WEYL’S THEOREM
FOR ALGEBRAICALLY $k$-QUASICLASS A OPERATORS

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Abstract. If $T$ or $T^*$ is an algebraically $k$-quasiclass A operator acting on an infinite dimensional separable Hilbert space and $F$ is an operator commuting with $T$, and there exists a positive integer $n$ such that $F^n$ has a finite rank, then we prove that Weyl’s theorem holds for $f(T) + F$ for every $f \in H(\sigma(T))$, where $H(\sigma(T))$ denotes the set of all analytic functions in a neighborhood of $\sigma(T)$. Moreover, if $T^*$ is an algebraically $k$-quasiclass A operator, then $\alpha$-Weyl’s theorem holds for $f(T)$. Also, we prove that if $T$ or $T^*$ is an algebraically $k$-quasiclass A operator then both the Weyl spectrum and the approximate point spectrum of $T$ obey the spectral mapping theorem for every $f \in H(\sigma(T))$.

Keywords: algebraically $k$-quasiclass A operator, Weyl’s theorem, $\alpha$-Weyl’s theorem.

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1. INTRODUCTION

We begin with some standard notation on Fredholm theory. Throughout this paper let $\mathcal{H}$ be a separable complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $B(\mathcal{H})$ and $K(\mathcal{H})$ denote respectively, the $C^*$-algebra of all bounded linear operators and the ideal of compact operators acting on $\mathcal{H}$. If $T \in B(\mathcal{H})$, we shall write $\ker T$ and $\text{ran} T$ for the null space and the range of $T$ respectively. Also let $\alpha(T) = \dim \ker T$, $\beta(T) = \dim \ker T^*$ and let $\sigma(T)$, $\sigma_a(T)$ denote the spectrum, approximate point spectrum of $T$, respectively. Let $p = p(T)$ be the ascent of $T$; i.e., the smallest nonnegative integer $p$ such that $\ker T^p = \ker T^{p+1}$. If such an integer does not exist, we put $p(T) = \infty$. Analogously, let $q = q(T)$ be the descent of $T$; i.e., the smallest nonnegative integer $q$ such that $\text{ran} T^q = \text{ran} T^{q+1}$, and if such an integer does not exist, we put $q(T) = \infty$. It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$. Moreover, $0 < p(\lambda - T) = q(\lambda - T) < \infty$ precisely when $\lambda$ is a pole of the resolvent of $T$, see Heuser [20, Proposition 50.2]. An operator $T \in B(\mathcal{H})$ is called Fredholm if $\text{ran} T$ is...
closed and both ker\(T\) and \(\mathcal{H}/\text{ran}T\) are finite dimensional. The index of a Fredholm operator \(T \in \mathcal{B}(\mathcal{H})\), denoted by \(i(T)\), is given by the integer
\[
i(T) = \alpha(T) - \beta(T).
\]
An operator \(T \in \mathcal{B}(\mathcal{H})\) is called Weyl if it is Fredholm of index zero and Brown if it is Fredholm of finite ascent and descent. The essential spectrum \(\sigma_e(T)\), the Weyl spectrum \(\sigma_w(T)\) and the Browder spectrum \(\sigma_b(T)\) of \(T \in \mathcal{B}(\mathcal{H})\) are defined by
\[
\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\},
\]
\[
\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\},
\]
\[
\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\}.
\]
Let iso\(K\) denote the isolated points of \(K \subseteq \mathbb{C}\). We write
\[
\pi_{00}(T) = \{\lambda \in \text{iso}\sigma(T) : 0 < \alpha(T - \lambda) < \infty\},
\]
\[
\pi_{00}^a(T) = \{\lambda \in \text{iso}\sigma_a(T) : 0 < \alpha(T - \lambda) < \infty\},
\]
and
\[
p_{00}(T) = \sigma(T) \setminus \sigma_b(T).
\]
It is evident that \(\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \text{acc}\sigma(T)\) and \(p_{00}(T) \subseteq \pi_{00}^a(T) \subseteq \pi_{00}^a(T)\), where \(\text{acc}\sigma(T) = \sigma(T) \setminus \text{iso}\sigma(T)\). It is well known that \(\sigma_w(T)\) is non-empty and
\[
\sigma_w(T) = \bigcap \{\sigma(T + K) : K \in \mathcal{K}(\mathcal{H})\}.
\]
It is interesting to note that \(\sigma_b(T)\) can be characterized in a way parallel to the definition of \(\sigma_w(T)\):
\[
\sigma_b(T) = \bigcap \{\sigma(T + K) : K \in \mathcal{K}(\mathcal{H}) \text{ and } KT = TK\}.
\]
We say that Weyl’s theorem holds for \(T \in \mathcal{B}(\mathcal{H})\) if
\[
\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),
\]
and that Browder’s theorem holds for \(T \in \mathcal{B}(\mathcal{H})\) if
\[
\sigma(T) \setminus \sigma_w(T) = p_{00}(T),
\]
that is,
\[
\sigma_w(T) = \sigma_b(T).
\]
By definition,
\[
\sigma_{ea}(T) = \bigcap \{\sigma_a(T + K) : K \in \mathcal{K}(\mathcal{H})\}
\]
is the essential approximate point spectrum, and
\[
\sigma_{ab}(T) = \bigcap \{\sigma_a(T + K) : K \in \mathcal{K}(\mathcal{H}) \text{ and } KT = TK\}
is the Browder approximate point spectrum. Let
\[ \Phi_+(\mathcal{H}) = \{ T \in B(\mathcal{H}) : \text{ran} T \text{ is closed and } \alpha(T) < \infty \}, \]
and
\[ \Phi_-(\mathcal{H}) = \{ T \in \Phi_+(\mathcal{H}) : i(T) \leq 0 \}. \]
In [29, Theorem 3.1], it was shown that
\[ \sigma_{ea}(T) = \{ \lambda \in \mathbb{C} : T - \lambda \notin \Phi_-^a(\mathcal{H}) \}. \]
We say that \( \alpha \)-Weyl's theorem holds for \( T \in B(\mathcal{H}) \) if
\[ \sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}(T), \]
and that \( \alpha \)-Browder’s theorem holds for \( T \in B(\mathcal{H}) \) if
\[ \sigma_{ea}(T) = \sigma_{ab}(T). \]
It is known [15,29] that if \( T \in B(\mathcal{H}) \) then we can express the implications between various Weyl’s theorems and Browder’s theorems in the following diagram.

