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WEYL'S THEOREM FOR ALGEBRAICALLY k-QUASICLASS A OPERATORS

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Abstract. If T or T^* is an algebraically k-quasiclass A operator acting on an infinite dimensional separable Hilbert space and F is an operator commuting with T, and there exists a positive integer n such that F^n has a finite rank, then we prove that Weyl's theorem holds for $f(T)+F$ for every $f \in H(\sigma(T))$, where $H(\sigma(T))$ denotes the set of all analytic functions in a neighborhood of $\sigma(T)$. Moreover, if T^* is an algebraically k-quasiclass A operator, then α-Weyl's theorem holds for $f(T)$. Also, we prove that if T or T^* is an algebraically k-quasiclass A operator then both the Weyl spectrum and the approximate point spectrum of T obey the spectral mapping theorem for every $f \in H(\sigma(T))$.

Keywords: algebraically k-quasiclass A operator, Weyl's theorem, α-Weyl's theorem.

Mathematics Subject Classification: 47B20.

1. INTRODUCTION

We begin with some standard notation on Fredholm theory. Throughout this paper let $\mathcal{H}$ be a separable complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $B(\mathcal{H})$ and $K(\mathcal{H})$ denote respectively, the C^*-algebra of all bounded linear operators and the ideal of compact operators acting on $\mathcal{H}$. If $T \in B(\mathcal{H})$, we shall write ker$T$ and ran$T$ for the null space and the range of T respectively. Also let $\alpha(T)$ =dim ker$T$, $\beta(T)$ =dim kerT^* and let $\sigma(T)$, $\sigma_a(T)$ denote the spectrum, approximate point spectrum of T, respectively. Let $p = p(T)$ be the ascent of T; i.e., the smallest nonnegative integer p such that $\ker T^p = \ker T^{p+1}$. If such an integer does not exist, we put $p(T) = \infty$. Analogously, let $q = q(T)$ be the descent of T; i.e., the smallest nonnegative integer q such that ran T^q =ranT^{q+1}, and if such an integer does not exist, we put $q(T) = \infty$. It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$. Moreover, $0 < p(\lambda - T) = q(\lambda - T) < \infty$ precisely when λ is a pole of the resolvent of T, see Heuser [20, Proposition 50.2]. An operator $T \in B(\mathcal{H})$ is called Fredholm if ran$T$ is closed and both ker$T$ and $\mathcal{H}/\text{ran}T$ are finite dimensional. The index of a Fredholm operator $T \in B(\mathcal{H})$, denoted by $i(T)$, is given by the integer

$$i(T) = \alpha(T) - \beta(T).$$

An operator $T \in B(\mathcal{H})$ is called Weyl if it is Fredholm of index zero and Brown if it is Fredholm of finite ascent and descent. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ of $T \in B(\mathcal{H})$ are defined by

$$\begin{aligned}\sigma_e(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\},\\ \sigma_w(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\},\\ \sigma_b(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\}.\end{aligned}$$

Let iso$\mathcal{K}$ denote the isolated points of $\mathcal{K} \subseteq \mathbb{C}$. We write

$$\begin{aligned}\pi_{00}(T) &= \{\lambda \in \text{iso}\sigma(T) : 0 < \alpha(T - \lambda) < \infty\},\\ \pi_{00}^a(T) &= \{\lambda \in \text{iso}\sigma_a(T) : 0 < \alpha(T - \lambda) < \infty\},\end{aligned}$$

and

$$p_{00}(T) = \sigma(T) \setminus \sigma_b(T).$$

It is evident that $\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \text{acc}\sigma(T)$ and $p_{00}(T) \subseteq \pi_{00}(T) \subseteq \pi_{00}^a(T)$, where $\text{acc}\sigma(T) = \sigma(T) \setminus \text{iso}\sigma(T)$. It is well known that $\sigma_w(T)$ is non-empty and

$$\sigma_w(T) = \bigcap\{\sigma(T + K) : K \in K(\mathcal{H})\}.$$

It is interesting to note that $\sigma_b(T)$ can be characterized in a way parallel to the definition of $\sigma_w(T)$:

$$\sigma_b(T) = \bigcap\{\sigma(T + K) : K \in K(\mathcal{H}) \ \text{ and } \ KT = TK\}.$$

