exists a positive constant $K_5(\lambda)$ such that

$$
\begin{aligned}
J_{\lambda,\mu}(u) \le\ & K_1(\|u\|^{(\varphi_1)_0} + \|u\|^{(\varphi_1)^0}) + K_2(\|u\|^{(\varphi_2)_0} + \|u\|^{(\varphi_2)^0}) + \\
&+ K_3(\mu)\left(\|u\|^{\alpha^-} + \|u\|^{\alpha^+}\right) + K_4(\lambda) - K_5(\lambda)\|u\|^{q^-},
\end{aligned}
$$

for all $u \in E_1$. Hence

$$
\begin{aligned}
&K_1(\|u\|^{(\varphi_1)_0} + \|u\|^{(\varphi_1)^0}) + \\
&+ K_2(\|u\|^{(\varphi_2)_0} + \|u\|^{(\varphi_2)^0}) + K_3(\mu)\left(\|u\|^{\alpha^-} + \|u\|^{\alpha^+}\right) + K_4(\lambda) - K_5(\lambda)\|u\|^{q^-} \ge 0,
\end{aligned}
$$

for all $u \in S$. And since $q^- > \max((\varphi_1)^0, (\varphi_2)^0, \alpha^+)$, we conclude that S is bounded in E. $\square$

Lemma 3.3. *If $\{u_n\} \subset E$ is a sequence which satisfies the properties*

$$|J_{\lambda,\mu}(u_n)| < C_4, \tag{3.13}$$

$$dJ_{\lambda,\mu}(u_n) \to 0 \quad as \quad n \to \infty, \tag{3.14}$$

where C_4 is a positive constant, then $\{u_n\}$ possesses a convergent subsequence.

Proof. First we show that $\{u_n\}$ is bounded in E. If not,we may assume that $\|u_n\| \to \infty$ as $n \to \infty$. Thus we may consider that $\|u_n\| > 1$ for any integer n. Using (3.14) it follows that there exists $N_1 > 0$ such that for any $n > N_1$ we have

$$\|dJ_{\lambda,\mu}(u_n)\| \leq 1.$$

On the other hand, for all $n > N_1$ fixed, the application $E \ni v \mapsto \langle dJ_{\lambda,\mu}(u_n), v\rangle$ is linear and continuous. The above information implies that

$$|\langle dJ_{\lambda,\mu}(u_n), v\rangle| \leq \|dJ_{\lambda,\mu}(u_n)\| \, \|v\| \leq \|v\|, \quad v \in E, \quad n > N_1.$$

Setting $v = u_n$ we have

$$-\|u_n\| \leq \int_\Omega \Phi_1(|\nabla u_n|)dx + \int_\Omega \Phi_2(|\nabla u_n|)dx - \lambda \int_\Omega |u_n|^{q(x)}dx + \mu \int_\Omega |u_n|^{\alpha(x)}dx \leq \|u_n\|$$

for all $n > N_1$. We obtain

$$-\|u_n\| - \int_\Omega \Phi_1(|\nabla u_n|)dx - \int_\Omega \Phi_2(|\nabla u_n|)dx - \mu \int_\Omega |u_n|^{\alpha(x)}dx \leq -\lambda \int_\Omega |u_n|^{q(x)}dx \tag{3.15}$$

for all $n > N_1$. Provided that $\|u_n\| > 1$ relation (3.1), (3.13) and (3.15) imply

$$\begin{aligned} C_4 > J_{\lambda,\mu}(u_n) &\geq \Big(1 - \frac{1}{q^-}\Big) \left[\int_\Omega \Phi_1(|\nabla u_n|)dx + \int_\Omega \Phi_2(|\nabla u_n|)dx\right] + \\ &\quad + \mu\Big(\frac{1}{\alpha^+} - \frac{1}{q^-}\Big) \int_\Omega |u_n|^{\alpha(x)}dx - \frac{1}{q^-}\|u_n\| \geq \\ &\geq \Big(1 - \frac{1}{q^-}\Big) \int_\Omega \Phi(|\nabla u_n|)dx - \frac{1}{q^-}\|u_n\| \geq \\ &\geq \Big(1 - \frac{1}{q^-}\Big) \|u_n\|^{\varphi_0} - \frac{1}{q^-}\|u_n\|. \end{aligned}$$

