A NOTE ON THE INDEPENDENT ROMAN DOMINATION IN UNICYCLIC GRAPHS

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Abstract. A Roman dominating function (RDF) on a graph $G = (V, E)$ is a function $f : V \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The weight of an RDF is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. An RDF $f$ in a graph $G$ is independent if no two vertices assigned positive values are adjacent. The Roman domination number $\gamma_R(G)$ (respectively, the independent Roman domination number $i_R(G)$) is the minimum weight of an RDF (respectively, independent RDF) on $G$. We say that $\gamma_R(G)$ strongly equals $i_R(G)$, denoted by $\gamma_R(G) \equiv i_R(G)$, if every RDF on $G$ of minimum weight is independent. In this note we characterize all unicyclic graphs $G$ with $\gamma_R(G) \equiv i_R(G)$.

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1. INTRODUCTION

We consider finite, undirected, and simple graphs $G$ with vertex set $V = V(G)$ and edge set $E = E(G)$. The open neighborhood of a vertex $v \in V$ is $N(v) = N_G(v) = \{u \in V : uv \in E\}$ and the degree of $v$, denoted by $d_G(v)$, is the cardinality of its open neighborhood. A vertex of degree one is called a leaf, and its neighbor is called a support vertex. If $v$ is a support vertex, then $v$ is called strong if $v$ is adjacent to at least two leaves.

For a graph $G$, let $f : V(G) \rightarrow \{0, 1, 2\}$ be a function, and let $(V_0; V_1; V_2)$ be the ordered partition of $V = V(G)$ induced by $f$, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i = 0, 1, 2$. There is a $1 \rightarrow 1$ correspondence between the functions $f : V(G) \rightarrow \{0, 1, 2\}$ and the ordered partitions $(V_0; V_1; V_2)$ of $V(G)$. So we will write $f = (V_0; V_1; V_2)$.

A function $f : V(G) \rightarrow \{0, 1, 2\}$ is a Roman dominating function (RDF) on $G$ if every vertex $u$ of $G$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ of $G$ for which $f(v) = 2$. The weight of an RDF is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. An RDF $f$ in a graph $G$ is independent if no two vertices assigned positive values
are adjacent. The Roman domination number \( \gamma_R(G) \) (respectively, the independent Roman domination number \( i_R(G) \)) is the minimum weight of an RDF (respectively, independent RDF) on \( G \). A function \( f = (V_0; V_1; V_2) \) is called a \( \gamma_R(G) \)-function or \( \gamma_R \)-function for \( G \) if it is a Roman dominating function on \( G \) and \( f(V(G)) = \gamma_R(G) \). An \( i_R(G) \)-function or \( i_R \)-function for \( G \) is defined similarly. Let \( f \) be a \( \gamma_R(G) \)-function, and \( f(x) = 0 \) for some vertex \( x \). Then we say that \( x \) is a private neighbor of a vertex \( y \) with \( f(y) = 2 \) if \( f \) is not an RDF for \( G - xy \). Roman domination has been introduced by Cockayne et al. [3] and has been studied for example in [7]. The study of independent Roman domination has been initiated in [1].

We say that \( \gamma_R(G) \) and \( i_R(G) \) are strongly equal for \( G \), denoted by \( \gamma_R(G) \equiv i_R(G) \), if every \( \gamma_R(G) \)-function is an \( i_R(G) \)-function. In [2] a constructive characterization of all trees \( T \) with \( \gamma_R(T) \equiv i_R(T) \) is provided. Note that strong equality between two parameters was considered first by Haynes and Slater [6]. Later Haynes, Henning and Slater gave in [4] and [5] constructive characterizations of trees with strong equality between some domination parameters.

In this note we characterize all unicyclic graphs \( G \) with \( \gamma_R(G) \equiv i_R(G) \).