We say that Weyl's theorem holds for $T \in B(\mathcal{H})$ if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),$$

and that Browder's theorem holds for $T \in B(\mathcal{H})$ if

$$\sigma(T) \setminus \sigma_w(T) = p_{00}(T),$$

that is,

$$\sigma_w(T) = \sigma_b(T).$$

By definition,

$$\sigma_{ea}(T) = \bigcap\{\sigma_a(T + K) : K \in K(\mathcal{H})\}$$

is the essential approximate point spectrum, and

$$\sigma_{ab}(T) = \bigcap\{\sigma_a(T + K) : K \in K(\mathcal{H}) \ \text{ and } \ KT = TK\}$$

is the Browder approximate point spectrum. Let

$$\Phi_+(\mathcal{H}) = \{T \in B(\mathcal{H}) : \text{ran}T \ \text{ is closed and } \ \alpha(T) < \infty\},$$

and

$$\Phi_+^-(\mathcal{H}) = \{T \in \Phi_+(\mathcal{H}) : \quad i(T) \leq 0\}.$$

In [29, Theorem 3.1], it was shown that

$$\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \Phi_+^-(\mathcal{H})\}.$$

We say that α-Weyl's theorem holds for $T \in B(\mathcal{H})$ if

$$\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}^a(T),$$

and that α-Browder's theorem holds for $T \in B(\mathcal{H})$ if

$$\sigma_{ea}(T) = \sigma_{ab}(T).$$

It is known [15,29] that if $T \in B(\mathcal{H})$ then we can express the implications between various Weyl's theorems and Browder's theorems in the following diagram.

$$\begin{tikzcd}
\alpha\text{-Weyl's theorem} \arrow[r] \arrow[d] & \text{Weyl's theorem} \arrow[d] \\
\alpha\text{-Browder's theorem} \arrow[r] & \text{Browder's theorem}
\end{tikzcd}$$

H. Weyl [32] discovered that Weyl's theorem holds for hermitian operators and it has been extended from hermitian operators to hyponormal operators and to Toeplitz operators by L.A. Coburn [8], to cohyponormal operators by V. Rakočević [29], and to seminormal operators by S.K. Berberian [6,7]. And this result was generalized for p-hyponormal operators by M. Chō, M. Iton and S. Ōshiro [9], for class A operators by A. Uchiyama [31], for algebraically hyponormal operators by Y.M. Han [22] and for algebraically paranormal operators by R.E. Curto and Y.M. Han [10]. Recently, II. J. An and Y.M. Han [2] showed that Weyl's theorem holds for algebraically quasi-class A operators. Recall that $T \in B(\mathcal{H})$ is p-hyponormal for $p > 0$ if $(T^*T)^p - (TT^*)^p \geq 0$ [1]; when $p = 1$, T is called hyponormal. T is called cohyponormal, if T^* is hyponormal. If T is either hyponormal or cohyponormal, then T is called seminormal. And T is called paranormal if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for all $x \in \mathcal{H}$ [16,17]. In order to discuss the relations between paranormal and p-hyponormal and log-hyponormal operators (T is invertible and $\log T^*T \geq \log TT^*$), T. Furuta, M. Ito and T. Yamazaki [18] introduced a very interesting class of operators: class A defined by $|T^2| - |T|^2 \geq 0$, where $|T| = (T^*T)^{\frac{1}{2}}$ which is called the absolute value of T and they showed that class A is a subclass of paranormal operators and contains p-hyponormal and log-hyponormal ones. I.H. Jeon and I. H. Kim [23] introduced quasi-class A (i.e., $T^*(|T^2| - |T|^2)T \geq 0$) operators as an extension of the notion of class A operators. In this paper, we extend this result to algebraically k-quasiclass A operators using different methods.

Definition 1.1. $T \in B(\mathcal{H})$ is called a k-quasiclass A operator for a positive integer k if

$$T^{*k}(|T^2| - |T|^2)T^k \geq 0.$$

For interesting properties of k-quasiclass A operators, see [20,30]. In [30], this class of operators is called quasi-class (A, k). We say that T is algebraically k-quasiclass A if there exists a nonconstant complex polynomial h such that $h(T)$ is k-quasiclass A.