Letting $n \to \infty$ we obtain a contradiction. It follows that $\{u_n\}$ is bounded in E. And we deduce that there exists a subsequence, again denoted by $\{u_n\}$, and $u \in E$ such that $\{u_n\}$ converges weakly to u in E. Since E is compactly embedded in $L^{q(x)}(\Omega)$ and $L^{\alpha(x)}(\Omega)$, then $\{u_n\}$ converges strongly to u in $L^{q(x)}(\Omega)$ and $L^{\alpha(x)}(\Omega)$, respectively. Similar arguments as those used on page 50 in [12] imply that $\{u_n\}$ converges strongly to u in E. The proof of Lemma 3.3 is complete. □

Proof of Theorem 2.1. It is clear that the functional $J_{\lambda,\mu}$ is even and verifies $J_{\lambda,\mu}(0) = 0$. Lemma 3.1, Lemma 3.2 and Lemma 3.3 implies that the mountain pass theorem can be applied to the functional $J_{\lambda,\mu}$. We conclude that problem (1.1) has infinitely many weak solutions in E. The proof of Theorem 2.1 is complete. □

4. PROOF OF THEOREM 2.2

First, we prove the assertion (i) in Theorem 2.2. We show that for any $\mu > 0$ there exists $\lambda_* > 0$ such that for every $\lambda \in (0, \lambda_*)$ the problem (1.1) has a nontrivial weak solution. The key argument in the proof is related to Ekeland's variational principle. In order to apply it we need the following lemmas:

Lemma 4.1. *For all $\mu > 0$ and all $\rho \in (0, 1)$ there exist $\lambda_* > 0$ and $b > 0$ such that, for all $u \in E$ with $\|u\| = \rho$, $J_{\lambda,\mu}(u) \geq b > 0$ for any $\lambda \in (0, \lambda_*)$.*

Proof. Since $q^+ < \dfrac{N\varphi_0}{N - \varphi_0}$ for all $x \in \overline{\Omega}$, we have the continuous embedding $E \hookrightarrow L^{q(x)}(\Omega)$. This implies that there exists a positive constant M such that

$$|u|_{q(x)} \leq M\|u\| \quad \forall u \in E. \tag{4.1}$$

We fix $\rho \in (0, 1)$ such that $\rho < \min(1, 1/M)$. Then for all $u \in E$ with $\|u\| = \rho$ we deduce that

$$|u|_{q(x)} < 1.$$

Furthermore, relations (1.14) yield for all $u \in E$ with $\|u\| = \rho$, we have

$$\int_\Omega |u|^{q(x)} dx \leq |u|_{q(x)}^{q^-}.$$

The above inequality and relations (4.1) imply, for all $u \in E$ with $\|u\| = \rho$, that

$$\int_\Omega |u|^{q(x)} dx \leq M^{q^-} \|u\|^{q^-}. \tag{4.2}$$

Using relations (1.10), (3.1) and (4.2) we deduce that, for any $u \in E$ with $\|u\| = \rho$, the following inequalities hold true:

$$\begin{aligned} J_{\lambda,\mu}(u) &\geq \int_\Omega \Phi_1(|\nabla u|) + \int_\Omega \Phi_2(|\nabla u|)dx - \frac{\lambda}{q^-}\int_\Omega |u|^{q(x)}dx + \\ &\quad + \frac{\mu}{\alpha^+}\int_\Omega |u|^{\alpha(x)}dx \geq \\ &\geq \int_\Omega \Phi(|\nabla u|) - \frac{\lambda}{q^-}\int_\Omega |u|^{q(x)}dx \geq \\ &\geq \|u\|^{\varphi^0} - \frac{\lambda}{q^-}M^{q^-}\|u\|^{q^-} \geq \\ &\geq \rho^{q^-}\left(\rho^{\varphi^0 - q^-} - \frac{\lambda}{q^-}M^{q^-}\right). \end{aligned}$$

