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A NOTE ON A RELATION BETWEEN THE WEAK AND STRONG DOMINATION NUMBERS OF A GRAPH

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Abstract. In a graph G = (V, E) a vertex is said to dominate itself and all its neighbors. A set $D \subseteq V$ is a weak (strong, respectively) dominating set of G if every vertex $v \in V - S$ is adjacent to a vertex $u \in D$ such that $d_G(v) \ge d_G(u)$ ($d_G(v) \le d_G(u)$, respectively). The weak (strong, respectively) domination number of G, denoted by $\gamma_w(G)$ ($\gamma_s(G)$, respectively), is the minimum cardinality of a weak (strong, respectively) dominating set of G. In this note we show that if G is a connected graph of order $n \ge 3$, then $\gamma_w(G) + t\gamma_s(G) \le n$, where $t = 3/(\Delta + 1)$ if G is an arbitrary graph, t = 3/5 if G is a block graph, and t = 2/3 if G is a claw free graph.

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1. INTRODUCTION

We consider finite, undirected, simple graphs. Let G be a graph, with vertex set V and edge set E. The open neighborhood of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood is $N[v] = N(v) \cup \{v\}$. For a subset $S \subseteq V$, the open neighborhood is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood is $N[S] = N(S) \cup S$. By G[S] we denote the subgraph induced by the vertices of S. If v is a vertex of V, then the degree of v denoted by $d_G(v)$, is the size of its open neighborhood. A tree is a connected graph that contains no cycle. A star $K_{1,q}$ is a tree of order q+1 with at least q vertices of degree 1. A subdivided star SS_q is obtained from a star $K_{1,q}$ by replacing each edge uv of the star by a vertex v and edges v and v. The claw is the star v, Given any graph v, a graph v is a graph in which every block (maximal 2-connected graph) is a clique. It is well-known that block graphs are exactly chordal graphs that do not contain v is induced subgraph.

In [5], Sampathkumar and Pushpa Latha have introduced the concept of weak and strong domination in graphs. A subset $D \subseteq V$ is a weak dominating set (wd-set) if every vertex $v \in V - S$ is adjacent to a vertex $u \in D$, where $d_G(v) \geq d_G(u)$. The subset D is a strong dominating set (sd-set) if every vertex $v \in V - S$ is adjacent to a vertex $u \in D$, where $d_G(u) \geq d_G(v)$. The weak (strong, respectively) domination number $\gamma_w(G)$ ($\gamma_s(G)$, respectively) is the minimum cardinality of a wd-set (an sd-set, respectively) of G. If D is an sd-set of G of size $\gamma_s(G)$, then we call D a $\gamma_s(G)$ -set. Strong and weak domination have been studied for example in [1–4].

In their paper introducing weak and strong domination in graphs, Sampathkumar and Pushpa Latha showed that a graph G of order n satisfies $\gamma_w(G) + \gamma_s(G) \leq n$ if G is a d-balanced graph (G has an sd-set D_1 and a wd-set D_2 such that $D_1 \cap D_2 = \emptyset$). However there exist graphs G for which $\gamma_w(G) + \gamma_s(G) > n$. For example if G is a subdivided star SS_q with $q \geq 3$, then $\gamma_w(SS_q) = \gamma_s(SS_q) = q + 1 = (n+1)/2$.

2. RESULTS

We begin by giving an observation and two useful lemmas.

Observation 2.1. 1) For a cycle C_n we have $\gamma_w(C_n) = \gamma_s(C_n) = \lceil n/3 \rceil$.

2) For a nontrivial path P_n we have

$$\gamma_s(P_n) = \lceil n/3 \rceil$$
 and $\gamma_w(P_n) = \begin{cases} \lceil n/3 \rceil, & \text{if } n \equiv 1 \pmod{3}, \\ \lceil n/3 \rceil + 1, & \text{otherwise.} \end{cases}$

Lemma 2.2. Let G = (V, E) be a nontrivial connected graph. Then G has a $\gamma_s(G)$ -set D such that for every vertex $x \in D$ having at least one neighbor in V - D, there is a vertex $y \in V - D$ adjacent to x such that $d_G(y) \leq d_G(x)$.

Proof. Among all $\gamma_s(G)$ -sets let D be a one such vertex such that $\sum_{u\in D} d_G(u)$ is maximum. Obviously the result is valid if |V|=2. Hence let $|V|\geq 3$ and assume that D contains a vertex x such that $N(x)\cap (V-D)\neq\emptyset$ and $d_G(y)>d_G(x)$ for every $y\in N(x)\cap (V-D)$. Then $\{y\}\cup D-\{x\}=D'$ is a $\gamma_s(G)$ -set such that $\sum_{u\in D'}d_G(u)>\sum_{u\in D}d_G(u)$, contradicting our choice of D.

Lemma 2.3. Let X be an independent set of a connected graph G such that every vertex of X has degree at least three. Then:

- (i) if G is a claw free graph, then $3|X| \le 2|N(X)|$,
- (ii) if G is a block graph, then $2|X| + 1 \le |N(X)|$.

Proof. (i) Let E' be the set of edges between X and N(X). Then $3|X| \leq |E'|$. Also since G is claw free and X is independent, every vertex of G has at most two neighbors in X, implying that $|E'| \leq 2|N(X)|$. Therefore, $3|X| \leq |E'| \leq 2|N(X)|$.