Note that algebraically k-quasiclass A is preserved under translation by scalars and restriction to closed invariant subspaces.

In general, the following inclusions hold:

p-hyponormal $\subseteq$ class A $\subseteq$ quasi-class A $\subseteq$ k-quasiclass A $\subseteq$ algebraically k-quasiclass A.

Definition 1.2. An operator $T \in B(\mathcal{H})$ is said to have the single valued extension property (abbrev. SVEP) at $\lambda \in \mathbb{C}$ if for every open neighborhood $\mathcal{G}$ of λ, the only function $f \in H(\mathcal{G})$ such that $(T - \mu)f(\mu) = 0$ on $\mathcal{G}$ is $0 \in H(\mathcal{G})$, where $H(\mathcal{G})$ means the space of all analytic functions on $\mathcal{G}$ having values in $\mathcal{H}$. Trivially, every operator $T \in B(\mathcal{H})$ has SVEP at points of the resolvent $\rho(T) = \mathbb{C}\backslash\sigma(T)$; moreover, from the identity theorem for analytic functions we have that every operator $T \in B(\mathcal{H})$ has SVEP at points of the boundary of $\sigma(T)$. In particular, every operator has SVEP at the isolated points of its spectrum. When T has SVEP at each $\lambda \in \mathbb{C}$, say that T has SVEP.

The single valued extension property dates back to the early days of local spectral theory and was introduced by N. Dunford [11,12]. This property plays a basic role in local spectral theory, see the recent monograph of K.B. Laursen and M.M. Neumann [26] or P. Aiena [3].

Definition 1.3. An operator $U \in B(\mathcal{H})$ is said to be a quasiaffinity if it is injective and has dense range. The operator $S \in B(\mathcal{H})$ is called a quasiaffine transform of $T \in B(\mathcal{H})$, notation $S \prec T$, if there exists a quasiaffinity $U \in B(\mathcal{H})$ such that $TU = US$. If both $S \prec T$ and $T \prec S$ hold, then S and T are called quasisimilar.

The quasinilpotent part $H_0(T - \lambda)$ and the analytic core $K(T - \lambda)$ of $T - \lambda$ are defined by

$$H_0(T - \lambda) = \{x \in \mathcal{H} : \lim_{n\to\infty} \|(T - \lambda)^n x\|^{\frac{1}{n}} = 0\}$$

and

$$\begin{aligned} K(T - \lambda) = \{x \in \mathcal{H} : & \text{ there exists a sequence } \{x_n\} \subseteq \mathcal{H} \text{ and } \delta > 0 \\ & \text{for which } x = x_0, (T - \lambda)x_{n+1} = x_n \text{ and } \|x_n\| \leq \delta^n \|x\| \\ & \text{for all } n = 1, 2, \ldots\}. \end{aligned}$$

We note that $H_0(T - \lambda)$ and $K(T - \lambda)$ are generally non-closed hyperinvariant subspaces of $T - \lambda$ such that $\ker(T - \lambda)^n \subseteq H_0(T - \lambda)$ for all $n = 0, 1, 2, \ldots$ and $(T - \lambda)K(T - \lambda) = K(T - \lambda)$ [26].

Definition 1.4. An operator $T \in B(\mathcal{H})$ is said to be semi-regular if $\mathrm{ran}T$ is closed and $\ker T \subseteq T^\infty(\mathcal{H}) = \bigcap_{n\in N} \mathrm{ran}T^n$.

Definition 1.5. An operator $T \in B(\mathcal{H})$ admits a generalized Kato decomposition, GKD for short, if there exists a pair of T-invariant closed subspaces (M, N) such that $\mathcal{H} = M \oplus N$, the restriction $T|_M$ is quasinilpotent and $T|_N$ is semi-regular. An operator $T \in B(\mathcal{H})$ has a GKD at every isolated point of $\sigma(T)$. We say that T is of Kato type at a point λ if $(T-\lambda)|_M$ is nilpotent in the GKD for $T-\lambda$.

See for details [3]. Recall that semi-Fredholm operators are of Kato type [24, Theorem 4]. For more information on semi-Fredholm operators, semi-regular operators and Kato type operators, see [3, 26].