By the above inequality, we remark that for

$$\lambda_* = \frac{q^-}{2M^{q^-}}\rho^{\varphi^0 - q^-} \tag{4.3}$$

and for any $\lambda \in (0, \lambda_*)$, there exists $b = \dfrac{\rho^{\varphi_0}}{2} > 0$ such that

$$J_{\lambda,\mu}(u) \geq b > 0, \quad \forall \mu > 0, \quad \forall u \in E \quad \text{with} \quad \|u\| = \rho.$$

The proof of Lemma 4.1 is complete. □

Lemma 4.2. *There exists $\varphi \in E$ such that $\varphi \geq 0$, $\varphi \neq 0$ and $J_{\lambda,\mu}(t\varphi) < 0$, for $t > 0$ small enough.*

Proof. Let $l = \min\{(\varphi_1)_0, (\varphi_2)_0, \alpha^-\}$. Since $q^- < l$, then let $\epsilon_0 > 0$ be such that $q^- + \epsilon_0 < l$. On the other hand, since $q \in C(\overline{\Omega})$, it follows that there exists an open set $\Omega_0 \subset\subset \Omega$ such that $|q(x) - q^-| < \epsilon_0$ for all $x \in \Omega_0$. Thus, we conclude that $q(x) \leq q^- + \epsilon_0 < l$ for all $x \in \overline{\Omega}_0$.

Let $\varphi \in C_0^\infty(\Omega)$ be such that $\mathrm{supp}(\varphi) \supset \overline{\Omega}_0$, $\varphi(x) = 1$ for all $x \in \overline{\Omega}_0$ and $0 \leq \varphi \leq 1$ in Ω. Then using the above information for any $t \in (0,1)$ we have

$$\begin{aligned}
J_{\lambda,\mu}(t\varphi) &= \int_\Omega \Phi_1(|\nabla(t\varphi)|)dx + \int_\Omega \Phi_2(|\nabla(t\varphi)|)dx - \\
&\quad - \lambda \int_\Omega \frac{t^{q(x)}}{q(x)}|\varphi|^{q(x)}dx + \mu \int_\Omega \frac{t^{\alpha(x)}}{\alpha(x)}|\varphi|^{\alpha(x)}dx \leq \\
&\leq \int_\Omega \Phi_1(|\nabla(t\varphi)|)dx + \int_\Omega \Phi_2(|\nabla(t\varphi)|)dx - \\
&\quad - \frac{\lambda}{q^+}\int_\Omega t^{q(x)}|\varphi|^{q(x)} + \mu\frac{t^{\alpha^-}}{\alpha^-}\int_\Omega |\varphi|^{\alpha(x)}dx \leq \\
&\leq t^{(\varphi_1)_0}\int_\Omega \Phi_1(|\nabla\varphi|)dx + t^{(\varphi_2)_0}\int_\Omega \Phi_2(|\nabla\varphi|)dx + \\
&\quad + \frac{\mu t^{\alpha^-}}{\alpha^-}\int_\Omega |\varphi|^{\alpha(x)}dx - \frac{\lambda t^{q^-+\epsilon_0}}{q^+}\int_{\Omega_0}|\varphi|^{q(x)}dx, \leq \\
&\leq t^l\left[\int_\Omega \Phi_1(|\nabla\varphi|)dx + \int_\Omega \Phi_2(|\nabla\varphi|)dx + \frac{\mu}{\alpha^-}\int_\Omega |\varphi|^{\alpha(x)}dx\right] - \frac{\lambda t^{q^-+\epsilon_0}}{q^+}|\Omega_0|.
\end{aligned}$$