2. MAIN RESULT

We first describe the procedure given in [2] to built trees \( T \) with \( \gamma_R(T) \equiv i_R(T) \). Let \( \mathcal{T} \) be the family of trees \( T \) that can be obtained from \( k \) \((k \geq 1)\) disjoint stars of centers \( x_1, x_2, ..., x_k \) where each star has order at least three, attached by edges from their center vertices either to a single vertex or to the same leaf of a path \( P_3 \). Such a vertex is called a special vertex of \( T \). Let \( \mathcal{F} \) be the collection of trees \( T \) that can be obtained from a sequence \( T_1, T_2, ..., T_k \) \((k \geq 1)\) of trees, where \( T_1 \) is a star \( K_{1,t} \) with \( t \geq 2 \), \( T = T_k \) and, if \( k \geq 2 \), \( T_{i+1} \) can be obtained recursively from \( T_i \) by one of the following operations:

- **Operation** \( O_1 \) : Assume \( y \) is a leaf of \( T_i \) with \( f_i(y) = 0 \) and whose support vertex \( z \) is either strong or satisfies \( \gamma_R(T_i - z) > \gamma_R(T_i) \). Then \( T_{i+1} \) is obtained from \( T_i \) by adding a new vertex \( x \) and adding the edge \( xy \).

- **Operation** \( O_2 \) : Assume \( y \) is a vertex of \( T_i \). Then \( T_{i+1} \) is obtained from \( T_i \) by adding a tree \( T \in \mathcal{T} \) of special vertex \( x \) and adding the edge \( xy \) with the condition that if \( x \) is a support vertex, then \( y \) satisfies \( \gamma_R(T_i - y) \geq \gamma_R(T_i) \).

- **Operation** \( O_3 \) : Assume \( y \) is a vertex of \( T_i \) assigned \( 0 \) or \( 1 \) for every \( \gamma_R(T_i) \)-function. Then \( T_{i+1} \) is obtained from \( T_i \) by adding a path \( P_3 = u-v-w \) and adding the edge \( wy \).

**Theorem 2.1** (Chellali and Jafari Rad [2]). Let \( T \) be a tree. Then \( \gamma_R(T) \equiv i_R(T) \) if and only if \( T = K_1 \) or \( T \in \mathcal{F} \).

Let \( \mathcal{H} \) be the class of all graphs \( G \) such that \( G \) is obtained from a tree \( T \in \mathcal{F} \) by joining two non-adjacent vertices \( v_1, v_2 \) such that:

(1) For every \( \gamma_R(T) \)-function \( f \), \( 0 \in \{f(v_1), f(v_2)\} \),
Theorem 2.2. Let $G$ be a unicyclic graph. Then $\gamma_R(G) \equiv i_R(G)$ if and only if $G \in \mathcal{H}$.

Proof. Let $G$ be a unicyclic graph, where $C$ is its unique cycle. Assume that $\gamma_R(G) \equiv i_R(G)$ and let $f = (V_0, V_1, V_2)$ be a $\gamma_R(G)$-function. By assumption $f$ is independent. Let $x \in V(C) \cap N_0$, and let $N(x) \cap V(C) = \{y, z\}$. Clearly $x$ cannot be a private neighbor for both $y$ and $z$. Hence we assume that $x$ is not a private neighbor of $y$ and let $T = G - xy$. Then $f$ is an RDF for $T$, and so $\gamma_R(T) \leq i_R(T) \leq \gamma_R(G) = i_R(G)$. If $\gamma_R(T) < i_R(G)$, and $f_1$ is a $\gamma_R(T)$-function, then $f_1$ is an RDF for $G$ with weight less than $\gamma_R(G)$, a contradiction. Thus $\gamma_R(T) = i_R(T) = i_R(G) = \gamma_R(G)$. Next we show that any $\gamma_R(T)$-function is independent. Assume to the contrary that $f$ is a $\gamma_R(T)$-function and $f$ is not independent. Since $f$ is an RDF for $G$ and $\gamma_R(G) = \gamma_R(T)$, we obtain that $f$ is a $\gamma_R(G)$-function, contradicting the fact that $\gamma_R(G) \equiv i_R(G)$. Thus $f$ is independent and consequently, $\gamma_R(T) \equiv i_R(T)$. We deduce that $T \in \mathcal{F}$.