2. WEYL'S THEOREM FOR ALGEBRAICALLY k-QUASICLASS A OPERATORS

We start with the following lemmas which summarizes some basic properties of k-quasiclass A operators.

Lemma 2.1 ([20, 30])**.** *Let $T \in B(\mathcal{H})$ be a k-quasiclass A operator for a positive integer k and*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad on \quad \mathcal{H} = \overline{ran(T^k)} \oplus \ker T^{*k}$$

be 2×2 matrix expression. Assume that $ran(T^k)$ is not dense, then T_1 is a class A operator on $\overline{ran(T^k)}$ and $T_3^k = 0$. Furthermore, $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Lemma 2.2 ([30])**.** *Let $T \in B(\mathcal{H})$ be a k-quasiclass A operator for a positive integer k and $\sigma(T) = \{\lambda\}$. Then $T = \lambda$ if $\lambda \neq 0$, and $T^{k+1} = 0$ if $\lambda = 0$.*

In [13], B.P. Duggal and S.V. Djordjević showed that quasinilpotent algebraically p-hyponormal operators are nilpotent. Recently, R.E. Curto and Y.M. Han in [10], II.J. An and Y.M. Han in [2] extended this result to algebraically paranormal operators and algebraically quasi-class A operators respectively. In the following theorem we show a similar result for algebraically k-quasiclass A operators.

Theorem 2.3. *Let $T \in B(\mathcal{H})$ be a quasinilpotent algebraically k-quasiclass A operator for a positive integer k. Then T is nilpotent.*

Proof. Let h be a complex nonconstant polynomial h such that $h(T)$ is k-quasiclass A. If $\mathrm{ran}(h(T)^k)$ is dense, then $h(T)$ is a class A operator. Therefore T is an algebraically paranormal operator. We have that T is nilpotent by [10, 10, Lemma 2.2]. If $\mathrm{ran}(h(T)^k)$ is not dense, then by Lemma 2.1 we can represent $h(T)$ as the upper triangular matrix

$$h(T) = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \overline{\mathrm{ran}(h(T)^k)} \oplus \ker h(T)^{*k},$$

where T_1 is a class A operator on $\overline{\mathrm{ran}(h(T)^k)}$, $T_3^k = 0$ and $\sigma(h(T)) = \sigma(T_1) \cup \{0\}$. In fact, $\sigma(h(T)) = h(\sigma(T)) = \{h(0)\}$ for T is quasinilpotent. Since $\sigma(h(T)) = \sigma(T_1) \cup \{0\}$, we have $h(0) = 0$. Hence $h(T)$ is quasinilpotent. Since $h(T)$ is a

k-quasiclass A operator, by Lemma 2.2 $h(T)$ is nilpotent. Since $h(0) = 0$, we have $h(T) = \alpha T^m \prod_{i=1}^{n}(\lambda_i - T)$ for some natural number m and complex number α. Since $h(T)$ is nilpotent and all $\lambda_i - T$ are invertible, it follows that T is nilpotent. This completes the proof. □

Recently, many excellent mathematicians have studied the Weyl type theorem of linear bounded operators defined on Banach spaces. Some of the basic results established for the Banach adjoint $T^{'}$ are also true for the T^*. For example, by means of the classical Fréchet-Riesz representation theorem we can deduce that the SVEP for $T^{'}$ and T^* are equivalent. In the following proofs, we shall apply this case.

An operator $T \in B(\mathcal{H})$ is said to be isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T and polaroid if every isolated point of $\sigma(T)$ is a pole of the resolvent of T. In general, if T is polaroid then it is isoloid. In [20], we proved that every k-quasiclass A operator is isoloid. Moveover we can prove the following result.

Theorem 2.4. *Let $T \in B(\mathcal{H})$ be an algebraically k-quasiclass A operator for a positive integer k. Then T has SVEP, and both T and T^* are polaroid. In particular, both T and T^* are isoloid.*

Proof. Let h be a nonconstant complex polynomial such that $h(T)$ is k-quasiclass A. By [20, Theorem 2.6], we know that $h(T)$ has finite ascent for all complex numbers. So we have that $h(T)$ has SVEP by [25, Proposition 1.8]. Hence T also has SVEP by [26, Theorem 3.3.9].