Therefore,

$$J_{\lambda,\mu}(t\varphi) < 0,$$

for $t < \delta^{1/(l-q^- -\epsilon_0)}$ with

$$0 < \delta < \min\left\{1, \frac{\lambda|\Omega_0|}{q^+\left[\int_\Omega \Phi_1(|\nabla\varphi|)dx + \int_\Omega \Phi_2(|\nabla\varphi|)dx + \frac{\mu}{\alpha^-}\int_\Omega |\varphi|^{\alpha(x)}dx\right]}\right\}.$$

Finally, we point out that $\int_\Omega \Phi_1(|\nabla\varphi|)dx + \int_\Omega \Phi_2(|\nabla\varphi|)dx + \frac{\mu}{\alpha^-}\int_\Omega |\varphi|^{\alpha(x)}dx > 0$. In fact, if $\int_\Omega \Phi_1(|\nabla\varphi|)dx + \int_\Omega \Phi_2(|\nabla\varphi|)dx + \frac{\mu}{\alpha^-}\int_\Omega |\varphi|^{\alpha(x)}dx = 0$, then $\int_\Omega |\varphi|^{\alpha(x)}dx = 0$. Using relation (1.14), we deduce that $|\varphi|_{\alpha(x)} = 0$ and consequently $\varphi = 0$ in Ω which is a contradiction. The proof of the lemma is complete. □

Proof of (i). Let $\mu > 0$, λ_* be defined as in (4.3) and $\lambda \in (0, \lambda_*)$. By Lemma 4.1, it follows that on the boundary of the ball centered at the origin and of radius ρ in E, denoted by $B_\rho(0)$, we have

$$\inf_{\partial B_\rho(0)} J_{\lambda,\mu} > 0. \tag{4.4}$$

On the other hand, by Lemma 4.2, there exists $\varphi \in E$ such that $J_{\lambda,\mu}(t\varphi) < 0$, for all $t > 0$ small enough. Moreover, relations (1.10), (3.1) and (4.2) imply that for any $u \in B_\rho(0)$, we have

$$J_{\lambda,\mu}(u) \geq \|u\|^{\varphi^0} - \frac{\lambda}{q^-} M^{q^-} \|u\|^{q^-}.$$

It follows that

$$-\infty < \underline{c} := \inf_{B_\rho(0)} J_{\lambda,\mu} < 0.$$

We let now $0 < \epsilon < \inf_{\partial B_\rho(0)} J_{\lambda,\mu} - \inf_{B_\rho(0)} J_{\lambda,\mu}$. Using the above information, the functional $J_{\lambda,\mu} : \overline{B_\rho(0)} \longrightarrow \mathbb{R}$, is lower bounded on $\overline{B_\rho(0)}$ and $J_{\lambda,\mu} \in C^1(\overline{B_\rho(0)}, \mathbb{R})$. Then by Ekeland's variational principle there exists $u_\epsilon \in \overline{B_\rho(0)}$ such that

$$\begin{cases} \underline{c} \leq J_{\lambda,\mu}(u_\epsilon) \leq \underline{c} + \epsilon, \\ 0 < J_{\lambda,\mu}(u) - J_{\lambda,\mu}(u_\epsilon) + \epsilon\|u - u_\epsilon\|, \quad u \neq u_\epsilon. \end{cases}$$

Since

$$J_{\lambda,\mu}(u_\epsilon) \leq \inf_{\overline{B_\rho(0)}} J_{\lambda,\mu} + \epsilon \leq \inf_{B_\rho(0)} J_{\lambda,\mu} + \epsilon < \inf_{\partial B_\rho(0)} J_{\lambda,\mu},$$

we deduce that $u_\epsilon \in B_\rho(0)$.