\[ \begin{array}{cc}
\text{\( \alpha \)-Weyl’s theorem} & \longrightarrow & \text{Weyl’s theorem} \\
\downarrow & & \downarrow \\
\text{\( \alpha \)-Browder’s theorem} & \longrightarrow & \text{Browder’s theorem}
\end{array} \]

H. Weyl [32] discovered that Weyl’s theorem holds for hermitian operators and it has been extended from hermitian operators to hyponormal operators and to Toeplitz operators by L.A. Coburn [8], to cohyponormal operators by V. Rakočević [29], and to seminormal operators by S.K. Berberian [6,7]. And this result was generalized for \( p \)-hyponormal operators by M. Chô, M. Iton and S. Ōshiro [9], for class A operators by A. Uchiyama [31], for algebraically hyponormal operators by Y.M. Han [22] and for algebraically paranormal operators by R.E. Curto and Y.M. Han [10]. Recently, H. J. An and Y.M. Han [2] showed that Weyl’s theorem holds for algebraically quasi-class A operators. Recall that \( T \in B(\mathcal{H}) \) is \( p \)-hyponormal for \( p > 0 \) if \( (T^*T)^p - (TT^*)^p \geq 0 \) [1]; when \( p = 1 \), \( T \) is called hyponormal. \( T \) is called cohyponormal, if \( T^* \) is hyponormal. If \( T \) is either hyponormal or cohyponormal, then \( T \) is called seminormal. And \( T \) is called paranormal if \( ||Tx||^2 \leq ||T^2x|| ||x|| \) for all \( x \in \mathcal{H} \) [16,17]. In order to discuss the relations between paranormal and \( p \)-hyponormal and log-hyponormal operators \( (T \text{ is invertible and } \log T^*T \geq \log TT^*) \), T. Furuta, M. Ito and T. Yamazaki [18] introduced a very interesting class of operators: class A defined by \( |T^2| - |T|^2 \geq 0 \), where \( |T| = (T^*T)^{1/2} \) which is called the absolute value of \( T \) and they showed that class A is a subclass of paranormal operators and contains \( p \)-hyponormal and log-hyponormal ones. I.H. Jeon and I. H. Kim [23] introduced quasi-class A (i.e., \( T^*(|T^2| - |T|^2)T \geq 0 \)) operators as an extension of the notion of class A operators. In this paper, we extend this result to algebraically \( k \)-quasiclass A operators using different methods.
Definition 1.1. \( T \in B(\mathcal{H}) \) is called a \( k \)-quasiclass A operator for a positive integer \( k \) if
\[
T^k(|T^2| - |T|^2)T^k \geq 0.
\]