Now, if λ is an isolated point of $\sigma(T)$, $M = K(T - \lambda)$ and $N = H_0(T - \lambda)$, then (M, N) is a GKD for $T - \lambda$. Since $(T - \lambda)|_N$ is quasinilpotent and algebraically k-quasiclass A, it follows that $(T - \lambda)|_N$ is nilpotent by Theorem 2.3. Hence $T - \lambda$ is of Kato type. The SVEP for T and T^* at λ implies that both $p(T - \lambda)$ and $q(T - \lambda)$ are finite by [4, Theorem 2.3]. Hence λ is a pole of the resolvent of T.

Analogously, we shall prove that T^* is polaroid. Let λ be an isolated point of $\sigma(T^*)$. Then $\overline{\lambda}$ is an isolated point of $\sigma(T)$ and hence by the first part of the proof we have that $\overline{\lambda}$ is a pole of the resolvent of T. Therefore there exists a natural number n such that $n = p(T - \overline{\lambda}) = q(T - \overline{\lambda})$. Hence we have $\mathcal{H} = \ker(T - \overline{\lambda})^n \oplus \operatorname{ran}(T - \overline{\lambda})^n$ and $\operatorname{ran}(T - \overline{\lambda})^n$ is closed. From this we have $\mathcal{H} = (\ker(T - \overline{\lambda})^n)^\perp \oplus (\operatorname{ran}(T - \overline{\lambda})^n)^\perp = \operatorname{ran}(T^* - \lambda)^n \oplus \ker(T^* - \lambda)^n$. Hence $p(T^* - \lambda) = q(T^* - \lambda) < \infty$, that is, λ is a pole of the resolvent of T^*. This completes the proof. □

Next we show that Weyl's theorem holds for $f(T)$, if T or T^* is an algebraically k-quasiclass A operator for a positive integer k.

Theorem 2.5. *Let T or T^* be an algebraically k-quasiclass A operator for a positive integer k. Then Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

Proof. Suppose that T is an algebraically k-quasiclass A operator for a positive integer k. We show first that Weyl's theorem holds for T. We use the fact [14, Theorem 2.2] that if T is polaroid then Weyl's theorem holds for T if and only if T has SVEP at points of $\lambda \notin \sigma_w(T)$. By Theorem 2.4, we have that T has SVEP and T is polaroid. Hence T satisfies Weyl's theorem.

We show that Weyl's theorem holds for $f(T)$. Since T is isoloid, by [27, Lemma] we have

$$\sigma(f(T))\backslash\pi_{00}(f(T)) = f(\sigma(T)\backslash\pi_{00}(T)) = f(\sigma_w(T)),$$

where the last equality holds since T satisfies Weyl's theorem. Since T has SVEP, by [4, Corollary 2.6], we have $f(\sigma_w(T)) = \sigma_w(f(T))$. Therefore we have

$$\sigma(f(T))\backslash\pi_{00}(f(T)) = \sigma_w(f(T)),$$

so Weyl's theorem holds for $f(T)$.

Suppose that T^* is an algebraically k-quasiclass A operator for a positive integer k. We show first that Weyl's theorem holds for T. We use the fact [4, Theorem 3.1] that if T or T^* has SVEP, then Weyl's theorem holds for T if and only if $\pi_{00}(T) = p_{00}(T)$. By Theorem 2.4, we have that T^* has SVEP. Hence it is sufficient to show that $\pi_{00}(T) = p_{00}(T)$. $p_{00}(T) \subseteq \pi_{00}(T)$ is clear, so we only need to prove $\pi_{00}(T) \subseteq p_{00}(T)$. Let $\lambda \in \pi_{00}(T)$. Then λ is an isolated point of $\sigma(T)$. Hence λ is a pole of the resolvent of T for T is polaroid by Theorem 2.4, that is, $p(\lambda-T) = q(\lambda-T) < \infty$. By assumption we have $\alpha(\lambda - T) < \infty$, so $\beta(\lambda - T) < \infty$. Hence we conclude that $\lambda \in p_{00}(T)$. Therefore Weyl's theorem holds for T. Since T^* has SVEP, by [4, Corollary 2.6], we have $f(\sigma_w(T)) = \sigma_w(f(T))$. Noting that T is isoloid, as in the proof of the first part, we have that Weyl's theorem holds for $f(T)$. This completes the proof. □