Next we prove (1). Suppose that there is a $\gamma_R(T)$-function $f$ such that $0 \notin \{f(x), f(y)\}$. If $\{f(x), f(y)\} = \{2, 1\}$ and $f(x) = 1$, then $g$ defined on $G$ by $g(x) = 0$ and $g(u) = f(u)$ if $u \neq x$ is an RDF for $G$ with weight less than $\gamma_R(G)$, a contradiction. Thus $\{f(x), f(y)\} \neq \{2, 1\}$ but then $f$ would be a non-independent $\gamma_R(G)$-function, a contradiction since $\gamma_R(G) \equiv i_R(G)$.

Finally, let us prove (2). Assume that there is a non-independent RDF $f$ for $T - x$ with weight $\gamma_R(T)$ such that $f(y) = 2$. Then $f$ is a $\gamma_R(G)$-function which is not independent, a contradiction.

Conversely, assume that $G \in \mathcal{H}$. Let $G$ be obtained from a tree $T \in \mathcal{F}$ by joining two vertices $x$ and $y$ such that (1) and (2) hold. First notice that $\gamma_R(G) \leq \gamma_R(T)$. Assume to the contrary that $\gamma_R(G) < \gamma_R(T)$, and let $f = (V_0, V_1, V_2)$ be a $\gamma_R(G)$-function. If $\{f(x), f(y)\} \neq \{0, 2\}$, then $f$ is an RDF for $T$ with weight less than $\gamma_R(T)$, a contradiction. Thus $\{f(x), f(y)\} = \{0, 2\}$. Suppose that $f(y) = 0$. Then $N(y) \cap V_2 = \{x\}$. Now $g$ defined on $T$ by $g(y) = 1$ and $g(u) = f(u)$ if $u \neq y$, is an RDF for $T$. Then $w(g) = \gamma_R(T)$ for otherwise $g$ is an RDF for $T$ with weight less than $\gamma_R(T)$ which is impossible. Hence $g$ is a $\gamma_R(T)$-function and $0 \notin \{g(x), g(y)\}$, contradicting (1). Therefore $\gamma_R(G) = \gamma_R(T)$. Now let $h$ be an $i_R(G)$-function. Note that $h$ is a $\gamma_R(T)$-function since $\gamma_R(T) \equiv i_R(T)$. If $h$ is not an RDF for $G$, then $0 \notin \{h(x), h(y)\}$, and $h$ is a $\gamma_R(T)$-function that does not satisfy (1), a contradiction. Thus $h$ is an RDF for $G$, and so $i_R(G) \leq \gamma_R(T) = \gamma_R(G) \leq i_R(G)$, implying that $i_R(G) = \gamma_R(G) = \gamma_R(T) = i_R(T)$. Thus $h$ is an $i_R(G)$-function. We next show that each $\gamma_R(G)$-function is independent. Assume to the contrary that $f = (V_0, V_1, V_2)$ is a $\gamma_R(G)$-function and $f$ is not independent. If $0 \notin \{f(x), f(y)\}$, then $f$ is a $\gamma_R(T)$-function which is not independent, contradicting the fact that $T \in \mathcal{F}$. Thus $0 \in \{f(x), f(y)\}$, and we may assume that $f(y) = 0$. Furthermore, $N(y) \cap V_2 = \{x\}$. Then $f_{T-y}$ is an RDF for $T - y$ with weight $\gamma_R(T)$ and $f(x) = 2$, a contradiction with (2). We deduce that $\gamma_R(G) \equiv i_R(G)$. \qed
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