For interesting properties of \( k \)-quasiclass A operators, see [20,30]. In [30], this class of operators is called quasi-class \( (A, k) \). We say that \( T \) is algebraically \( k \)-quasiclass A if there exists a nonconstant complex polynomial \( h \) such that \( h(T) \) is \( k \)-quasiclass A.

Note that algebraically \( k \)-quasiclass A is preserved under translation by scalars and restriction to closed invariant subspaces.

In general, the following inclusions hold:
\[
p\text{-hyponormal} \subseteq \text{class } A \subseteq \text{quasi-class } A \subseteq k\text{-quasiclass } A \subseteq \text{algebraically } k\text{-quasiclass } A.
\]

Definition 1.2. An operator \( T \in B(\mathcal{H}) \) is said to have the single valued extension property (abbrev. SVEP) at \( \lambda \in \mathbb{C} \) if for every open neighborhood \( \mathcal{G} \) of \( \lambda \), the only function \( f \in H(\mathcal{G}) \) such that \( (T - \mu)f(\mu) = 0 \) on \( \mathcal{G} \) is \( 0 \in H(\mathcal{G}) \), where \( H(\mathcal{G}) \) means the space of all analytic functions on \( \mathcal{G} \) having values in \( \mathcal{H} \). Trivially, every operator \( T \in B(\mathcal{H}) \) has SVEP at points of the resolvent \( \rho(T) = \mathbb{C} \setminus \sigma(T) \); moreover, from the identity theorem for analytic functions we have that every operator \( T \in B(\mathcal{H}) \) has SVEP at points of the boundary of \( \sigma(T) \). In particular, every operator has SVEP at the isolated points of its spectrum. When \( T \) has SVEP at each \( \lambda \in \mathbb{C} \), say that \( T \) has SVEP.

The single valued extension property dates back to the early days of local spectral theory and was introduced by N. Dunford [11,12]. This property plays a basic role in local spectral theory, see the recent monograph of K.B. Laursen and M.M. Neumann [26] or P. Aiena [3].

Definition 1.3. An operator \( U \in B(\mathcal{H}) \) is said to be a quasiaffinity if it is injective and has dense range. The operator \( S \in B(\mathcal{H}) \) is called a quasiaffine transform of \( T \in B(\mathcal{H}) \), notation \( S \prec T \), if there exists a quasiaffinity \( U \in B(\mathcal{H}) \) such that \( TU = US \). If both \( S \prec T \) and \( T \prec S \) hold, then \( S \) and \( T \) are called quasisimilar.

The quasinilpotent part \( H_0(T - \lambda) \) and the analytic core \( K(T - \lambda) \) of \( T - \lambda \) are defined by
\[
H_0(T - \lambda) = \{ x \in \mathcal{H} : \lim_{n \to \infty} \|(T - \lambda)^n x\|^\frac{1}{n} = 0 \}
\]
and
\[
K(T - \lambda) = \{ x \in \mathcal{H} : \text{there exists a sequence } \{x_n\} \subseteq \mathcal{H} \text{ and } \delta > 0 \text{ for which } x = x_0, (T - \lambda)x_{n+1} = x_n \text{ and } \|x_n\| \leq \delta^n \|x\| \text{ for all } n = 1, 2, \ldots \}.
\]

We note that \( H_0(T - \lambda) \) and \( K(T - \lambda) \) are generally non-closed hyperinvariant subspaces of \( T - \lambda \) such that \( \ker(T - \lambda)^n \subseteq H_0(T - \lambda) \) for all \( n = 0, 1, 2, \ldots \) and \( (T - \lambda)K(T - \lambda) = K(T - \lambda) \) [26].