Corollary 2.6. *Let T or T^* be an algebraically k-quasiclass A operator for a positive integer k. If F is an operator commuting with T and for which there exists a positive integer n such that F^n has a finite rank, then Weyl's theorem holds for $f(T) + F$ for every $f \in H(\sigma(T))$.*

Proof. Suppose T or T^* is an algebraically k-quasiclass A operator for a positive integer k. By Theorem 2.4 and Theorem 2.5, we have that T is isoloid and Weyl's theorem holds for $f(T)$. Observe that if T is isoloid then $f(T)$ is isoloid. The result follows from [28, Theorem 2.4]. □

From the proof of Theorem 2.5, we have that the Weyl spectrum obeys the spectral mapping theorem for algebraically k-quasiclass A operators.

Corollary 2.7. *Let T or T^* be an algebraically k-quasiclass A operator for a positive integer k. Then for every $f \in H(\sigma(T))$, we have*

$$f(\sigma_w(T)) = \sigma_w(f(T)).$$

3. α-WEYL'S THEOREM FOR ALGEBRAICALLY k-QUASICLASS A OPERATORS

For $T \in B(\mathcal{H})$, it is well known that $\sigma_{ea}(f(T)) \subseteq f(\sigma_{ea}(T))$ is always true for every $f \in H(\sigma(T))$. We have the following result that the essential approximate point spectrum obeys the spectral mapping theorem for k-quasiclass A operators.

Theorem 3.1. *Let T or T^* be an algebraically k-quasiclass A operator for a positive integer k. Then*

$$\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$$

for every $f \in H(\sigma(T))$.

Proof. We only need to prove that $f(\sigma_{ea}(T)) \subseteq \sigma_{ea}(f(T))$ since $\sigma_{ea}(f(T)) \subseteq f(\sigma_{ea}(T))$ is always true for any operator.

Suppose first that T is an algebraically k-quasiclass A operator for a positive integer k and let $f \in H(\sigma(T))$. Assume that $\lambda \notin \sigma_{ea}(f(T))$. Then we have $f(T)-\lambda \in \Phi_+^-(\mathcal{H})$ and $f(T)-\lambda = g(T)\prod_{i=1}^n (T-\lambda_i)$, where $\lambda_i \in \mathbb{C}$, $i = 1, 2, \ldots, n$, and $g(T)$ is invertible.

Obviously, $\lambda \in f(\sigma_{ea}(T))$ if and only if $\lambda_i \in \sigma_{ea}(T)$ for some i. If $\lambda_i \notin \sigma_{ea}(T)$ for every $\lambda_i \in \{\lambda_i\}_{i=1}^n$, then we have $\lambda \notin f(\sigma_{ea}(T))$ and $f(\sigma_{ea}(T)) \subseteq \sigma_{ea}(f(T))$. In the following we shall prove that $\lambda_i \notin \sigma_{ea}(T)$ for every $\lambda_i \in \{\lambda_i\}_{i=1}^n$.

In fact, we have that every $T-\lambda_i \in \Phi_+(\mathcal{H})$ for $f(T)-\lambda \in \Phi_+(\mathcal{H})$. Since T is an algebraically k-quasiclass A operator, we have that T has SVEP by Theorem 2.4. Hence for each i we have $i(T-\lambda_i) \leq 0$ by [5, Theorem 2.6]. Therefore $T-\lambda_i \in \Phi_+^-(\mathcal{H})$, that is, $\lambda_i \notin \sigma_{ea}(T)$ for each i. As a consequence, $f(\sigma_{ea}(T)) \subseteq \sigma_{ea}(f(T))$.

