SOME PROPERTIES OF SET-VALUED SINE FAMILIES

Ewelina Mainka-Niemczyk

Abstract. Let \( \{F_t: t \geq 0\} \) be a family of continuous additive set-valued functions defined on a convex cone \( K \) in a normed linear space \( X \) with nonempty convex compact values in \( X \). It is shown that (under some assumptions) a regular sine family associated with \( \{F_t: t \geq 0\} \) is continuous and \( \{F_t: t \geq 0\} \) is a continuous cosine family.

Keywords: set-valued sine and cosine families, continuity of sine families, Hukuhara differences, concave set-valued functions.

Mathematics Subject Classification: 26E25, 47H04, 47D09, 39B52.

1. INTRODUCTION

Our primary objective in this paper is to introduce some basic properties of families of set-valued functions satisfying the functional equation

\[ G_{t+s}(x) = G_{t-s}(x) + 2F_t(G_s(x)), \]

which are called here sine families and refer to the trigonometric functional equation

\[ g(t + s) - g(t - s) = 2f(t)g(s) \]

considered e.g. in [1, p. 138], [2, p. 365].

Sine families are strongly connected with cosine families, which have been considered by various authors. Cosine families of continuous linear operators were investigated e.g. in [4–7] and [16], whereas the set-valued case in [14], [10,11] and [12].

A set-valued regular sine family appeared (non-explicitly) in the paper [10] as a Hukuhara derivative of a cosine family of continuous additive set-valued functions.

2. PRELIMINARIES

Throughout the paper, we assume that all linear spaces are real. Let \( X \) be a normed linear space. \( n(X) \) denotes the set of all nonempty subsets of \( X \), whereas \( b(X) \) stands
for the set of all bounded members of $n(X)$ and $c(X)$ stands for the set of all compact members of $n(X)$. Moreover, by $bcl(X)$ we denote all closed members of $b(X)$, by $bccl(X)$ all convex members of $bcl(X)$ and by $cc(X)$ all convex members of $c(X)$.

By $B(x_0, r)$ we denote the open ball of the radius $r$ centered at a point $x_0$.

A subset $K$ of the space $X$ is called a cone if $tK \subset K$ for all $t \in [0, \infty)$. We say that a cone is convex if it is a convex set.

Let $K$ be a convex cone in $X$. Assume that $\{F_t : t \geq 0\}$ is a family of set-valued functions $F_t : K \to n(X), t \geq 0$.

A family $\{G_t : t \geq 0\}$ of set-valued functions $G_t : K \to n(K), t \geq 0$, is called a sine family associated with the family $\{F_t : t \geq 0\}$, if

$$G_{t+s}(x) = G_{t-s}(x) + 2F_t(G_s(x))$$  \hspace{1cm} (2.1)

for $0 \leq s \leq t$ and $x \in K$, where $F_t(G_s(x)) := \bigcup\{F_t(y) : y \in G_s(x)\}$.

**Example 2.1.** Let $K = [0, \infty)$, $G_t(x) = \{\sin t\}$ and $F_t(x) = \{\cos t\}$ for $t \geq 0$. Then $\{G_t : t \geq 0\}$ is a sine family associated with the family $\{F_t : t \geq 0\}$.

**Example 2.2.** Let $K = [0, \infty)$, $G_t(x) = \{0, \sinh |x|\}$ and $F_t(x) = \{1, \cosh |x|\}$ for $t \geq 0$. Then $\{G_t : t \geq 0\}$ is a sine family associated with the family $\{F_t : t \geq 0\}$.

A family $\{F_t : t \geq 0\}$ of set-valued functions $F_t : K \to n(K), t \geq 0$, is called a cosine family, if

$$F_0(x) = \{x\}$$  \hspace{1cm} (2.2)

for all $x \in K$ and

$$F_{t+s}(x) + F_{t-s}(x) = 2F_t(F_s(x))$$  \hspace{1cm} (2.3)

whenever $0 \leq s \leq t$ and $x \in K$.