(ii) Assume now that G is a block graph and let A = N(X). Consider the graph G[(X, A)] induced by the vertices of X and A. We can suppose that G[(X, A)] is connected, for otherwise we can repeat the procedure below for each component. Let

 v_1, v_2, \ldots, v_t be the vertices of X and A_1, A_2, \ldots, A_t the subsets of A ordering as follow: $A_1 = N(v_1) \cap A$ and for $2 \le k \le t$, x_k is a vertex of X adjacent to a vertex of $\bigcup_{j=1}^{k-1} A_j$ with $A_k = N(v_k) \cap (A - \bigcup_{j=1}^{k-1} A_j)$. Since every vertex of X has degree at least three, we have $|A_1| \ge 3$. Also, since G[(X, A)] is a connected block graph, each vertex x_k for $k \ge 2$ has exactly one neighbor in $\bigcup_{j=1}^{k-1} A_j$. Using this fact and the fact that every vertex of X has degree at least three, it follows that $|A_k| \ge 2$ for $2 \le k \le t$. Therefore, $|N(X)| = |A| = |A_1| + |A_2| + \ldots + |A_t| \ge 3 + 2(t-1) = 2|X| + 1$. \square

Now we are ready to state our main result.

Theorem 2.4. Let G be a connected graph of order $n \geq 3$ and maximum degree Δ . Then $\gamma_w(G) + 3\gamma_s(G)/(\Delta + 1) \leq n$. Moreover,

- (i) if G is a claw free graph, then $\gamma_w(G) + 3\gamma_s(G)/5 \le n$, and
- (ii) if G is a block graph, then $\gamma_w(G) + 2\gamma_s(G)/3 \leq (3n-1)/3$.

Proof. Clearly since $n \geq 3$, we have $\Delta \geq 2$. If $\Delta = 2$, then G is either a cycle C_n or a path P_n , and by Observation 2.1 the result holds. Thus we may assume that $\Delta \geq 3$. Let D be a $\gamma_s(G)$ -set satisfying the conditions of Lemma 2.2. Let $A = \{x \in D: N(x) \cap (V-D) \neq \emptyset\}$ and X = D-A. Observe that by our choice of D, the set V-D weakly dominates A. If $X = \emptyset$, then A = D, and consequently, $\gamma_w(G) \leq |V-D| = n - \gamma_s(G)$. Hence the result is valid even for (i) and (ii) when G is claw free or a block graph, respectively. From now on we will assume that $X \neq \emptyset$. If X contains two adjacent vertices u and v, then one of $D - \{u\}$ or $D - \{v\}$ is a strong dominating set of G, a contradiction. Hence X is an independent set. Note that every vertex of D has degree at least two, otherwise n = 2 or G is not connected. Also since $N(X) \subseteq A$ we have $d_G(u) \geq 3$ for every $u \in X$; otherwise $D - \{u\}$ is an sd-set of G, a contradiction. Now since V - D weakly dominates A, the set $(V - D) \cup X$ weakly dominates G, and therefore

$$\gamma_w(G) \le |(V - D) \cup X| = n - |D| + |X|.$$

Now let us show how to bound |X| by |D| when G is an arbitrary graph, claw free, or a block graph. Note that $|D| = |X| + |A| \ge |X| + |N(X)|$. Let E(X, N(X)) be the set of edges between X and N(X). Since $d_G(u) \ge 3$ for every $u \in X$ and $N(X) \subset D$ we have $3|X| \le |E(X,N(X))|$. Also each vertex y of N(X) has degree at most $\Delta - 1$, otherwise $D - N(y) \cap X$ would be an sd-set of G, a contradiction. It follows that every vertex of N(X) has at most $\Delta - 2$ neighbors in X, thus $|E(X,N(X))| \le (\Delta - 2)|N(X)|$. This implies that $3|X| \le |E(X,N(X))| \le (\Delta - 2)|N(X)|$, and consequently, $|N(X)| \ge 3|X|/(\Delta - 2)$. Since $|D| \ge |X| + |N(X)|$, we obtain $|X| \le (\Delta - 2)|D|/(\Delta + 1)$. Now we get $\gamma_w(G) \le n - |D| + |X| = n - 3|D|/(\Delta + 1)$. Using Lemma 2.3, one can improve the previous result when G is a claw free graph

or a block graph. Hence we obtain (i) and (ii), respectively. We omit the details.

Since the class of trees is contained in the class of block graphs we obtain the following corollary.

Corollary 2.5. If T is a tree of order $n \ge 3$, then $\gamma_w(T) + 2\gamma_s(T)/3 \le (3n-1)/3$.

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REFERENCES

- J.H. Hattingh, M.A. Henning, On strong domination in graphs, J. Combin. Math. Combin. Comput. 26 (1998) 73–92.
- [2] J.H. Hattingh, R.C. Laskar, On weak domination in graphs, Ars Combinatoria 49 (1998).
- [3] D. Rautenbach, Bounds on the weak domination number, Austral. J. Combin. 18 (1998), 245–251.
- [4] D. Rautenbach, Bounds on the strong domination number, Discrete Math. 215 (2000), 201–212.
- [5] E. Sampathkumar, L. Pushpa Latha, Strong, weak domination and domination balance in graphs, Discrete Math. **161** (1996), 235–242.

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