Assume now that T^* is an algebraically k-quasiclass A operator. Then T^* has SVEP by Theorem 2.4. Then we have that $i(T-\lambda_i) \geq 0$ for each i by [5, Theorem 2.8]. Since

$$0 \leq \sum_{i=1}^n i(T-\lambda_i) = i(f(T)-\lambda) \leq 0,$$

we have $i(T-\lambda_i) = 0$ for each i. Since every $T-\lambda_i \in \Phi_+(\mathcal{H})$, we have $T-\lambda_i \in \Phi_+^-(\mathcal{H})$, that is, $\lambda_i \notin \sigma_{ea}(T)$ for each i. Hence $f(\sigma_{ea}(T)) \subseteq \sigma_{ea}(f(T))$. This completes the proof. $\square$

Theorem 3.2. *Let T^* be an algebraically k-quasiclass A operator for a positive integer k. Then α-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

Proof. Suppose T^* is an algebraically k-quasiclass A operator for a positive integer k. We first prove that α-Weyl's theorem holds for T. We use the fact [4, Theorem 3.6] that if T^* has SVEP, then T satisfies α-Weyl's theorem if and only if T satisfies Weyl's theorem. Since T^* is an algebraically k-quasiclass A operator, we have that T^* has SVEP by Theorem 2.4 and T satisfies Weyl's theorem by Theorem 2.5. So T satisfies α-Weyl's theorem.

Next, we shall prove that α-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$. Let $f \in H(\sigma(T))$. Since T satisfies α-Weyl's theorem, we have that α-Browder's theorem holds for T. Hence $\sigma_{ea}(T) = \sigma_{ab}(T)$. Since T^* is an algebraically k-quasiclass A operator, it follows from Theorem 3.1 that

$$\sigma_{ea}(f(T)) = f(\sigma_{ea}(T)) = f(\sigma_{ab}(T)) = \sigma_{ab}(f(T)),$$

and so α-Browder's theorem holds for $f(T)$. We use the fact [15, Theorem 3.8] that if T satisfies α-Browder's theorem then α-Weyl's theorem holds for T if $\text{ran}(T-\lambda)$ is closed for each $\lambda \in \pi_{00}^a(T)$. So it suffices to show that if $\lambda \in \pi_{00}^a(f(T))$, then $\mathrm{ran}(f(T)-\lambda)$ is closed. Let $\lambda \in \pi_{00}^a(f(T))$. Then λ is an isolated point of $\sigma_a(f(T)) = f(\sigma_a(T))$ and $0 < \alpha(f(T)-\lambda) < \infty$. Since λ is an isolated point of $f(\sigma_a(T))$, if $\alpha_i \in \sigma_a(T)$, then α_i is an isolated point of $\sigma_a(T)$ by $f(T)-\lambda = g(T)\prod_{i=1}^n (T-\lambda_i)$ in Theorem 3.1. Since T^* has SVEP, we have that $\sigma_a(T) = \sigma(T)$ by [19, Corollary 7]. But T is isoloid by Theorem 2.4, so we have $0 < \alpha(T-\alpha_i) < \infty$ for each $i = 1, 2, \ldots, n$. Since T satisfies α-Weyl's theorem, we have $\alpha_i \notin \sigma_{ea}(T)$ for each $i = 1, 2, \ldots, n$. Hence $\mathrm{ran}(f(T)-\lambda)$ is closed. This completes the proof. □

Theorem 3.3. *Let T be an algebraically k-quasiclass A operator for a positive integer k and $S \prec T$. Then α-Browder's theorem holds for $f(S)$ for every $f \in H(\sigma(S))$.*

Proof. Since T is an algebraically k-quasiclass A operator for a positive integer k, we have that T has SVEP by Theorem 2.4. Hence α-Browder's theorem holds for $f(S)$ for every $f \in H(\sigma(S))$ by [10, Theorem 3.3]. This completes the proof. □

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REFERENCES

[1] A. Aluthge, *On p-hyponormal operators for* $0 < p < 1$, Integral Equations Operator Theory **13** (1990), 307–315.

[2] II.J. An, Y.M. Han, *Weyl's theorem for algebraically quasi-class A operators*, Integral Equations Operator Theory **62** (2008), 1–10.

[3] P. Aiena, *Fredholm and Local Spectral Theory with Applications to Multipliers*, Kluwer, 2004.

[4] P. Aiena, *Classes of operators satisfying α-Weyl's theorem*, Studia Math. **169** (2005), 105–122.