Take a set-valued function $\phi : K \to n(Y)$, where $Y$ is a normed linear space. We say that $\phi$ is lower semi-continuous at a point $t_0 \in K$ if for every neighbourhood $V$ of zero in $Y$ there exists a neighbourhood $U$ of zero in $X$ such that

$$\phi(t_0) \subset \phi(t) + V$$

for all $t \in (t_0 + U) \cap K$. We say that $\phi$ is upper semi-continuous at a point $t_0 \in K$ if for every neighbourhood $V$ of zero in $Y$ there exists a neighbourhood $U$ of zero in $X$ such that

$$\phi(t) \subset \phi(t_0) + V$$

for all $t \in (t_0 + U) \cap K$. $\phi$ is continuous at $t_0 \in K$ if it is both lower semi-continuous and upper semi-continuous at $t_0$. It is continuous on $K$ if it is continuous at each point of $K$. It is easy to prove that a set-valued function $\phi : K \to bcl(Y)$ is continuous if and only if a single valued function $K \ni x \mapsto \phi(x) \in bcl(Y)$ is continuous with respect to the Hausdorff metric derived from the norm in $Y$.

A sine family $\{G_t : t \geq 0\}$ is continuous if the function $t \mapsto G_t(x)$ is continuous for every $x \in K$.

A set-valued function $F : K \to n(X)$ is said to be additive if

$$F(x + y) = F(x) + F(y)$$  \hspace{1cm} (2.4)
for all \( x, y \in X \). \( F \) is linear if (2.4) holds true and it is homogeneous, i.e.

\[
F(\lambda x) = \lambda F(x)
\]  

for all \( x \in K, \lambda \geq 0 \). An additive and continuous set-valued function with values in \( cc(X) \) is linear (cf. Theorem 5.3 in [9]). We say \( F \) is midconcave if

\[
F\left[\frac{1}{2}(x + y)\right] \subset \frac{1}{2}[F(x) + F(y)]
\]

for all \( x, y \in K \) (cf. [9]).

**Proposition 2.3.** Let \( X \) be a normed linear space and let \( K \) be a convex cone in \( X \). Assume that \( \{ F_t : t \geq 0 \} \) is a family of set-valued functions \( F_t : K \to n(X) \), such that \( F_0 \) is upper semi-continuous linear with compact values and \( x \in F_0(x) \) for \( x \in K \). If \( \{ G_t : t \geq 0 \} \) is a sine family associated with the family \( \{ F_t : t \geq 0 \} \) and \( G_0(x) \in cc(K) \) for \( x \in K \), then \( G_0(x) = \{ 0 \} \) for \( x \in K \).

Indeed, putting \( t = 0 \) and \( s = 0 \) in (2.1), by the cancellation law (cf. [13]) we obtain the equality \( \{ 0 \} = F_0(G_0(x)) \), \( x \in K \). Since \( y \in F_0(y) \) for all \( y \in K \), this equality yields \( G_0(x) = \{ 0 \} \) for \( x \in K \).

A family \( \{ G_t : t \geq 0 \} \) is increasing if \( G_s(x) \subset G_t(x) \) for every \( x \in K \) and \( 0 \leq s \leq t \).

The two following propositions are easy to prove.

**Proposition 2.4.** Let \( X \) be a normed linear space and let \( K \) be a convex cone in \( X \). Assume that \( \{ F_t : t \geq 0 \} \) is a family of set-valued functions \( F_t : K \to n(X) \), such that \( x \in F_t(x) \) for \( x \in K \), \( t \geq 0 \). If \( \{ G_t : t \geq 0 \} \) is a sine family associated with the family \( \{ F_t : t \geq 0 \} \), then the inclusion

\[
G_u(x) + 2G_v(x) \subset G_{u + 2v}(x)
\]

holds for every \( u, v \geq 0, x \in K \).

**Proposition 2.5.** Let \( X \) be a normed linear space and let \( K \) be a convex cone in \( X \). If a family \( \{ G_t : t \geq 0 \} \) of set-valued functions \( G_t : K \to n(X) \), such that \( 0 \in G_t(x) \) for \( t \geq 0, x \in K \), fulfils inclusion (2.6), then it is increasing.