Now, we define $I_{\lambda,\mu} : \overline{B_\rho(0)} \longrightarrow \mathbb{R}$ by $I_{\lambda,\mu}(u) = J_{\lambda,\mu}(u) + \epsilon \cdot \|u - u_\epsilon\|$. It is clear that u_ϵ is a minimum point of $I_{\lambda,\mu}$ and thus

$$\frac{I_{\lambda,\mu}(u_\epsilon + t \cdot v) - I_{\lambda,\mu}(u_\epsilon)}{t} \geq 0,$$

for small $t > 0$ and any $v \in B_1(0)$. The above relation yields

$$\frac{J_{\lambda,\mu}(u_\epsilon + tv) - J_{\lambda,\mu}(u_\epsilon)}{t} + \epsilon \cdot \|v\| \geq 0.$$

Letting $t \to 0$ it follows that $\langle dJ_{\lambda,\mu}(u_\epsilon), v\rangle + \epsilon\|v\| \geq 0$ and we infer that $\|dJ_{\lambda,\mu}(u_\epsilon)\| \leq \epsilon$.

We deduce that there exists a sequence $\{w_n\} \subset B_\rho(0)$ such that

$$J_{\lambda,\mu}(w_n) \longrightarrow \underline{c} \quad \text{and} \quad dJ_{\lambda,\mu}(w_n) \longrightarrow 0_{E^*}. \tag{4.5}$$

It is clear that $\{w_n\}$ is bounded in E. Thus, there exists a subsequence again denoted by $\{w_n\}$, and w in E such that, $\{w_n\}$ converges weakly to w in E.

Since E is compactly embedded in $L^{q(x)}(\Omega)$ and in $L^{\alpha(x)}(\Omega)$, then $\{w_n\}$ converges strongly in $L^{q(x)}(\Omega)$ and $L^{\alpha(x)}(\Omega)$. Using similar arguments as those used in the proof of Lemma 3.3 we deduce that $\{w_n\}$ converges strongly to w in E. Since $J_{\lambda,\mu} \in C^1(E, \mathbb{R})$, we conclude that

$$dJ_{\lambda,\mu}(w_n) \to dJ_{\lambda,\mu}(w) \quad \text{as} \quad n \to \infty. \tag{4.6}$$

Relations (4.4) and (4.5) show that $dJ_{\lambda,\mu}(w) = 0$ and thus w is a weak solution for problem (1.1). Moreover, by relation (4.5) it follows that $J_{\lambda,\mu}(w) < 0$ and thus, w is a nontrivial weak solution for (1.1).

The proof of (i) in Theorem 2.2 is complete. □

Now we need to prove (ii) in Theorem 2.2. For this purpose, we will show that $J_{\lambda,\mu}$ possesses a nontrivial global minimum point in E. With that end in view we start by proving two auxiliary results.

Lemma 4.3. *The functional $J_{\lambda,\mu}$ is coercive on E.*

Proof. For any a, $b > 0$ and $0 < k < l$ the following inequality holds

$$at^k - bt^l \le a\left(\frac{a}{b}\right)^{k/l-k}, \quad \forall t \ge 0.$$

Using the above inequality we deduce that for any $x \in \Omega$ and $u \in E$ we have

$$\begin{aligned}\frac{\lambda}{q^-}|u|^{q(x)} - \frac{\mu}{\alpha^+}|u|^{\alpha(x)} &\le \frac{\lambda}{q^-}\left(\frac{\lambda\alpha^+}{\mu q^-}\right)^{q(x)/\alpha(x)-q(x)} \le \\ &\le \frac{\lambda}{q^-}\left[\left(\frac{\lambda\alpha^+}{\mu q^-}\right)^{q^+/\alpha^- - q^+} + \left(\frac{\lambda\alpha^+}{\mu q^-}\right)^{q^-/\alpha^+ - q^-}\right] = C,\end{aligned}$$

where C is a positive constant independent of u and x. Integrating the above inequality over Ω we obtain

$$\frac{\lambda}{q^-}\int_\Omega |u|^{q(x)}dx - \frac{\mu}{\alpha^+}\int_\Omega |u|^{\alpha(x)}dx \le D, \tag{4.7}$$

where D is a positive constant independent of u.