Definition 1.4. An operator \( T \in B(\mathcal{H}) \) is said to be semi-regular if \( \ker T \) is closed and \( \ker T \subseteq T^\infty(\mathcal{H}) = \bigcap_{n \in \mathbb{N}} \ker T^n \).
Definition 1.5. An operator $T \in B(H)$ admits a generalized Kato decomposition, GKD for short, if there exists a pair of $T$-invariant closed subspaces $(M, N)$ such that $H = M \oplus N$, the restriction $T|_M$ is quasinilpotent and $T|_N$ is semi-regular. An operator $T \in B(H)$ has a GKD at every isolated point of $\sigma(T)$. We say that $T$ is of Kato type at a point $\lambda$ if $(T - \lambda)|_M$ is nilpotent in the GKD for $T - \lambda$.

See for details [3]. Recall that semi-Fredholm operators are of Kato type [24, Theorem 4]. For more information on semi-Fredholm operators, semi-regular operators and Kato type operators, see [3, 26].

2. WEYL’S THEOREM

FOR ALGEBRAICALLY k-QUASICLASS A OPERATORS

We start with the following lemmas which summarizes some basic properties of $k$-quasiclass A operators.

Lemma 2.1 ([20, 30]). Let $T \in B(H)$ be a $k$-quasiclass A operator for a positive integer $k$ and

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad H = \overline{\text{ran}(T^k)} \oplus \ker T^{*k}$$

be $2 \times 2$ matrix expression. Assume that $\text{ran}(T^k)$ is not dense, then $T_1$ is a class A operator on $\overline{\text{ran}(T^k)}$ and $T^k_3 = 0$. Furthermore, $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Lemma 2.2 ([30]). Let $T \in B(H)$ be a $k$-quasiclass A operator for a positive integer $k$ and $\sigma(T) = \{\lambda\}$. Then $T = \lambda$ if $\lambda \neq 0$, and $T^{k+1} = 0$ if $\lambda = 0$.

In [13], B.P. Duggal and S.V. Djordjević showed that quasinilpotent algebraically $p$-hyponormal operators are nilpotent. Recently, R.E. Curto and Y.M. Han in [10], H.J. An and Y.M. Han in [2] extended this result to algebraically paranormal operators and algebraically quasi-class A operators respectively. In the following theorem we show a similar result for algebraically $k$-quasiclass A operators.

Theorem 2.3. Let $T \in B(H)$ be a quasinilpotent algebraically $k$-quasiclass A operator for a positive integer $k$. Then $T$ is nilpotent.

Proof. Let $h$ be a complex nonconstant polynomial $h$ such that $h(T)$ is $k$-quasiclass A. If $\text{ran}(h(T)^k)$ is dense, then $h(T)$ is a class A operator. Therefore $T$ is an algebraically paranormal operator. We have that $T$ is nilpotent by [10, 10, Lemma 2.2]. If $\text{ran}(h(T)^k)$ is not dense, then by Lemma 2.1 we can represent $h(T)$ as the upper triangular matrix

$$h(T) = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad H = \overline{\text{ran}(h(T)^k)} \oplus \ker h(T)^{*k},$$

where $T_1$ is a class A operator on $\overline{\text{ran}(h(T)^k)}$, $T^k_3 = 0$ and $\sigma(h(T)) = \sigma(T_1) \cup \{0\}$. In fact, $\sigma(h(T)) = \sigma(h(T)) = \{h(0)\}$ for $T$ is quasinilpotent. Since $\sigma(h(T)) = \sigma(T_1) \cup \{0\}$, we have $h(0) = 0$. Hence $h(T)$ is quasinilpotent. Since $h(T)$ is a
By Theorem 2.6, we know that $\sigma$ are finite by Theorem 2.3. Hence, Theorem 3.3.9.

Proof. Let $h$ be a nonconstant complex polynomial such that $h(T)$ is $k$-quasiclass A. By [20, Theorem 2.6], we know that $h(T)$ has finite ascent for all complex numbers. So we have that $h(T)$ has SVEP by [25, Proposition 1.8]. Hence $T$ also has SVEP by [26, Theorem 3.3.9].