Let \( \{ F_t : t \geq 0 \} \) be a family of set-valued functions \( F_t : K \to n(K) \). We write

\[
\lim_{t \to 0^+} F_t(x) = \{ x \}
\]

if

\[
\lim_{t \to 0^+} d(F_t(x), \{ x \}) = 0,
\]

where \( d \) is the Hausdorff distance derived from the norm in \( X \).

A cosine family \( \{ F_t : t \geq 0 \} \) is regular if the above equality is satisfied for each \( x \in K \) (cf. [14]).

A sine family \( \{ G_t : t \geq 0 \} \) is regular if \( \lim_{t \to 0^+} \frac{G_t(x)}{t} = \{ x \} \).

**Example 2.6.** Let \( K = (-\infty, \infty) \) and \( F_t(x) = [1, \cosh t]x \) for \( t \geq 0 \). Then \( \{ F_t : t \geq 0 \} \) is a regular cosine family.
The sine family from Example 2.1 is regular, whereas the sine family given in Example 2.2 is not regular. Indeed, since \( \lim_{t \to 0^+} \frac{\sin t}{t} = 1 \) and \( \lim_{t \to 0^+} \frac{\sinh t}{t} = 1 \) we have

\[
\lim_{t \to 0^+} \frac{\{x \sin t\}}{t} = \{x\}
\]

and

\[
\lim_{t \to 0^+} \frac{[0, \sinh t] \cdot x}{t} = [0, x].
\]

Let \( A, B, C \) be sets of \( cc(X) \). We say that a set \( C \) is the Hukuhara difference of \( A \) and \( B \), i.e., \( C = A - B \) if \( B + C = A \). If this difference exists, then it is unique (see Lemma 1 in [13]).

The next lemma follows directly from the definition of Hukuhara difference.

**Lemma 2.7.** Let \( X \) be a normed linear space and let \( A, B, C, D \) be sets of \( cc(X) \). Then:

(a) \( A - A \) exists and \( A - A = \{0\} \);
(b) \( A - \{0\} \) exists and \( A - \{0\} = A \);
(c) if the differences \( A - C, C - D, D - B \) exist, then the differences \( A - B, (A - B) - (C - D) \) exist and \( (A - B) - (C - D) = (A - C) + (D - B) \).

From the definition of a sine family we obtain

**Lemma 2.8.** Let \( X \) be a normed linear space, \( K \) be a convex cone in \( X \) and let \( G_t : K \to cc(K), F_t : K \to cc(X) \) for \( t \geq 0 \). If \( \{G_t : t \geq 0\} \) is a sine family associated with the family \( \{F_t : t \geq 0\} \), then for all \( u, v \in [0, \infty) \) with \( u \leq v \) and all \( x \in K \) there exist Hukuhara differences

\[
G_v(x) - G_u(x).
\]

In the next section we will make use of the following lemma.

**Lemma 2.9** ([15, Lemma 3]). Let \( X \) be a normed linear space and \( K \) be a convex cone in \( X \). Assume that \( F : K \to cc(K) \) is a continuous additive set-valued function and \( A, B \in cc(K) \). If there exists the difference \( A - B \), then there exists \( F(A) - F(B) \) and \( F(A) - F(B) = F(A - B) \).

3. MAIN RESULTS

We give some interesting properties of sine families, in particular continuity and a connection with cosine families.