Using inequalities (1.11), (3.1) and (4.7) we obtain that, for any $u \in E$ with $\|u\| > 1$, we have

$$J_{\lambda,\mu}(u) \ge \int_\Omega \Phi(|\nabla u|)dx - \frac{\lambda}{q^-}\int_\Omega |u|^{q(x)}dx + \frac{\mu}{\alpha^+}\int_\Omega |u|^{\alpha(x)}dx \ge \|u\|^{\varphi_0} - D.$$

Then $J_{\lambda,\mu}$ is coercive and the proof of lemma is complete. □

Lemma 4.4. *The functional $J_{\lambda,\mu}$ is weakly lower semi-continuous.*

Proof. Since the functionals $\Lambda_i : E \to \mathbb{R}$,

$$\Lambda_i = \int_\Omega \Phi_i(|\nabla u|)dx, \quad \forall i \in \{1, 2\},$$

are convex, it follows that $\Lambda_1 + \Lambda_2$ is convex. Thus to show that the functional $\Lambda_1 + \Lambda_2$ is weakly lower semi-continuous on E, it is enough to show that $\Lambda_1 + \Lambda_2$ is strongly lower semi-continuous on E (see Corollary III. 8 in [11]).

We fix $u \in E$ and $\epsilon > 0$ and let $v \in E$ be arbitrary. Since $\Lambda_1 + \Lambda_2$ is convex and inequality (1.2) holds true, we have

$$\begin{aligned}
&\Lambda_1(v) + \Lambda_2(v) \geq \\
&\geq \Lambda_1(u) + \Lambda_2(u) + \left\langle \Lambda_1^{'}(u) + \Lambda_2^{'}(u), v - u \right\rangle \geq \\
&\geq \Lambda_1(u) + \Lambda_2(u) - \int_\Omega a_1(|\nabla u|)\,|\nabla(v-u)|\,dx - \int_\Omega a_2(|\nabla u|)\,|\nabla(v-u)|\,dx \geq \\
&\geq \Lambda_1(u) + \Lambda_2(u) - \int_\Omega \varphi_1(|\nabla u|)\,|\nabla(v-u)|\,dx - \int_\Omega \varphi_2(|\nabla u|)\,|\nabla(v-u)|\,dx \geq \\
&\geq \Lambda_1(u) + \Lambda_2(u) - 2\,\|v-u\|_1\,\|\varphi_1(\nabla u)\|_{L_{\Phi_1^*}} - 2\,\|v-u\|_2\,\|\varphi_2(\nabla u)\|_{L_{\Phi_2^*}} \geq \\
&\geq \Lambda_1(u) + \Lambda_2(u) - 2\,\|u-v\|\,(||\varphi(\nabla u)||_{L_{\Phi_1^*}} + ||\varphi(\nabla u)||_{L_{\Phi_2^*}}) \geq \\
&\geq \Lambda_1(u) + \Lambda_2(u) - \epsilon
\end{aligned}$$

for all $v \in E$ with $\|u-v\| < \epsilon/2\left[||\varphi_1(|\nabla u|)||_{\Phi_1^*} + ||\varphi_2(|\nabla u|)||_{\Phi_2^*}\right]$. It follows that $\Lambda_1 + \Lambda_2$ is strongly lower semi-continuous and since it is convex we obtain that $\Lambda_1 + \Lambda_2$ is weakly lower semi-continuous.