Now, if $\lambda$ is an isolated point of $\sigma(T)$, $M = K(T - \lambda)$ and $N = H_0(T - \lambda)$, then $(M, N)$ is a GK for $T - \lambda$. Since $(T - \lambda)|_N$ is quasinilpotent and algebraically $k$-quasiclass A, it follows that $(T - \lambda)|_N$ is nilpotent by Theorem 2.3. Hence $T - \lambda$ is of Kato type. The SVEP for $T$ and $T^*$ at $\lambda$ implies that both $p(T - \lambda)$ and $q(T - \lambda)$ are finite by [4, Theorem 2.3]. Hence $\lambda$ is a pole of the resolvent of $T$.

Analogously, we shall prove that $T^*$ is polaroid. Let $\lambda$ be an isolated point of $\sigma(T^*)$. Then $\lambda$ is an isolated point of $\sigma(T)$ and hence by the first part of the proof we have that $\lambda$ is a pole of the resolvent of $T$. Therefore there exists a natural number $n$ such that $n = p(T - \lambda) = q(T - \lambda)$. Hence we have $\mathcal{H} = \ker(T - \lambda)^n \oplus \text{ran}(T - \lambda)^n$ and $\text{ran}(T - \lambda)^n$ is closed. From this we have $\mathcal{H} = (\ker(T - \lambda)^n)^\perp \oplus (\text{ran}(T - \lambda)^n)^\perp = \text{ran}(T^* - \lambda)^n \oplus \ker(T^* - \lambda)^n$. Hence $p(T^* - \lambda) = q(T^* - \lambda) < \infty$, that is, $\lambda$ is a pole of the resolvent of $T^*$. This completes the proof.

Next we show that Weyl’s theorem holds for $f(T)$, if $T$ or $T^*$ is an algebraically $k$-quasiclass A operator for a positive integer $k$.

Theorem 2.5. Let $T$ or $T^*$ be an algebraically $k$-quasiclass A operator for a positive integer $k$. Then Weyl’s theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.

Proof. Suppose that $T$ is an algebraically $k$-quasiclass A operator for a positive integer $k$. We show first that Weyl’s theorem holds for $T$. We use the fact [14, Theorem 2.2] that if $T$ is polaroid then Weyl’s theorem holds for $T$ if and only if $T$ has SVEP at points of $\lambda \notin \sigma_w(T)$. By Theorem 2.4, we have that $T$ has SVEP and $T$ is polaroid. Hence $T$ satisfies Weyl’s theorem.
We show that Weyl’s theorem holds for $f(T)$. Since $T$ is isoloid, by [27, Lemma] we have
\[
\sigma(f(T))\setminus \pi_{00}(f(T)) = f(\pi(T)\setminus \pi_{00}(T)) = f(\sigma_w(T)),
\]
where the last equality holds since $T$ satisfies Weyl’s theorem. Since $T$ has SVEP, by [4, Corollary 2.6], we have $f(\sigma_w(T)) = \sigma_w(f(T))$. Therefore we have
\[
\sigma(f(T))\setminus \pi_{00}(f(T)) = \sigma_w(f(T)),
\]
so Weyl’s theorem holds for $f(T)$.

Suppose that $T^*$ is an algebraically $k$-quasiclass A operator for a positive integer $k$. We show first that Weyl’s theorem holds for $T$. We use the fact [4, Theorem 3.1] that if $T$ or $T^*$ has SVEP, then Weyl’s theorem holds for $T$ if and only if $\pi_{00}(T) = p_{00}(T)$. By Theorem 2.4, we have that $T^*$ has SVEP. Hence it is sufficient to show that $\pi_{00}(T) = p_{00}(T)$. $p_{00}(T) \subseteq \pi_{00}(T)$ is clear, so we only need to prove $\pi_{00}(T) \subseteq p_{00}(T)$. Let $\lambda \in \pi_{00}(T)$. Then $\lambda$ is an isolated point of $\sigma(T)$. Hence $\lambda$ is a pole of the resolvent of $T$ for $T$ is polaroid by Theorem 2.4, that is, $p(\lambda - T) = q(\lambda - T) < \infty$. By assumption we have $\alpha(\lambda - T) < \infty$, so $\beta(\lambda - T) < \infty$. Hence we conclude that $\lambda \in p_{00}(T)$. Therefore Weyl’s theorem holds for $T$. Since $T^*$ has SVEP, by [4, Corollary 2.6], we have $f(\sigma_w(T)) = \sigma_w(f(T))$. Noting that $T$ is isoloid, as in the proof of the first part, we have that Weyl’s theorem holds for $f(T)$. This completes the proof. 