**Theorem 3.1.** Let \( X \) be a normed linear space and \( K \) be a convex cone in \( X \). Assume that \( \{F_t : t \geq 0\} \) is a family of upper semi-continuous at zero set-valued functions \( F_t : K \to n(X), t \geq 0 \), such that \( x \in F_t(x) \) for \( x \in K, t \geq 0 \), \( F_0 \) is upper semi-continuous linear with compact values and \( F_t(0) = \{0\} \) for \( t \geq 0 \). Then a sine family \( \{G_t : t \geq 0\} \) of set-valued functions \( G_t : K \to b(K) \) associated with the family \( \{F_t : t \geq 0\} \), such that \( G_0 \) has convex compact values and \( 0 \in G_t(x) \) for \( x \in K, t \geq 0 \) is continuous.
**Proof.** Let us fix \( x \in K \) arbitrarily and put \( \phi(t) := G_t(x) \). From (2.6) we have
\[
\phi(u) + 2\phi(v) \subset \phi(u + 2v)
\]
for \( u \geq 0, v \geq 0 \). Putting \( u = v \) we get
\[
3\phi(u) \subset \phi(3u),
\]
and therefore
\[
\phi\left(\frac{u}{3}\right) \subset \frac{1}{3}\phi(u).
\]
Thus
\[
\phi\left(\frac{u}{3^n}\right) \subset \frac{1}{3^n}\phi(u)
\]
for \( u \geq 0 \) and \( n \in \mathbb{N} \). Taking \( u = 1 \) we obtain \( \phi\left(\frac{1}{3^n}\right) \subset \frac{1}{3^n}\phi(1) \) for \( n \in \mathbb{N} \). Let \( \varepsilon > 0 \). There exists \( n \in \mathbb{N} \) such that \( \frac{1}{3^n}\phi(1) \subset B(0, \varepsilon) \). By the monotonicity of \( \phi \)
\[
\phi(w) \subset B(0, \varepsilon)
\]
for \( 0 \leq w < \frac{1}{3^n} \). Since \( \phi(0) = \{0\} \) (Proposition 2.3), \( \phi \) is upper semi-continuous at 0.

Let us fix \( u \in (0, \infty) \) arbitrarily. We shall prove that \( \phi \) is upper semi-continuous at \( u \). It is easily seen, that it suffices to show that \( \phi \) is upper semi-continuous on the right. Suppose that \( V \) is a neighbourhood of zero in \( X \). Setting \( t = u \) in (2.1) and using the monotonicity of \( \phi \), we obtain
\[
\phi(u + s) = \phi(u - s) + 2F_u(\phi(s)) \subset \phi(u) + 2F_u(\phi(s))
\]
for all \( s \in (0, u) \). Since \( F_u \) is upper semi-continuous at 0 and \( F_u(0) = \{0\} \), there exists \( \varepsilon > 0 \) such that
\[
F_u(y) \subset \frac{1}{2} V
\]
for \( y \in B(0, \varepsilon) \cap K \). By (3.1) there is some positive integer \( n \) such that
\[
F_u(\phi(s)) \subset \frac{1}{2} V \quad \text{for} \quad s \in \left[0, \frac{1}{3^n}\right).
\]
Hence, for \( w \in (u, u + \frac{1}{3^n}) \) we have
\[
\phi(w) \subset \phi(u) + V,
\]
which shows that \( \phi \) is upper semi-continuous at \( u \).

Now it remains to show that \( \phi \) is lower semi-continuous. Let us fix \( u \in [0, \infty) \).
It is easily seen, that it suffices to show that \( \phi \) is lower semi-continuous on the left at \( u \in (0, \infty) \). Let us fix a neighbourhood \( V \) of zero in \( X \). Using (3.2) and the monotonicity of \( \phi \), we get
\[
\phi(u) \subset \phi(u + s) = \phi(u - s) + 2F_u(\phi(s))
\]
for all \( s \in (0, u) \). A similar reasoning as before shows that there is some positive integer \( n \) such that \( \phi(u) \subset \phi(w) + V \), for all \( w \in (u - \frac{1}{3^n}, u) \), thus \( \phi \) is lower semi-continuous in \( u \). This completes the proof. \( \square \)
Lemma 3.2. Let $X$ be a normed linear space, $K$ be a convex cone in $X$, $G_t: K \rightarrow \text{cc}(K)$, $F_t: K \rightarrow \text{cc}(X)$, $t \geq 0$ and let $F_0$ be upper semi-continuous linear. If $\{G_t : t \geq 0\}$ is a regular sine family associated with the family $\{F_t : t \geq 0\}$ and $x \in F_t(x)$, $x \in K$, $t \geq 0$, then
\[ x \in G_s(x) \] (3.3)
for all $x \in K$ and $s > 0$.