[5] P. Aiena, O. Monsalve, *Operators which do not have single valued extension property*, J. Math. Anal. **250** (2000), 435–448.

[6] S.K. Berberian, *An extension of Weyl's theorem to a class of not necessarily normal operators*, Michigan Math. J. **16** (1969), 273–279.

[7] S.K. Berberian, *The Weyl's Spectrum of an operator*, Indiana Univ. Math. J. **20** (1970), 529–544.

[8] L.A. Coburn, *Weyl's theorem for non-normal operators*, Michigan Math. J. **13** (1966), 285–288.

[9] M. Chō, M. Iton, S. Ōshiro, *Weyl's theorem holds for p-hyponormal operators*, Glasg. Math. J. **39** (1997), 217–220.

[10] R.E. Curto, Y.M. Han, *Weyl's theorem for algebraically paranormal operators*, Integral Equations Operator Theory **47** (2003), 307–314.

[11] N. Dunford, *Spectral theory I*, Pacific J. Math. **2** (1952), 559–614.

[12] N. Dunford, *Spectral operator*, Pacific J. Math. **4** (1954), 321–354.

[13] B.P. Duggal, S.V. Djordjević, *Weyl's theorem in the class of algebraically p-hyponormal operators*, Comment. Math. Prace Mat. **40** (2000), 49–56.

[14] B.P. Duggal, *Polaroid operators satisfying Weyl's theorem*, Linear Algebra Appl. **414** (2006), 271–277.

[15] S.V. Djordjević, Y. M. Han, *Browder's theorems and spectral continuity*, Glasg. Math. J. **42** (2000), 479–486.

[16] T. Furuta, *On the class of paranormal operators*, Proc. Japan Acad. **43** (1967), 594–598.

[17] T. Furuta, *Invitation to Linear Operators*, Taylor and Francis, London, 2001.

[18] T. Furuta, M. Ito, T. Yamazaki, *A subclass of paranormal operators including class of log-hyponormal and several classes*, Sci. Math. **1** (1998) 3, 389–403.

[19] J.K. Finch, *The single valued extension property on a Banach space*, Pacific J. Math. **58** (1975), 61–69.

[20] F.G. Gao, X.C. Fang, *On k-quasiclass A operators*, J. Inequal. Appl. **2009** (2009), Article ID 921634, 1–10.

[21] H. Heuser, *Functional Analysis*, John Wiley and Sons, 1982.

[22] Y.M. Han, W.Y. Lee, *Weyl's theorem holds for algebraically hyponormal operators*, Proc. Amer. Math. Soc. **128** (2000), 2291–2296.

[23] I.H. Jeon, I.H. Kim, *On operators satisfying* $T^*|T^2|T \geq T^*|T|^2T$, Linear Algebra Appl. **418** (2006), 854–862.

[24] T. Kato, *Perturbation theory for nullity, deficiency and other quantities of linear operators*, J. Math. Anal. **6** (1958), 261–322.

[25] K.B. Laursen, *Operators with finite ascent*, Pacific J. Math. **152** (1992), 323–336.

[26] K.B. Laursen, M.M. Neumann, *Introduction to Local Spectral Theory*, Clarendon Press, Oxford, 2000.

[27] W.Y. Lee, S.H. Lee, *A spectral mapping theorem for the Weyl spectral*, Glasg. Math. J. **38** (1996), 61–64.

[28] M. Oudghiri, *Weyl's theorem and purturbations*, Integral Equations Operator Theory **53** (2005), 535–545.

[29] V. Rakočević, *On the essential approximate point spectral II*, Mat. Vesnik **36** (1984), 89–97.

[30] K. Tanahashi, I.H. Jeon, I.H. Kim, A. Uchiyama, *Quasinilpotent part of class A or* (p,k)*-quasihyponormal operators*, Oper. Theory Adv. Appl. **187** (2008), 199–210.

[31] A. Uchiyama, *Weyl's theorem for class A operators*, Math. Inequal. Appl. **4** (2001), 143–150.

[32] H. Weyl, *Über beschränkte quadratische Formen, deren diffierenz vollsteig ist*, Rend. Circ. Mat. Palermo **27** (1909), 373–392.

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