**Corollary 2.6.** Let $T$ or $T^*$ be an algebraically $k$-quasiclass A operator for a positive integer $k$. If $F$ is an operator commuting with $T$ and for which there exists a positive integer $n$ such that $F^n$ has a finite rank, then Weyl’s theorem holds for $f(T) + F$ for every $f \in H(\sigma(T))$.

**Proof.** Suppose $T$ or $T^*$ is an algebraically $k$-quasiclass A operator for a positive integer $k$. By Theorem 2.4 and Theorem 2.5, we have that $T$ is isoloid and Weyl’s theorem holds for $f(T)$. Observe that if $T$ is isoloid then $f(T)$ is isoloid. The result follows from [28, Theorem 2.4].

From the proof of Theorem 2.5, we have that the Weyl spectrum obeys the spectral mapping theorem for algebraically $k$-quasiclass A operators.

**Corollary 2.7.** Let $T$ or $T^*$ be an algebraically $k$-quasiclass A operator for a positive integer $k$. Then for every $f \in H(\sigma(T))$, we have
\[
f(\sigma_w(T)) = \sigma_w(f(T)).
\]

3. $\alpha$-WEYL’S THEOREM 

FOR ALGEBRAICALLY $k$-QUASICLASS A OPERATORS

For $T \in B(H)$, it is well known that $\sigma_{\alpha}(f(T)) \subseteq f(\sigma_{\alpha}(T))$ is always true for every $f \in H(\sigma(T))$. We have the following result that the essential approximate point spectrum obeys the spectral mapping theorem for $k$-quasiclass A operators.
Theorem 3.1. Let $T$ or $T^*$ be an algebraically $k$-quasiclass $A$ operator for a positive integer $k$. Then
$$\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$$
for every $f \in H(\sigma(T))$.

Proof. We only need to prove that $f(\sigma_{ea}(T)) \subseteq \sigma_{ea}(f(T))$ since $\sigma_{ea}(f(T)) \subseteq f(\sigma_{ea}(T))$ is always true for any operator.

Suppose first that $T$ is an algebraically $k$-quasiclass $A$ operator for a positive integer $k$ and let $f \in H(\sigma(T))$. Assume that $\lambda \notin \sigma_{ea}(f(T))$, then we have $f(T) - \lambda = g(T)\prod_{i=1}^{n}(T - \lambda_i)$, where $\lambda_i \in \mathbb{C}$, $i = 1, 2, \ldots, n$, and $g(T)$ is invertible.

Obviously, $\lambda \in f(\sigma_{ea}(T))$ if and only if $\lambda_i \in \sigma_{ea}(T)$ for some $i$. If $\lambda_i \notin \sigma_{ea}(T)$ for every $\lambda_i \in \{\lambda_i\}_{i=1}^{n}$, then we have $\lambda \notin f(\sigma_{ea}(T))$ and $f(\sigma_{ea}(T)) \subseteq \sigma_{ea}(f(T))$. In the following we shall prove that $\lambda_i \notin \sigma_{ea}(T)$ for every $\lambda_i \in \{\lambda_i\}_{i=1}^{n}$.

In fact, we have that every $T - \lambda_i \in \Phi_+(H)$ for $f(T) - \lambda \in \Phi_+(H)$. Since $T$ is an algebraically $k$-quasiclass $A$ operator, we have that $T$ has SVEP by Theorem 2.4. Hence for each $i$ we have $i(T-\lambda_i) \leq 0$ by [5, Theorem 2.6]. Therefore $T - \lambda_i \in \Phi_+(H)$, that is, $\lambda_i \notin \sigma_{ea}(T)$ for each $i$. As a consequence, $f(\sigma_{ea}(T)) \subseteq \sigma_{ea}(f(T))$. Assume now that $T^*$ is an algebraically $k$-quasiclass $A$ operator. Then $T^*$ has SVEP by Theorem 2.4. Then we have that $i(T-\lambda_i) \geq 0$ for each $i$ by [5, Theorem 2.8]. Since
$$0 \leq \sum_{i=1}^{n} i(T-\lambda_i) = i(f(T) - \lambda) \leq 0,$$
we have $i(T-\lambda_i) = 0$ for each $i$. Since every $T - \lambda_i \in \Phi_+(H)$, we have $T - \lambda_i \in \Phi_-(H)$, that is, $\lambda_i \notin \sigma_{ea}(T)$ for each $i$. Hence $f(\sigma_{ea}(T)) \subseteq \sigma_{ea}(f(T))$. This completes the proof.