Proof. From (2.1), Proposition 2.3 and by $x \in F_t(x)$ we have
\[ G_s(x) = G_0(x) + 2F_s(G_2(x)) \supset 2G_2(x), \]
thus
\[ \frac{G_s(x)}{s} \subset \frac{G_s(x)}{s} \quad \text{for } n \in \mathbb{N}. \]
Regularity of $\{G_t : t \geq 0\}$ implies
\[ \frac{G_s(x)}{s} \rightarrow \{x\} \quad \text{as } n \rightarrow \infty, \]
therefore
\[ x \in \frac{G_s(x)}{s} \]
for all $x \in K$ and $s > 0$. \qed

Theorem 3.3. Let $X$ be a normed linear space and $K$ be a convex cone in $X$. Assume that $\{F_t : t \geq 0\}$ is a family of upper semi-continuous at zero additive set-valued functions $F_t: K \rightarrow \text{cc}(X)$, $t \geq 0$, such that $x \in F_t(x)$ for $x \in K$, $t \geq 0$ and $F_0$ is upper semi-continuous linear. If a sine family $\{G_t : t \geq 0\}$ of set-valued functions $G_t: K \rightarrow \text{cc}(K)$ associated with the family $\{F_t : t \geq 0\}$ is regular, then it is continuous.

Proof. Let us fix $x \in K$ arbitrarily and put $\psi(t) := G_t(x) - tx$, $t \geq 0$. Then $0 \in \psi(x)$, $t \geq 0$. Indeed, by Lemma 3.2 and Proposition 2.3 we have
\[ tx \in G_t(x) \]
for $t \geq 0$. Hence
\[ 0 \in G_t(x) - tx = \psi(t), \quad t \geq 0. \]
From (2.6) we have
\[ \psi(u) + 2\psi(v) = G_u(x) - ux + 2G_v(x) - 2vx = \]
\[ = G_u(x) + 2G_u(x) - (u + 2v)x \subset G_u+2v(x) - (u + 2v)x = \psi(u + 2v), \]
i.e.,
\[ \psi(u) + 2\psi(v) \subset \psi(u + 2v) \]
for \( u \geq 0, v \geq 0 \). In the same way as in the proof of Theorem 3.1 we obtain that for each \( \varepsilon > 0 \) there is \( n \in \mathbb{N} \) such that
\[
\psi(w) \subset B(0, \varepsilon)
\] (3.4)
for all \( w \in [0, \frac{1}{3n}) \), and that \( \psi \) is upper semi-continuous at 0.

Let us fix \( u \in (0, \infty) \) arbitrarily. We shall prove that \( \psi \) is upper semi-continuous at \( u \). Since \( \psi \) is increasing (Proposition 2.5), it suffices to show that \( \psi \) is upper semi-continuous on the right at \( u \). Suppose that \( V \) is a symmetric convex neighbourhood of zero in \( X \). Setting \( t = u \) in (2.1) we obtain
\[
\psi(u + s) = G_{u+s}(x) - (u + s)x = [G_{u-s}(x) - (u-s)x] + 2F_u(G_s(x)) - 2sx = \\
= \psi(u - s) + 2F_u(\psi(s) + sx) - 2sx = \\
= \psi(u - s) + 2F_u(\psi(s)) + 2F_u(sx) - 2sx
\]
i.e.,
\[
\psi(u + s) = \psi(u - s) + 2F_u(\psi(s)) + 2F_u(sx) - 2sx
\] (3.5)
for all \( s \in (0, u) \). Hence, by the monotonicity of \( \psi \)
\[
\psi(u + s) \subset \psi(u) + 2F_u(\psi(s)) + 2F_u(sx) - 2sx
\]
for \( s \in (0, u) \). Since \( F_u \) is upper semi-continuous at zero and \( F_u(0) = \{0\} \), there exists \( \varepsilon > 0 \) such that
\[
F_u(y) \subset \frac{1}{6}V
\]
for \( y \in B(0, \varepsilon) \cap K \). By (3.4) there is some positive integer \( n \) such that
\[
F_u(\psi(s)) \subset \frac{1}{6}V \quad \text{for} \quad s \in \left[0, \frac{1}{3n}\right).
\]
Moreover, we can assume that \( n \) is large enough in order that
\[
F_u(sx) \subset \frac{1}{6}V, \quad sx \in \frac{1}{6}V
\]
for \( s \in \left[0, \frac{1}{3n}\right) \). Hence, for \( w \in (u, u + \frac{1}{3n}) \) we have
\[
\psi(w) \subset \psi(u) + V,
\]
which shows that \( \psi \) is upper semi-continuous at \( u \).