Finally, if $\{w_n\} \subset E$ is a sequence which converges weakly to w in E then $\{w_n\}$ converges strongly to w in $L^{q(x)}(\Omega)$ and $L^{\alpha(x)}(\Omega)$ thus, $J_{\lambda,\mu}$ is weakly lower semi-continuous. The proof of Lemma 4.4 is complete. □

Proof of (ii). By Lemmas 4.3 and 4.4, we deduce that $J_{\lambda,\mu}$ is coercive and weakly lower semi-continuous on E. Then Theorem 1.2 in [31] implies that there exists $u_{\lambda,\mu} \in E$ a global minimizer of $J_{\lambda,\mu}$ and thus a weak solution of problem.

We show that $u_{\lambda,\mu}$ is not trivial for λ large enough. Indeed, letting $t_0 > 1$ be a fixed real and Ω_1 be an open subset of Ω with $|\Omega_1| > 0$ we deduce that there exists $u_0 \in C_0^\infty(\Omega) \subset E$ such that $u_0(x) = t_0$ for any $x \in \overline{\Omega}_1$ and $0 \leq u_0(x) \leq t_0$ in $\Omega \backslash \Omega_1$. We have

$$\begin{aligned}
J_{\lambda,\mu}(u_0) &= \int_\Omega \Phi_1(|\nabla u_0|)dx + \int_\Omega \Phi_2(|\nabla u_0|)dx - \\
&\quad - \lambda \int_\Omega \frac{1}{q(x)}|u_0|^{q(x)}dx + \mu \int_\Omega \frac{1}{\alpha(x)}|u_0|^{\alpha(x)}dx \leq \\
&\leq L(\mu) - \frac{\lambda}{q^+} t_0^{q^-}\,|\Omega_1|\,,
\end{aligned}$$

where $L(\mu)$ is a positive constant.

Thus there exists $\lambda^* > 0$ such that $J_{\lambda,\mu}(u_0) < 0$ for any $\lambda \in [\lambda^*, \infty)$. It follows that $J_{\lambda,\mu}(u_0) < 0$ for any $\lambda \geq \lambda^*$ and thus $u_{\lambda,\mu}$ is a nontrivial weak solution of problem (1.1) for λ large enough. The proof of the assertion (ii) is complete. □

REFERENCES

[1] R. Adams, L. Hedberg, *Function Spaces and Potential Theory*, Grundlehren Math. Wiss. **314**, Springer-Verlag, Berlin, 1996.

[2] R. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[3] Ph. Clement, M. García-Huidobro, R. Manásevich, K. Schmitt, *Mountain pass type solutions for quasilinear elliptic equations*, Calc. Var. Partial Differential Equations **11** (2000), 33–62.

[4] Ph. Clement, B. de Pagter, G. Sweers, F. de Thélin, *Existence of solutions to a semilinear elliptic system through Orlicz-Sobolev spaces*, Mediterr. J. Math **1** (2004), 241–267.

[5] M. Garciá-Huidobro, V.K. Le, R. Manásevich, K. Schmitt, *On principal eigenvalues for quasilinear elliptic differential operators: an Orlicz-Sobolev space setting*, Nonlinear Differential Equations Appl. (NoDEA) **6** (1999), 207–225.

[6] J.P. Gossez, *Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients*, Trans. Amer. Math. Soc. **190** (1974), 163–205.

[7] O. Allegue, M. Bezzarga, *A multiplicity of solutions for a nonlinear degenerate problem involving a $p(x)$-Laplace type operator*, Complex Var. Elliptic Equ. **55** (2010), 417–429.

[8] O. Allegue, A. Karoui Souayah, *Compactness results for quasilinear problems with variable exponent on the whole space*, EJDE, (2011), 1–15.

[9] K. Ben Ali, M. Bezzarga, *On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent*, Potential Theory and Stochastics in Albac, Aurel Cornea Memorial Volume, Conference Proceedings Albac, September 4–8, 2007, Theta 2008, 21–34.

[10] K. Ben Ali, K. Kefi, *Mountain pass and Ekeland's principle for eigenvalue problem with variable exponent*, Complex Var. Elliptic Equ. **54** (2009), 795–809.

[11] H. Brezis, *Analyse fonctionnelle: théorie et applications*, Masson, Paris, 1992.