Theorem 3.2. Let $T^*$ be an algebraically $k$-quasiclass $A$ operator for a positive integer $k$. Then $\alpha$-Weyl’s theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.

Proof. Suppose $T^*$ is an algebraically $k$-quasiclass $A$ operator for a positive integer $k$. We first prove that $\alpha$-Weyl’s theorem holds for $T$. We use the fact [4, Theorem 3.6] that if $T^*$ has SVEP, then $T$ satisfies $\alpha$-Weyl’s theorem if and only if $T$ satisfies Weyl’s theorem. Since $T^*$ is an algebraically $k$-quasiclass $A$ operator, we have that $T^*$ has SVEP by Theorem 2.4 and $T$ satisfies Weyl’s theorem by Theorem 2.5. So $T$ satisfies $\alpha$-Weyl’s theorem.

Next, we shall prove that $\alpha$-Weyl’s theorem holds for $f(T)$ for every $f \in H(\sigma(T))$. Let $f \in H(\sigma(T))$. Since $T$ satisfies $\alpha$-Weyl’s theorem, we have that $\alpha$-Browder’s theorem holds for $T$. Hence $\sigma_{ea}(T) = \sigma_{ab}(T)$. Since $T^*$ is an algebraically $k$-quasiclass $A$ operator, it follows from Theorem 3.1 that
$$\sigma_{ea}(f(T)) = f(\sigma_{ea}(T)) = f(\sigma_{ab}(T)) = \sigma_{ab}(f(T)),$$
and so $\alpha$-Browder’s theorem holds for $f(T)$. We use the fact [15, Theorem 3.8] that if $T$ satisfies $\alpha$-Browder’s theorem then $\alpha$-Weyl’s theorem holds for $T$ if ran$(T-\lambda)$ is closed.
Weyl’s theorem for algebraically $k$-quasiclass $A$ operators

for each $\lambda \in \pi^a_{00}(T)$. So it suffices to show that if $\lambda \in \pi^a_{00}(f(T))$, then $\text{ran}(f(T) - \lambda)$ is closed. Let $\lambda \in \pi^a_{00}(f(T))$. Then $\lambda$ is an isolated point of $\sigma_a(f(T)) = f(\sigma_a(T))$ and $0 < \alpha(f(T) - \lambda) < \infty$. Since $\lambda$ is an isolated point of $f(\sigma_a(T))$, if $\alpha_i \in \sigma_a(T)$, then $\alpha_i$ is an isolated point of $\sigma_a(T)$ by $f(T) - \lambda = g(T) \prod_{i=1}^{n} (T - \lambda_i)$ in Theorem 3.1. Since $T^*$ has SVEP, we have that $\sigma_a(T) = \sigma(T)$ by [19, Corollary 7]. But $T$ is isoloid by Theorem 2.4, so we have $0 < \alpha(T - \alpha_i) < \infty$ for each $i = 1, 2, \ldots, n$. Since $T$ satisfies $\alpha$-Weyl’s theorem, we have $\alpha_i \notin \sigma_{ea}(T)$ for each $i = 1, 2, \ldots, n$. Hence $\text{ran}(f(T) - \lambda)$ is closed. This completes the proof.

Theorem 3.3. Let $T$ be an algebraically $k$-quasiclass $A$ operator for a positive integer $k$ and $S \prec T$. Then $\alpha$-Browder’s theorem holds for $f(S)$ for every $f \in H(\sigma(S))$.

Proof. Since $T$ is an algebraically $k$-quasiclass $A$ operator for a positive integer $k$, we have that $T$ has SVEP by Theorem 2.4. Hence $\alpha$-Browder’s theorem holds for $f(S)$ for every $f \in H(\sigma(S))$ by [10, Theorem 3.3]. This completes the proof.

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