It remains to show that \( \psi \) is lower semi-continuous. Let us fix \( u \in [0, \infty) \). It is easily seen, that it suffices to show that \( \psi \) is lower semi-continuous on the left at \( u \in (0, \infty) \). Let us fix a symmetric convex neighbourhood \( V \) of zero in \( X \). Using the monotonicity of \( \psi \) and (3.5), we get
\[
\psi(u) \subset \psi(u + s) = \psi(u - s) + 2F_u(\psi(s)) + 2F_u(sx) - 2sx
\]
for all \( s \in (0, u) \). A similar reasoning as before shows that there is a positive integer \( n \) such that \( \psi(u) \subset \psi(w) + V \) for all \( w \in \left(u - \frac{1}{3n}, u\right) \). Therefore \( \psi \) is lower semi-continuous in \( u \), which completes the proof.
Remark 3.4. Let $X$ be a normed linear space, $K$ be a convex cone in $X$, $G_t: K \to \text{cc}(K)$, $F_t: K \to \text{cc}(X)$ for $t \geq 0$. If $\{G_t : t \geq 0\}$ is a regular sine family associated with the family $\{F_t : t \geq 0\}$ and all $F_t$ are continuous and additive, then the family $\{F_t : t \geq 0\}$ is unique.

Assume that $\{F_t : t \geq 0\}$ and $\{H_t : t \geq 0\}$ are two families of continuous and additive set-valued functions such that

$$G_{t+s}(x) = G_{t-s}(x) + 2F_t(G_s(x))$$

and

$$G_{t+s}(x) = G_{t-s}(x) + 2H_t(G_s(x)).$$

Then

$$G_{t-s}(x) + 2F_t(G_s(x)) = G_{t-s}(x) + 2H_t(G_s(x))$$

and by the cancellation law $F_t(G_s(x)) = H_t(G_s(x))$ for all $0 \leq s \leq t$. Using (2.5) we get

$$F_t\left(\frac{G_s(x)}{s}\right) = H_t\left(\frac{G_s(x)}{s}\right).$$

Letting $s$ tend to $0$ from the right, by regularity of $\{G_t : t \geq 0\}$ we obtain

$$F_t(x) = H_t(x).$$

Example 3.5. Let $K = [0, \infty)$, $G_t(x) = t[0, x]$, $F_t(x) = \{x\}$ and $H_t(x) = [0, x]$ for $t \geq 0$, $x \in K$. Then $\{G_t : t \geq 0\}$ is a sine family associated with the family $\{F_t : t \geq 0\}$ and with the family $\{H_t : t \geq 0\}$.