[12] Ph. Clément, M. Garcia-Huidobro, R. Manasevich, K. Schmitt, *Mountain pass type solutions for quasilinear elliptic equations*, Calc. Var. Partial Differential Equations **11** (2000), 33–62.

[13] Ph. Clément, B. de Pagter, G. Sweers, F. de Thélin, *Existence of solutions to a semilinear elliptic system through Orlicz-Sobolev spaces*, Mediterr. J. Math. **1** (2004), 241–267.

[14] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Math., 2017, Springer, Heidelberg, 2011.

[15] R. Filippucci, P. Pucci, V. Rădulescu, *Existence and non-existence results for quasilinear elliptic exterior problems with nonlinear boundary conditions*, Comm. Partial Differential Equations **33** (2008), 706–717.

[16] M. Ghergu, V. Rădulescu, *Singular Elliptic Problems: Bifurcation and Asymptotic Analysis*, Oxford Lecture Series in Mathematics and its Applications, 37, Oxford University Press, New York, 2008.

[17] T.C. Halsey, *Electrorheological fluids*, Science **258** (1992), 761–766.

[18] A. Karoui Souayah, K. Kefi, *On a class of nonhomogenous quasilinear problem involving Sobolev spaces with variable exponent*, An. Şt. Univ. Ovidius Constanţa **18** (2010) 1, 309–328.

[19] A. Kristály, V. Rădulescu, C. Varga, *Variational Principles in Mathematical Physics, Geometry, and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems*, Cambridge University Press, Cambridge, 2010.

[20] K. Kefi, *Nonhomogeneous boundary value problems in Sobolev spaces with variable exponent*, Int. J. Appl. Math. Sci. **3** (2006), 103–115.

[21] O. Kovàčik, J. Ràkosnìk, *On spaces $L^{p(x)}$ and $W^{1,p(x)}$*, Czechoslovak Math. J. **41** (1991), 592–618.

[22] M. Mihăilescu, V. Rădulescu, *A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids*, Proc. Roy. Soc. London Ser. A **462** (2006), 2625–2641.

[23] M. Mihăilescu, V. Rădulescu, *On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent*, Proc. Amer. Math. Soc. **135** (2007), 2625–2641.

[24] M. Mihăilescu, V. Rădulescu, *Existence and multiplicity of solutions for quasilinear nonhomogeneous problems: an Orlicz-Sobolev space setting*, J. Math. Anal. Appl. **330** (2007), 416–432.

[25] M. Mihăilescu, V. Rădulescu, *Neumann problems associated to nonhomogeneous differential operations in Orlicz-Sobolev spaces*, Ann. Inst. Fourier **58** (2008), 2087–2111.

[26] M. Mihăilescu, V. Rădulescu, D. Repovš, *On a non-homogeneous eigenvalue problem involving a potential: an Orlicz-Sobolev space setting*, J. Math. Pures Appliquées **93** (2010), 132–148.

[27] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, 1034, Springer, Berlin, 1983.

[28] P. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, Expository Lectures from the CBMS Regional Conference held at the University of Miami, American Mathematical Society, Providence, RI, 1984.

[29] V. Rădulescu, *Singular phenomena in nonlinear elliptic problems. From blow-up boundary solutions to equations with singular nonlinearities*, Handbook of Differential Equations: Stationary Par. Diff. Equa. 4 (Michel Chipot, ed.), North Holland, Elsevier, Amsterdam, 2007, 483–591.

[30] M.M. Rao, Z.D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, New York, 1991.

[31] M. Struwe, *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer, Heidelberg, 1996.

Asma Karoui Souayah
asma.souayah@yahoo.fr

Institut Préparatoire aux Etudes d'ingénieurs de Gafsa
Campus Universitaire Sidi Ahmed Zarrouk - 2112 Gafsa, Tunisia

Received: January 9, 2012.
Accepted: January 18, 2012.