Indeed, we have

$$G_{t+s}(x) = (t + s)[0, x] = (t - s)[0, x] + 2s[0, x] = G_{t-s}(x) + 2G_s(x) = G_{t-s}(x) + 2F_t(G_s(x))$$

and

$$G_{t+s}(x) = (t + s)[0, x] = (t - s)[0, x] + 2s[0, x] = G_{t-s}(x) + 2H_t(s[0, x]) = G_{t-s}(x) + 2H_s(G_s(x)).$$

Observe that all $F_t$ and $H_t$ are continuous and additive, but the sine family $\{G_t : t \geq 0\}$ is not regular, since

$$\lim_{t \to 0^+} \frac{G_t(x)}{t} = [0, x].$$

Theorem 3.6. Let $X$ be a real normed additive space, $K$ a convex cone in $X$ and let $\{F_t : t \geq 0\}$ be a family of continuous additive set-valued functions $F_t: K \to \text{cc}(K)$, such that $F_0(x) = \{x\}$, $x \in K$. Assume that $\{G_t : t \geq 0\}$ is a regular sine family of set-valued functions $G_t: K \to \text{cc}(K)$ associated with the family $\{F_t : t \geq 0\}$. Then:

(a) $\{F_t : t \geq 0\}$ is a cosine family.
Some properties of set-valued sine families

167

(b) if moreover

\[ x \in F_t(x) \] (3.6)

for \( x \in K \) and \( t \geq 0 \), then \( \{F_t : t \geq 0\} \) is a continuous cosine family. In particular it is regular.

Proof. (a) Let us take \( s, u, v \) such that \( 0 \leq s \leq v - u \), \( 0 \leq s \leq u \) and \( 0 \leq u \leq v \). From (2.1) we get

\[
G_{v+u+s}(x) = G_{v+u-s}(x) + 2F_{v+u}(G_s(x)), \\
G_{v-u+s}(x) = G_{v-u-s}(x) + 2F_{v-u}(G_s(x)), \\
G_{v+u-s}(x) = G_{v-u-s}(x) + 2F_v(G_{u+s}(x)), \\
G_{v+u-s}(x) = G_{v-u-s}(x) + 2F_v(G_{u-s}(x)),
\]

for all \( x \in K \). By Lemma 2.7 and Lemma 2.9, we have therefore

\[
2F_v(2F_u(G_s(x))) = 2F_v(G_{u+s}(x) - G_{u-s}(x)) = 2F_v(G_{u+s}(x)) - 2F_v(G_{u-s}(x)) = \\
= [G_{v+u+s}(x) - G_{v-u+s}(x)] - [G_{v+u-s}(x) - G_{v-u-s}(x)] = \\
= [G_{v+u+s}(x) - G_{v+u-s}(x)] + [G_{v-u+s}(x) - G_{v-u-s}(x)] = \\
= 2F_{v+u}(G_s(x)) + 2F_{v-u}(G_s(x)).
\]

Since \( F_t \) are linear, we can write

\[
2F_v\left(F_u\left(\frac{G_s(x)}{s}\right)\right) = F_{v+u}\left(\frac{G_s(x)}{s}\right) + F_{v-u}\left(\frac{G_s(x)}{s}\right).
\]

Letting \( s \) tend to 0 we obtain from continuity of \( F_t \)

\[
2F_v(F_u(x)) = F_{v+u}(x) + F_{v-u}(x).
\]

(b) The proof will be divided into three steps.

Step 1. From (2.3) and (3.6) follows the inclusion

\[
F_{t+s}(x) + F_{t-s}(x) \supset 2F_t(x)
\]

for \( 0 \leq s \leq t \), which implies that set-valued functions \( u \mapsto F_u(x) \ (x \in K) \) are midconcave in \([0, \infty)\) (cf. [11, the proof of Theorem 3]).

For fixed \( s > 0 \) and \( t > 0 \) from (2.1) and Lemma 3.2 we obtain

\[
F_t(x) \subset F_t\left(\frac{G_s(x)}{s}\right) = \frac{G_{t+s}(x) - G_{t-s}(x)}{2s}
\]

for all \( x \in K \). Since set-valued functions

\[
t \mapsto \frac{G_{t+s}(x) - G_{t-s}(x)}{2s}
\]

are continuous in \((s, \infty)\) (cf. Theorem 3.3), from Theorem 4.3 in [9] set-valued functions

\[
t \mapsto F_t(x)
\]
for \( x \in K \) are continuous in \((s, \infty)\), thus also in \((0, \infty)\). Continuity and midconcavity of set-valued functions \( t \mapsto F_t(x) \) imply their concavity, i.e.,

\[
F_{\lambda t+(1-\lambda)s}(x) \subset \lambda F_t(x) + (1-\lambda)F_s(x), \quad \lambda \in [0, 1], \; s, t > 0, \; x \in K
\]

(cf. Theorem 4.1 in [9]). We get therefore convexity of functions

\[
\psi(t) := \text{diam}(F_t(x))
\]
in \((0, \infty)\) for all \( x \in K \).

Indeed, let \( \lambda \in [0, 1] \) and \( s, t \in (0, \infty) \). By the concavity of the functions \( t \mapsto F_t(x) \) we have

\[
\psi(\lambda t + (1-\lambda)s) = \text{diam}[F_{\lambda t+(1-\lambda)s}(x)] \leq \text{diam}[\lambda F_t(x) + (1-\lambda)F_s(x)] \leq \\
\leq \text{diam}[\lambda F_t(x)] + \text{diam}[(1-\lambda)F_s(x)] = \\
= \lambda \text{diam}[F_t(x)] + (1-\lambda)\text{diam}[F_s(x)] = \lambda \psi(t) + (1-\lambda)\psi(s).
\]

**Step 2.** For \( t > 0 \) and \( x \in K \) we have

\[
F_t(x) + x = 2F_{\frac{1}{2}}(x).
\]

From (3.6) we obtain

\[
F_t(x) + x = F_{\frac{2}{2}}(x) + F_{\frac{2}{2}}(x) \supset F_{\frac{1}{2}}(x) + x,
\]
and therefore

\[
F_{\frac{1}{2}}(x) \subset F_t(x).
\]

Hence the sequence \( (F_{\frac{1}{2^n}}(x)) \) is descending. Put

\[
H_t(x) := \bigcap_{n=0}^{\infty} F_{\frac{1}{2^n}}(x).
\]

From the inclusion

\[
F_{\frac{1}{2^n}}(x) + x = 2F_{\frac{2}{2^n}}(x) \supset F_{\frac{1}{2^n}}(x) + F_{\frac{1}{2^n}}(x) \supset 2H_t(x)
\]

and Lemma 2 in [8] it follows that

\[
H_t(x) + x = \bigcap_{n=0}^{\infty} F_{\frac{1}{2^n}}(x) + x = \bigcap_{n=0}^{\infty} [F_{\frac{1}{2^n}}(x) + x] \supset 2H_t(x).
\]

Therefore, by the cancellation law we get

\[
H_t(x) = \{x\}.
\]
for $t > 0$ and $x \in K$. Thus $\lim_{n \to \infty} F_{\frac{t}{2^n}}(x) = \{x\}$ (cf. Lemma 3 in [8]), whence $\lim_{n \to \infty} \psi\left(\frac{t}{2^n}\right) = 0$. Since $\psi$ is convex, we have

$$\lim_{s \to 0^+} \psi(s) = 0.$$  

**Step 3.** Fix $\varepsilon > 0$. There is $\eta > 0$ such that

$$\psi(s) < \varepsilon \quad \text{for} \quad s \in (0, \eta).$$

Let $s \in (0, \eta)$ and $y \in F_s(x)$. We have then

$$\|y - x\| \leq \text{diam}(F_s(x)) = \psi(s) < \varepsilon.$$  

Hence

$$F_s(x) \subset B(x, \varepsilon)$$

and

$$\lim_{s \to 0^+} F_s(x) = \{x\}. \quad \square$$

**Acknowledgements**

The author thanks Dr Magdalena Piszczek for her remarks which improved Theorem 3.6.

**REFERENCES**


Ewelina Mainka-Niemczyk
ewelina.mainka@polsl.pl

Silesian University of Technology
Institute of Mathematics
Kaszubska 23, 44-100 Gliwice, Poland

Received: November 2, 2010.
Revised: February 2, 2011.
Accepted: March 5, 2011.